Neutron - Proton Scattering

Neutron - Proton scattering
(Low energy)

The relative motion of two particles having masses M_1 and M_2 can be described by the wave equation given by

$$-\frac{\hbar^2}{2\mu}\nabla^2\psi + V(r)\psi = E\psi$$

where μ is the rest mass, E is the internal energy of the system $\boxed{E = E_L - E_c}$

E_L — energy in the lab system
E_c — Kinetic energy of the centre of mass

$$E_c = \frac{M_1}{M_1+M_2}E_L$$

for n-p scattering $M_1 = M_2 = M$ (say)

so that
$$E_c = \frac{E_L}{2}$$

So only half the lab energy is available for scattering in the centre of mass system.

Relation between θ_c & θ_L

$$\theta_c = 2\theta_L$$

where θ_L is the angle of scattering in L system and θ_c is the angle of scattering in COM system.

Now the wave equation can be written as

$$\nabla^2 \psi + \frac{M}{\hbar^2}\left[E - V(r)\right]\psi = 0$$

where $\psi = \psi(r, \theta, \phi)$; θ and ϕ are the COM angles, r is the distance between neutron and proton, $E > 0$.

Method of Partial Waves

$$\psi_{inc} = \frac{1}{2iKr} \sum_{l=0}^{\infty} (2l+1) i^l \Big[\overbrace{\text{Exp}\, i\left(Kr - l\tfrac{\pi}{2}\right)}^{\text{outgoing wave}} - \underbrace{\text{Exp}\,{-i}\left(Kr - l\tfrac{\pi}{2}\right)}_{\text{Incoming wave}} \Big] P_l(\cos\theta)$$

$P_l(\cos\theta)$ is Legendre polynomial of order l

and $K^2 = \dfrac{2ME}{\hbar^2}$

when scatterer is present, the sph outgoing wave is affected either in Phase or in Amplitude or both. If elastic scattering is taking place (no reaction), then only phase is affected.

Total wave function when scatterer is present-

$$\psi(r) = \frac{1}{2ikr}\sum_{l=0}^{\infty}(2l+1)i^l\left[\eta_l e^{i(kr - l\frac{\pi}{2})} - e^{-i(kr - l\frac{\pi}{2})}\right] P_l(\cos\theta)$$

$$= \frac{1}{2ikr}\sum_{l=0}^{\infty}(2l+1)i^l\left[\eta_l e^{i(kr - l\frac{\pi}{2})} + e^{i(kr - l\frac{\pi}{2})} - e^{i(kr - l\frac{\pi}{2})} - e^{-i(kr - l\frac{\pi}{2})}\right] P_l(\cos\theta)$$

$$= \frac{1}{2ikr}\sum_{l=0}^{\infty}(2l+1)i^l\left[\eta_l e^{i(kr - l\frac{\pi}{2})} - e^{i(kr - l\frac{\pi}{2})}\right] P_l(\cos\theta)$$

$$+ \frac{1}{2ikr}\sum_{l}(2l+1)i^l\left[e^{i(kr - l\frac{\pi}{2})} - e^{-i(kr - l\frac{\pi}{2})}\right] P_l(\cos\theta)$$

$$= \psi_{inc} + \psi_{sc}$$

where $\psi_{sc} = \frac{1}{2ikr}\sum(2l+1)i^l\left[\eta_l e^{i(kr - l\frac{\pi}{2})} - e^{+i(kr - l\frac{\pi}{2})}\right] P_l(\cos\theta)$

$$\psi_{sc} = \frac{1}{r} f(\theta)\, e^{ikr}$$

$$\psi_{sc} = \frac{1}{r} \sum \frac{(\eta_l - 1)(2l+1)}{2ik} i^l e^{-il\frac{\pi}{2}} P_l(\cos\theta)\, e^{ikr}$$

$$f(\theta) = \frac{1}{2ik} \sum (2l+1) \left\{ e^{2i\delta_l} - 1 \right\} P_l(\cos\theta)$$

$$= \frac{1}{k} \sum_{l=0}^{\infty} (2l+1)\, e^{i\delta_l} \sin\delta_l\, P_l(\cos\theta)$$

$$\sin\delta_l = \frac{e^{i\delta_l} - e^{-i\delta_l}}{2i}$$

So the differential cross section will be

$$\sigma(\theta) = |f(\theta)|^2$$

$$= \frac{1}{K^2} \left| \sum_{l=0}^{\infty} (2l+1)\, e^{i\delta_l} \sin\delta_l \, P_l(\cos\theta) \right|^2$$

$$\sigma_{total} = \int \sigma(\theta)\, d\Omega = \int \sigma(\theta)\, 2\pi \sin\theta \, d\theta$$

$$= \frac{2\pi}{K^2} \int_0^{\pi} \left| \sum (2l+1)\, e^{i\delta_l} \sin\delta_l \, P_l(\cos\theta) \right|^2 \sin\theta \, d\theta$$

$$= \frac{2\pi}{K^2} \sum (2l+1)^2 \sin^2\delta_l \frac{2}{(2l+1)}$$

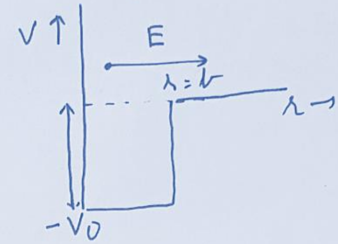
$$\boxed{\sigma_t = \frac{4\pi}{K^2} \sum (2l+1) \sin^2\delta_l}$$

Thus if we know phase shifts, we can calculate total cross section.

Nucleon-nucleon Scattering

The nucleon - nucleon scattering problem will be solved in centre of mass frame.

To solve nucleon - nucleon scattering problem using quantum mechanics we assume that interaction can be represented by square well potential, as we did in previous section for the deuteron. The only difference in this calculation and that of deuteron is that we are concerned with free incident particles with $E > 0$. We will simplify Schrodinger equation by assuming $l = 0$.



Energy criteria for energy for s wave scattering by neutrons in COM system.

If the incident particle has velocity v, it's angular momentum relative to the target is Mvb, where M is the mass of incident particle and b is the nuclear range. The relative ang. momenta between nucleons must be quantized in the units of $\hbar$; that $Mvb = l\hbar$ in semi-classical notation.

If $Mvb \ll \hbar$, then only $l = 0$ interactions are likely to occur. Thus $v \ll \frac{\hbar}{Mb}$ and the corresponding K.E is estimated as

$$T = \frac{1}{2} M v^2 \ll \frac{\hbar^2}{2 M b^2} = \frac{\hbar^2 c^2}{2 M c^2 b^2} = \frac{200 \times 200}{2 \times 1000 \times 4}$$

$$\boxed{T_{COM} < 5 \text{ MeV}}$$

$$T_{Lab} < 10 \text{ MeV}$$

Thus if the incident energy of projectiles is less than 10 MeV in Lab system (5 MeV in COM system), then only s wave scattering occurs.

The nucleon-nucleon prob will be solved in COM frame. Mass appearing in Schrodinger equation is the reduced mass, which is in the present case, half the nucleon mass.

Solution to the square well problem for $r < b$ as well as $r > b$

$$\frac{d^2u}{dr^2} + \frac{M}{\hbar^2}\left[E - V(r) - \frac{l(l+1)\hbar^2}{Mr^2}\right]u = 0$$

but $l = 0$

$$\frac{d^2u}{dr^2} + \frac{M}{\hbar^2}\left[E - V(r)\right]u = 0 \quad \Big\} \quad \begin{aligned} V(r) &= -V_0 \text{ for } r<b \\ &= 0 \text{ for } r>b \end{aligned}$$

$$u''_{in} + K_2^2 u_{in} = 0 \qquad \text{for } r < b$$

$$u''_{out} + K^2 u_{out} = 0 \qquad \text{for } r > b$$

where

$$K_2^2 = \frac{M}{\hbar^2}(E + V_0) \;\; \& \;\; K^2 = \frac{ME}{\hbar^2}$$

Since

$$\frac{d^2u}{dr^2} + \frac{M}{\hbar^2}\left[E - V(r)\right]u = 0$$

for $r < b$, $V = -V_0$

so $$\frac{d^2 u_{in}}{dr^2} + \frac{M}{\hbar^2}\left[E + V_0\right]u_{in} = 0$$

$$\Rightarrow \frac{d^2 u_{in}}{dr^2} + K_2^2 u_{in} = 0$$

$$u_{in} = A \sin K_2 r + A' \cos K_2 r$$

as $r \longrightarrow 0$, $u_{in} \longrightarrow 0$, hence $A' = 0$

$$\boxed{u_{in} = A \sin K_2 r}$$

Now for $r > b$, $V(r) = 0$

$$\frac{d^2 u_{out}}{dr^2} + \frac{ME}{\hbar^2} u_{out} = 0$$

$$\frac{d^2 u_{out}}{dr^2} + K^2 u_{out} = 0$$

$$u_{out} = B' \sin Kr + B'' \cos Kr$$

as

$$u_{out} = B \sin(Kr + \delta_l)$$

$$u_{out} = B \sin(Kr + \delta_0) \quad \text{for } l=0 \text{ case}$$

Hence

$$u_{in} = A \sin K_2 r$$

$$u_{out} = B \sin(Kr + \delta_0)$$

$$u_{in} = A \sin k_2 r$$

$$u_{out} = B \sin(kr + \delta_0)$$

Using boundary conditions on u and $\frac{du}{dr}$ at $r = b$

$$A \sin k_2 b = B \sin(kb + \delta_0)$$

$$A k_2 \cos k_2 b = B k \cos(kb + \delta_0)$$

$$\boxed{k_2 \cot k_2 b = k \cot(kb + \delta_0)}$$

Since E is known, so k and k_2 are known for a given b, we can estimate δ_0, which may be used to calculate σ_{total}.

$$k^2 = \frac{ME}{\hbar^2}, \quad k_2^2 = \frac{M}{\hbar^2}(E + V_0)$$

$$\sigma = \frac{4\pi \sin^2 \delta_0}{k^2}$$

Calculation of scattering cross section

$$\sigma = \frac{4\pi}{k^2} \sin^2\delta_0$$

since

$$k_2 \cot k_2 b = k \cot(kb + \delta_0)$$

$$\Longrightarrow \quad \sin\delta_0 = \frac{\sin kb\,(k \cot kb - k_2 \cot k_2 b)}{\sqrt{k^2 + k_2^2 \cot^2 k_2 b}}$$

for $E = 10\ \text{keV}$

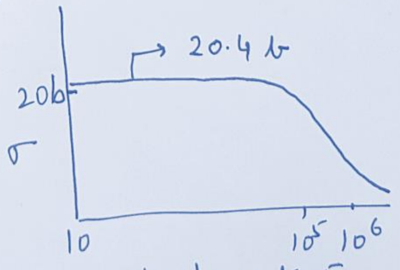
$$k_2 = \frac{\sqrt{M(V_0 + E)}}{\hbar} \simeq 0.92\ \text{fm}^{-1} \quad \text{for } V_0 = 35\ \text{MeV}$$

$$k = \sqrt{ME}/\hbar \simeq 0.016\ \text{fm}^{-1}$$

$$\sigma \simeq 4.6\ \text{b}$$

σ is in the range 4.6 to 5 b.

$(1\ b = 10^{-28}\ m^2)$



The low energy experimental values ($\simeq 20$ b) is not in agreement with our calculated value 5 barns for s wave scattering cross setion.

Solution to the discrepancy lies in the valative orientations of spins of incident and scattered nucleons.

The proton and neutron both spin $\frac{1}{2}$ particles can combine to give either 0 or 1 spin. $S=1$ combination has 3 substates, while $S=0$ has only one.

$$\sigma = \frac{3}{4}\sigma_t + \frac{1}{4}\sigma_s$$

for $\sigma_t = 4.6\,b$ & $\sigma = 20.4\,b$

we get $\sigma_s = 67.8\,b$

This result indicates that there is an enormous difference in singlet and triplet state — that is the nuclear force must be spin dependent.

Scattering length

Scattering amplitude $f(\theta)$ and total scattering cross section are given by

$$f_l(\theta) = \frac{1}{K} e^{i\delta_l} \sin\delta_l$$

$$\sigma = \int |f_l(\theta)|^2 d\Omega$$

for $l = 0$ i.e. s wave scattering

$$f(\theta) = \frac{1}{K} e^{i\delta_0} \sin\delta_0$$

hence total scattering cross section for $l=0$

$$\sigma = \frac{4\pi}{K^2} \sin^2\delta_0$$

In the limit of zero energy $(k \to 0)$
for scattering amplitude to remain finite,
δ should approach zero as fast as k
such that

$$\boxed{\lim_{k \to 0} |f(\theta)| = \lim_{k \to 0} \frac{\sin\delta}{k} = -a}$$

$$\lim_{k \to 0} \frac{\sin\delta}{k} = -a$$

Above equation defines the scattering length 'a' and it is negative of the absolute value of the scattering amplitude $f(\theta)$ in the limiting case of zero-energy scattering.

negative sign is necessary in order to make the scattering length a positive parameter for bound states.

Scattering length can also be defined as following

$$\sigma = \frac{4\pi}{K^2} \sin^2\delta_0 = \frac{4\pi}{K^2 + K^2 \cot^2\delta_0}$$

We can see that as $K \longrightarrow 0$, $\sigma \longrightarrow \infty$

To make σ finite, we assume that

$$K \cot \delta_0 = (-) \frac{1}{a_K}$$

So $\sigma = \dfrac{4\pi}{k^2 + \frac{1}{a_k^2}}$ $a_k \rightarrow$ scattering length

for σ to remain finite as $k \rightarrow 0$

$$\lim_{k \rightarrow 0} k \cot \delta_0 = \lim_{k \rightarrow 0} \left(-\frac{1}{a_k} \right) = -\frac{1}{a}$$

$a \longrightarrow$ Fermi scattering length

$$\sigma = \lim_{k \rightarrow 0} \frac{4\pi}{(k^2 + k^2 \cot^2 \delta_0)}$$

$$\boxed{\sigma = 4\pi a^2}$$

Hence

$$\lim_{k \to 0} |f(\theta)| = \frac{\delta_0}{k} = -a$$

$$\sigma = 4\pi a^2 = \pi (2a)^2$$

Hence Fermi scattering length is half the radius of a non penetrable sphere.

The modified radial wave function i.e. when the potential is included and the effect of the potential is to introduce a phase shift δ in the asymptotic wave function. The phase shift arises due to the distortion of the wave in the nuclear potential region $0 < r < b$.

Beyond the potential region

$$u(r) = \frac{e^{i\delta}}{k} \sin(kr+\delta)$$

as $k \longrightarrow 0$

$$u(r) = \lim_{k\to 0} \frac{e^{i\delta}}{k} \sin(kr+\delta)$$

as $k \longrightarrow 0$, $\delta \longrightarrow 0$

and $\sin(kr+\delta) \cong kr+\delta$

hence

$$u(r) = \frac{1}{k}(kr+\delta)$$

$$= r + \frac{\delta}{k}$$

but $\lim\limits_{k\to 0} \frac{\sin\delta}{k} \cong \frac{\delta}{k} = -a$

$$\Rightarrow \frac{\delta}{a} = -k \quad \text{or} \quad \frac{\delta}{k} = -a$$

Hence

$u(r) = (r-a)$ as $k \longrightarrow 0$.

In the limiting case when $k \to 0$

$$u(r) = r - a.$$

So the scattering length 'a' can also be defined as the distance from origin to the point of intersection of the modified radial function with r axis.

The scattering length can either be positive or negative for an attractive potential, but it's always positive for repulsive potential.

We can get the absolute value of scattering length but not the sign, from total scattering cross section in the limit of zero energy

Reference Books

- Nuclear Physics by S. N. Ghoshal
- Introductory Nuclear Physics by K. S. Krane

THANK

YOU