(j) If $f(\alpha, \beta)$ is a symmetric bilinear form and $q(\alpha) = f(\alpha, \alpha)$ then prove that $q(\alpha + \beta) - q(\alpha - \beta) = 4 f(\alpha, \beta)$.

Ùeeo $f(\alpha, \beta)$ Skeâ meceecele eÉjeKeue meceleele ne leLee $q(\alpha) = f(\alpha, \beta)$ lee emeæ keâeepeÙe ekeâ $q(\alpha + \beta) - q(\alpha - \beta) = 4 f(\alpha, \beta)$

Unit-I / FkeâeF-I 5/11

2. (a) Prove that the set of all automorphisms of a group forms a group.

emeæ keâeepeÙe ekeâ ekeâmee mecen keâe mecemle mJeekeâejekeâj leeDee keâe meceÛÛeÙe Yee Skeâ mecen nelee n~

(b) State and prove Sylow's Second Theorem.

meeFuee keâe eÉleeÙe ØeceÙe keâe GuueKe keâeepeÙe leLee Gme emeæ keâeepeÙe~

3. (a) Prove that the centre Z of a group is a normal subgroup of G.

emeæ keâeepeÙe ekeâ ekeâmee mecen G keâe mecen keâvõ Z mecen G keâe Skeâ ØeemeeceevÙe Ghemecen nelee n~

(b) If p is a prime number and p divides D(G), the order of the group G, then prove that G has an element of order p.

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Roll No. ____________

S-684

B.A./B.Sc. (Part-III) Examination, 2015

(Old Course)

MATHEMATICS

Second Paper

(*Abstract Algebra*)

Time Allowed : Three Hours] ***[Maximum Marks :*** B.A. : 35, B.Sc. : 75

Note : Attempt five questions in all, selecting one question from each unit. Question No. 1 is compulsory.

ØelÙekeâ FkeâeF me Skeâ ØeMve Ûevele nS, keâue heeÜe ØeMvee keâ Gòej oeepeS~ ØeMve me.1 DeeveJeeÙe n~

1. (a) Show that the normalizer N(a) of the element 'a' of a group G is subgroup of G.

oMeeFÙe ekeâ ekeâmee mecen G keâ DeJeÙeJe a keâe ØeemeeceevÙekeâ N(a) mecen G keâe Ghemecen nelee n~ 15/30

(b) If a is a fixed element of a group G, then show that the mapping $f_a : G \to G$ defined by $f_a(x) = x^{-1}\, a\, x$ is an automorphism on G.

Ùeeo a mecen G keâe Skeâ eveeMÛeleâ DeJeÙeJe ne lee oMeeFÙe ekeâ ØeeleeÛe$eCe $f_a : G \to G$, pee $f_a(x) = x^{-1}\, a\, x$ Éeje heejYeeef<ele n, G hej Skeâ mJeekeâeejekeâjlee n~

(c) If $\phi : R \to R'$ be a homomorphism on the ring R into the ring R' with kernel K, then show that K is a subgroup of R under addition.

Ùeeo $\phi : R \to R'$ JeueÙe R me JeueÙe R' ce Skeâ meceekeâeejlee ne epemekeâe Dee° K ne lee oMeeFÙe ekeâ K JeueÙe R keâe Skeâ Ùeeieelcekeâ Ghemecen n~

(d) Prove that a commutative ring with unity is a field if it has no proper ideals.

emeæ keâeepeÙe ekeâ Skeâ FkeâeF Oeejkeâ ›eâceeJeeveceÙe JeueÙe Skeâ #e$e neiee Ùeeo Gmekeâe keâeF GeÛele ieCepeeJeuee ve ne~

(e) Prove that every Euclidean Ring possesses unity element.

emeæ keâeepeÙe ekeâ ØelÙekeâ Ùeekeue[eÙeve JeueÙe ce FkeâeF DeJeÙeJe eJeoceeve nelee n~

(f) Let w_1 and w_2 be subspaces of dimensions p and q, respectively, of a vector space V(F) such that $w_1 \cap w_2 = \{0\}$, then find the dimension of the subspace $w_1 + w_2$.

Ùeeo w_1 leLee w_2 ekeâmee meeoMe mecee° V(F) keâ ›eâceMe: p leLee q eJecee Jeeue Ghemecee° ne leLee $w_1 \cap w_2 = \{0\}$ ne lee Ghemecee° $w_1 + w_2$ keâe eJecee %eele keâeepeÙe~

(g) Check wether the set $\{(1,0,0),(1,1,0,),(1,1,1)\}$ is a basis of $R^3(R)$.

heje#eCe keâeepeÙe ekeâ keÙee meceÛÛeÙe $\{(1,0,0),(1,1,0,),(1,1,1)\}$ $R^3(R)$ keâe Skeâ DeeOeej n~

(h) Show that the mapping $T : R^3 \to R^2$ given by $T(x,y,z) = (x+y,\ 2z - x)$ is a linear transformation.

oMeeFS ekeâ $T(x,y,z) = (x+y,\ 2z - x)$ Éeje eve™ehele ØeeleeÛe$eCe $T : R^3 \to R^2$ Skeâ jeKekeâ ™heelejCe n~

(i) If α, β are vectors is an inner product space then prove that $\| \alpha + \beta \| \le \| \alpha \| + \| \beta \|$.

Ùeeo α, β Skeâ Deelej ieCeve mecee° keâ DeJeÙeJe ne lee emeæ keâeepeÙe ekeâ $\| \alpha + \beta \| \le \| \alpha \| + \| \beta \|$

Ùeeo B leLee B' Skeâ n eJeceeÙe heejecele mecee° V(F) keâ oe ›eâecele DeeOeej ne, f mecee° v hej Skeâ eÉjeKekeâ meceleele ne leLee p #e$e f hej heejYeee<ele n × n Skeâ Smee DeeJÙen n ekeâ $[\alpha]_B = P[\alpha]_{B'}, \forall \alpha \in V$, lee emeæ keâeepeÙe ekeâ $[f]_{B'} = P^t[f]_B P$.

(b) Let v be a complex vector space and f be a form on v such that f(α, α) is real for every $\alpha \in v$ then prove that f is Hermitian.

Ùeeo f Skeâ meecceße meeoMe mecee° v hej Skeâ Smee meceleele ne ekeâ v keâ mecemle DeJeÙeJee α keâ eueÙe f(α, α) JeemleeJekeâ ne lee emeæ keâeepeÙe ekeâ f necešeÙe neiee~

9. (a) State and prove the polarization identity for the complex inner product space.

meecceße Deelej ieCeve mecee° keâ eueÙe OeJeCe lelmecekeâ keâe GuueKe keâeepeÙe leLee Gme emeæ keâeepeÙe~

(b) Apply Gram-Schmidt process to the set of vectors {(1,0,1), (1,0,–1), (0,3,4)} to obtain an orthonormal basis of R^3 with respect to the standard inner product defined on R^3.

meeoMe keâ meceÛÛeÙe {(1,0,1), (1,0,–1), (0,3,4)} hej «eece-eMeceš uecyekeâekeâjCe Øe›eâce keâe ØeÙeeie keâjkeâ R^3 hej heejYeee<ele ceevekeâ Deelej ieCeveHeâue keâ meehe#e R^3 keâe Skeâ ØemeeceevÙe ueeeyekeâ DeeOeej keâe ieCevee keâeepeÙe~

Ùeeo p Skeâ DeYeepÙe meKÙee n leLee p mecen G keâe keâeeš D(G) keâe eJeYeeepele keâjlee n lee emeæ keâeepeÙe ekeâ G ce p keâeeš keâe Skeâ DeJeÙeJe eJeÅeceeve neiee~

Unit-II / FkeâeF-II 5/11

4. (a) If F is a field, prove that its only ideals are {0} and F. Deduce that a homomorphism of a field is either an isomorphism or maps each of its element into its zero.

Ùeeo F Skeâ #e$e ne lee emeæ keâeepeÙe ekeâ Fmekeâ i peeJeuee keâJeue {0} leLee F ne neie~ Dele: eveieceve keâeepeÙe ekeâ ekeâmee #e$e keâe meceekeâeejlee Ùee lee leuÙeekeâeejlee nelee n Ùee ØelÙekeâ DeJeÙeJe keâe Gmekeâ MevÙe hej ØeeleæÛe ele keâjlee n~

(b) Let R be an Euclidean Ring, then prove that every nonzero element in R is either a unit in R or can be written as a product of a finite number of elements of R.

Ùeeo R Skeâ Ùeekeue[eÙeve JeueÙe ne emeæ keâeepeÙe ekeâ R keâe ØelÙekeâ DeJeÙeJe Ùee lee R lelmecekeâ neiee Ùee Fme R keâ heejecele meKÙee ce DeJeÙeJee keâ ieCeveHeâue keâ ™he ce eueKee pee mekeâlee n~

5. (a) Prove that if R is a unique factorization domain, then the product of two primitives polynomials in R[x] is again a primitive polynomial in R[x].

emeæ keâeepeÙe ekeâ Ùeeo R Skeâ DeeÉleeÙe ieCeveKeC[ve Øeevle ne lee R[x] keâ oe heJeie yenheoe keâe ieCeveHeâue R[x] ce Skeâ heJeie yenheo neiee~

(b) Prove that the polynomial $f(x) = 1 + x + x^2 + \text{----} + x^{p-1}$, where p is a prime number, is irreducible over the field of rational numbers.

emeæ keâeepeÙe ekeâ yenheo $f(x) = 1 + x + x^2 + \text{----} + x^{p-1}$, pene p Skeâ DeYeepÙe meKÙee n, heejceÙe meKÙeeDee keâ #e$e ce DeKe[veeÙe yenheo neiee~

Unit-III / FkeâeF-III 5/11

6. (a) If w is a subspace of a finite dimensional vector space v than prove that w is finite dimensional and dim w < dim v.

Ùeeo w Skeâ heejcele eJeceeÙe meeoMe mecee° v keâe Ghemecee° ne lee emeæ keâeepeÙe ekeâ w heejcele eJeceeÙe neiee leLee dim w < dim v.

(b) Prove that any set of linearly independent vectors in a finite dimensional vector space V(F) can be extended to a basis of V(F).

emeæ keâeepeÙe ekeâ ekeâmee heejcele eJeceeÙe meeoMe mecee° V(F) keâe keâeF Yee Skeâleelele: mJele$e meceÛÛeÙe V(F) keâ Skeâ DeeOeej ce eJemlele ekeâÙee pee mekeâlee n~

7. (a) Let V(F) and W(F) finite dimensiond be vector spaces and $T : V \rightarrow W$ be a linear transformation, then prove that the range of T is a subspace of W(F) and $N = \{\alpha \in V \,/\, T\alpha = 0\}$ is a subspace of V(F).

Ùeeo V(F) leLee W(F) heejcele eJeceeÙe meeoMe mecee° ne Deej $T : V \rightarrow W$ Skeâ jeKeÙe Øeeleeeeˆ$eCe ne lee emeæ keâeepeÙe ekeâ T keâe heejmej W(F) keâe Ghemecee° leLee $N = \{\alpha \in V \,/\, T\alpha = 0\}$ V(F) keâe Ghemecee° neiee~

(b) For the basis B={(1,–2,3),(1,–1,1),(2,–4,7)} of $V^3(c)$, find the dual basis of B.

$V^3(c)$ keâ DeeOeej B={(1,–2,3),(1,–1,1),(2,–4,7)} keâe Éle DeeOeej %eele keâeepeÙe~

Unit-IV / FkeâeF-IV 5/12

8. (a) Let B and B' be two ordered bases of a finite n dimensional vector space V(F), f be a bilinear form on V and P be the $n \times n$ invertible matrix over F such that $[\alpha]_B = P[\alpha]_{B'}, \forall\, \alpha \in V$, then prove that $[f]_{B'} = P^t [f]_B P$.

 P.T.O.