Unit-I 5/11

FkeâeF-I

2. (a) Define limit point of a set in a metric space. Show that a subset A is closed if and only if A contains all of its limit points.

ekeâmee oj ekeâ mecee° ce ekeâmee meceÛÛeÙe keâ meecee eyevo keâe hee j Yeee<e l e keâeepeS~ oMeeFS ekeâ keâeF meceÛÛeÙe A meJe l e n Ùeeo Deej keâJeue Ùeeo A ce Gmekeâ meYee meecee eyevo meceeenl e n~

(b) Define a complete metric space. Prove that if X be a complete metric space, then a subspace Y of x is complete if and only if it is closed.

ekeâmee heCe oj ekeâ mecee° keâe hee j Yeee<e l e keâeepeS~ emeæ keâeepeS ekeâ Ùeeo X Skeâ heCe oj ekeâ mecee° ne, l ee Gmekeâe Ghemecee° Y Yee Skeâ heCe oj ekeâ mecee° neiee Ùeeo Deej keâJeue Ùeeo Jen meJe l e ne~

3. (a) Prove that if a function f(x) is bounded and integrable-R in [a,b], then |f(x)| is also integrate-R in [a,b].

emeæ keâeepeS ekeâ Ùeeo keâeF Heâueve f(x) Devl e j eue [a,b]

A (Printed Pages 8)

Roll No. ____________

S-677

B.A. & B.Sc. (Part-III) Examination, 2015

MATHEMATICS

First Paper

(Analysis)

Time Allowed : Three Hours] ***[Maximum Marks :*** B.A. : 35, B.Sc. : 75

Note : Attempt five questions in all, choosing one question from each unit. Question No. 1 is compulsory.

Øel Ùekeâ FkeâeF me Skeâ ØeMve Ûevel e nS, keâue heeÜe ØeMvee keâe nue keâeepeS~ ØeMve me. 1 DeeveJeeÙe n~

1. Attempt all parts : 15/30

meYee Yeei e nue keâeepeS :

(a) Define an open sphere and a closed sphere in a metric space.

ekeâmee oj ekeâ mecee° ce Skeâ eJeJe l e ieeue Skeâ meJeòe ieeue keâe hee j Yeee<e l e keâeepeS~

(b) Define Dirichlet's test for alternating series and show that the series $\sum_{n=2}^{\infty} \frac{\sin n}{\log n}$ is convergent.

Skeâevle jCe ßeCee Jeeuee e[ejÛeueš peeÛe keâe jYeee<ele keâeepeS leLee emeæ keâeepeS ekeâ ßeCee $\sum_{n=2}^{\infty} \frac{\sin n}{\log n}$ DeeYemeeje n~

(c) Find the radius of convergence of the following power series.

evecve leele ßeCee keâe DeeYemeejle e$epÙee %eele keâeepeS :

$$\sum \frac{(2n)!}{(n!)^2} x^n$$

(d) Test the convergence of the following improper integral.

evecve DeveeÛele meceekeâue keâe DeeYemeeejlee keâe hej#eCe keâeepeS :

$$\int_1^{\infty} \frac{x^3}{(a^3+x^3)^2} dx$$

(e) Show that an even function can have no sine terms in its Fourier expansion.

oMeeFS ekeâ ekeâmee mece Heâueve keâe Heâeej Ùej ßeCee ce keâeF pÙee hej vene nelee n~

(f) Find all values of z such that $e^z = 1 + i\sqrt{3}$.

z keâ meYee ceeve %eele keâeepeS peye ekeâ $e^z = 1 + i\sqrt{3}$ n~

(g) Show that the following function is analytic everywhere :

oMeeFS ekeâ evecve Heâueve meJe$e JeMuee<ekeâ n :

$e^x (\cos y + i \sin y)$

(h) Prove that :

emeæ keâeepeS : $\int_C \bar{z}\, dz = -\pi i$

C is the upper half of circle |z|=1 from z=−1 to z = 1.

pene C Jele |z|=1 keâe z=−1 me z = 1 lekeâ keâe Theje DeeOee Yeeie n~

(i) Find the residue of the following function at z= 0 :

evecve Heâueve keâe z= 0 hej DeJeMe<e %eele keâeepeS :

$$f(\tau) = \frac{e^z}{z^4}$$

(j) Find the fixed points of the following transformation :

evecve ¤heevlejCe keâ Dehee jJeleveeÙe eyevo %eele keâeepeS :

$$w = \frac{3z+4}{z-1}$$

n :

$$\int_C \frac{\sin 3z}{z+\pi/2}\,dz$$

Unit-IV 5/12

FkeâeF-IV

8. (a) Expand in Laurent Series valid for $1<|z|<3$ the following function :

evecve Heâueve keâe ueejeve ßeCee ce eJemleej keâeepeS, pee $1<|z|<3$ keâ eueS he° n :

$$f(z) = \frac{1}{(z+1)\,(z+3)}$$

(b) Define the zeros of an analytic function and prove that they are isolated.

ekeâmee JeMuee<ekeâ Heâueve keâ eueS MevÙekeâ heejYeee<ele keâeepeS leLee emeæ keâeepeS ekeâ Je eJeÙeòeâ nele n~

9. (a) State and prove Cauchy's residue theorem.

keâeMee keâ DeJeMe<e ØeceÙe keâe keâLeve keâeepeS SJe emeæ keâeepeS~

(b) Show that : oMeeFS ekeâ :

$$\int_0^\infty \frac{x^2}{(x^2+1)\,(x^2+4)}\,dx = \frac{\pi}{6}$$

ce heejyeæ SJe jeceeve-meceekeâueveeÙe ne lee |f(x)| Yee [a,b] ce jeceeve-meceekeâueveeÙe neiee~

(b) Calculate the value of upper and lower Riemann integrals for f(x) in the interval [0,1] :

Heâueve f(x) keâ GÛÛe Deej evecve jeceeve-meceekeâuevee keâe ieCevee Devlejeue [0,1] ce keâeepeS :

$$f(x) = \sqrt{1-x^2}\ ,\ x \text{ rational (heejceÙe)}$$

$$= 1 - x\ ,\ x \text{ irrational (DeheejceÙe)}$$

Unit-II 5/11

FkeâeF-II

4. (a) Define uniform convergence of a sequence of function. Show that the following sequence is not uniformly convergent on R :

ekeâmee Deve›eâce keâ Skeâ meceeve-DeeYemejCe keâe heejYeee<ele keâeepeS~ oMeeFS ekeâ evecve Deve›eâce, R hej Skeâ meceeve DeeYemeejle vene n :

$$f_n(x) = \left\{\frac{nx}{1+n^2x^2}\right\}$$

(b) Show by giving sufficient reasons for any x :

heÙeehle keâejCee keâe ole nS oMeeFS ekeâ ØelÙ keâ x keâ euеS -

$$\frac{d}{dx}\left[\sum_{n=1}^{\infty}\frac{1}{n^3(1+nx^2)}\right] = -2x\sum_{n=1}^{\infty}\frac{1}{n^2(1+nx^2)^2}$$

5. (a) Prove that :

emeæ keâeepeS ekeâ :

$$\int_0^{\pi/2}\log\left(\frac{a+b\sin\theta}{a-b\sin\theta}\right)\operatorname{cosec}\theta\, d\theta = \pi\sin^{-1}\left(\frac{b}{a}\right) \quad (a > b)$$

(b) Prove that $f_{xy} \neq f_{yx}$ at the origin for the following function :

emeæ keâeepeS ekeâ ceue eyevo hej Heâueve keâe $f_{xy} \neq f_{yx}$:

$$f(x,y) = x^2\tan^{-1}\left(\frac{y}{x}\right) - y^2\tan^{-1}\left(\frac{x}{y}\right), \quad x \neq 0, y \neq 0$$

$= 0$ else where

Unit-III 5/11

FkeâeF-III

6. (a) Define an analytic function of a complex variable.

If $f(z) = u(x,y) + iv(x,y)$ is analytic in a domain D, then show that at each point $z = x+iy$ in D :

meecceße Ûej keâ JeMuee<ekeâ Heâueve keâe heejYee<ee oeepeS~ Ùeeo $f(z) = u(x,y) + iv(x,y)$ Skeâ #e$e D ce JeMuee<ekeâ

n, lee oMeeFS ekeâ ØelÙekeâ eyevo $z = x+iy$ hej :

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

(b) Define elementary transformation. Determine the region in w-plane of the area of z-plane bounded by the lines $x=0$, $y=0$, $x=1$, $y=2$ mapped under the transformation given below :

ØeejecYekeâ ¤heevlejCe keâe heejYee<ele keâeepeS~ evecve ¤heevlejCe keâe ØeÙeeie keâjle nS z-leue keâ #e$e pee jKeeDee $x=0$, $y=0$, $x=1$, $y=2$ Éeje mecyeæ n w-leue ce Øeleeyeeyele keâeepeS Ùeeo :

$$w = z + (2 - i)$$

7. (a) If f(z) is an analytic function within a closed contour c and z_0 is any point within c, prove the following :

Ùeeo Heâueve f(z) Skeâ mejue meJele keâvšj c keâ Devoj Deej Gmekeâ Thej JeMuee<ekeâ ne Deej c keâ Devoj z_0 keâeF eyevo ne lee emeæ keâeepeS :

$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z - z_0}\, dz$$

(b) Evaluate the following if c is a circle $|z| = 5$.

evecve keâe ceeve %eele keâeepeS pene c Skeâ Jeòe $|z| = 5$