Department of Physics
M.Sc. (Physics), Semester-IV
Elective Paper (PHYE – 401)
Physics at LHC and Beyond
Unit – III

by

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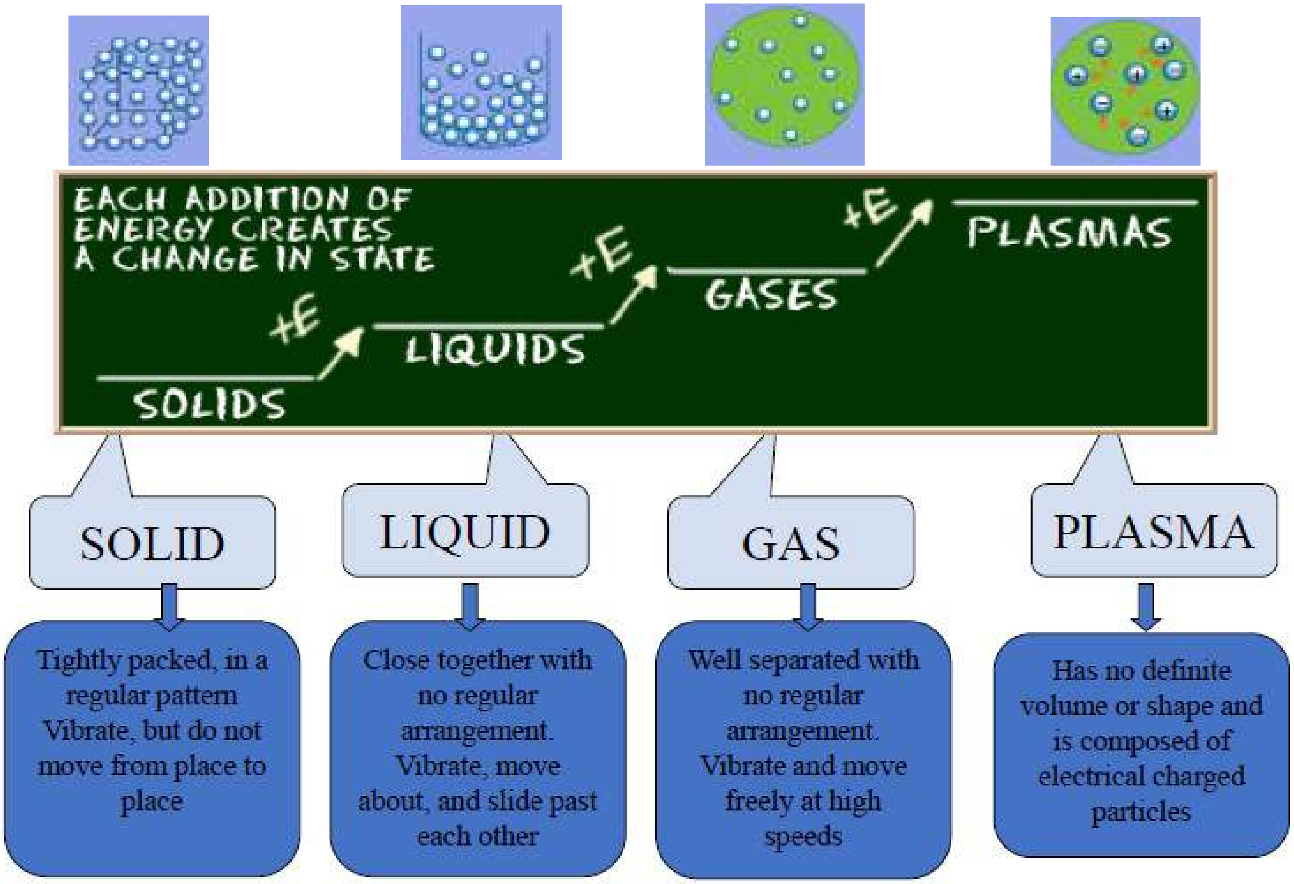
Syllabus

Need of the hour: Plasma-based accelerators. Definition and characteristics of plasma, collective behavior, plasma oscillations, plasma frequency. Propagation of electromagnetic waves in plasma, plasma electron quiver velocity, current density, linear dispersion relation, phase and group velocity, refractive index. Under dense, critically dense and over dense plasmas. Interaction of plasma with intense laser radiation fields, nonlinear interaction-relativistic quiver velocity, dispersion, ponderomotive forces.

Note : This powerpoint presentation is followed by notes.

What is PLASMA ?



• When gas becomes ionized, its dynamical behaviour is influenced by external electrical and magnetic fields.

• The separation of charges within the ionized gas, brings in new forces and its properties become different from those of neutral atoms and molecules.

Sir William Crookes

• Was the first to discover 'Plasma' in 1879 in a *Crookes Tube.*

• called it 'radiant matter'

Crooke's Tube

History of the term “PLASMA”

- 19th Century – Jan Evangelista Purkinje (Czech physiologist) – used the Greek word plasma (meaning “formed” or “molded”) – the clear fluid that remains after removal of all the corpuscular material in blood.

- 1922 – Irving Langmuir – proposed that the electrons, ions, and neutrals in an ionized gas could similarly be considered as corpuscular material entertained in some kind of fluid medium and called this entraining medium plasma.

- Ever since, plasma scientists have had to explain friends and acquaintances that they are not studying blood.

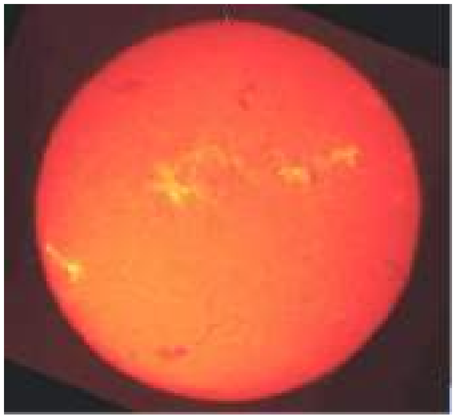
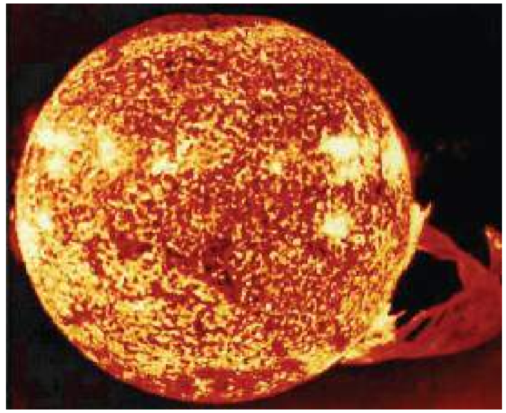
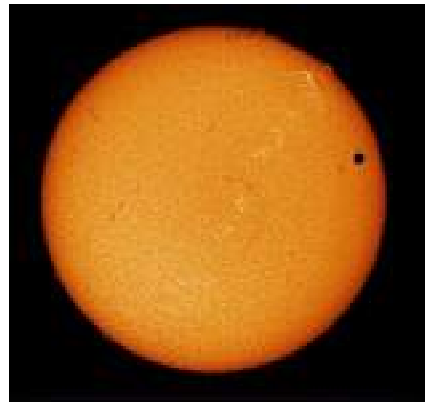
Definition of a PLASMA

The word *plasma* is used to describe a wide variety of macroscopically neutral substances containing many interacting free electrons and ionized atoms or molecules, which exhibit collective behavior due to the long-range coulomb forces.

- Not all media containing charged particles, however, can be classified as plasma. For a collection of interacting charged and neutral particles to exhibit plasma behavior it must satisfy certain conditions, or criteria, for plasma existence.

Where do we find Plasma ?

❖ All stars, including the Sun, are in Plasma state

❖ The interplanetary medium is in Plasma state

❖ The interstellar, intergalactic medium is in Plasma state.



Where else do we find Plasma ?

- Everywhere ! In fact, > 99% of the Universe consists of Plasma

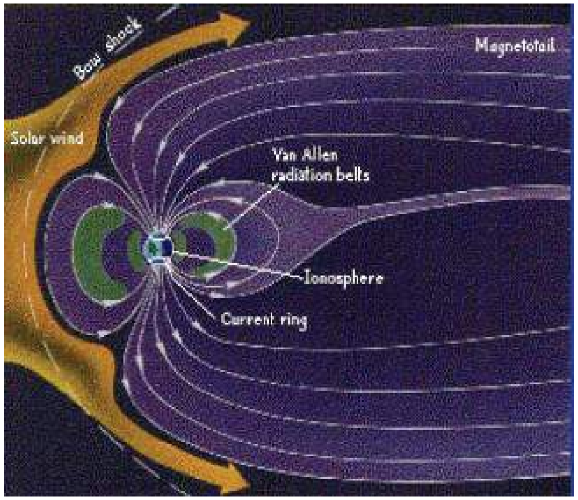
- Our Planet Earth is one of the few places in the Universe where the other three states of matter dominate over the fourth state : Plasma

- Where do we find plasma near Earth?

- A) Solar wind:

  This is plasma from the Sun, travelling outwards with a speed of 500 km/s.

  It gets deflected by the Earth's magnetic field.

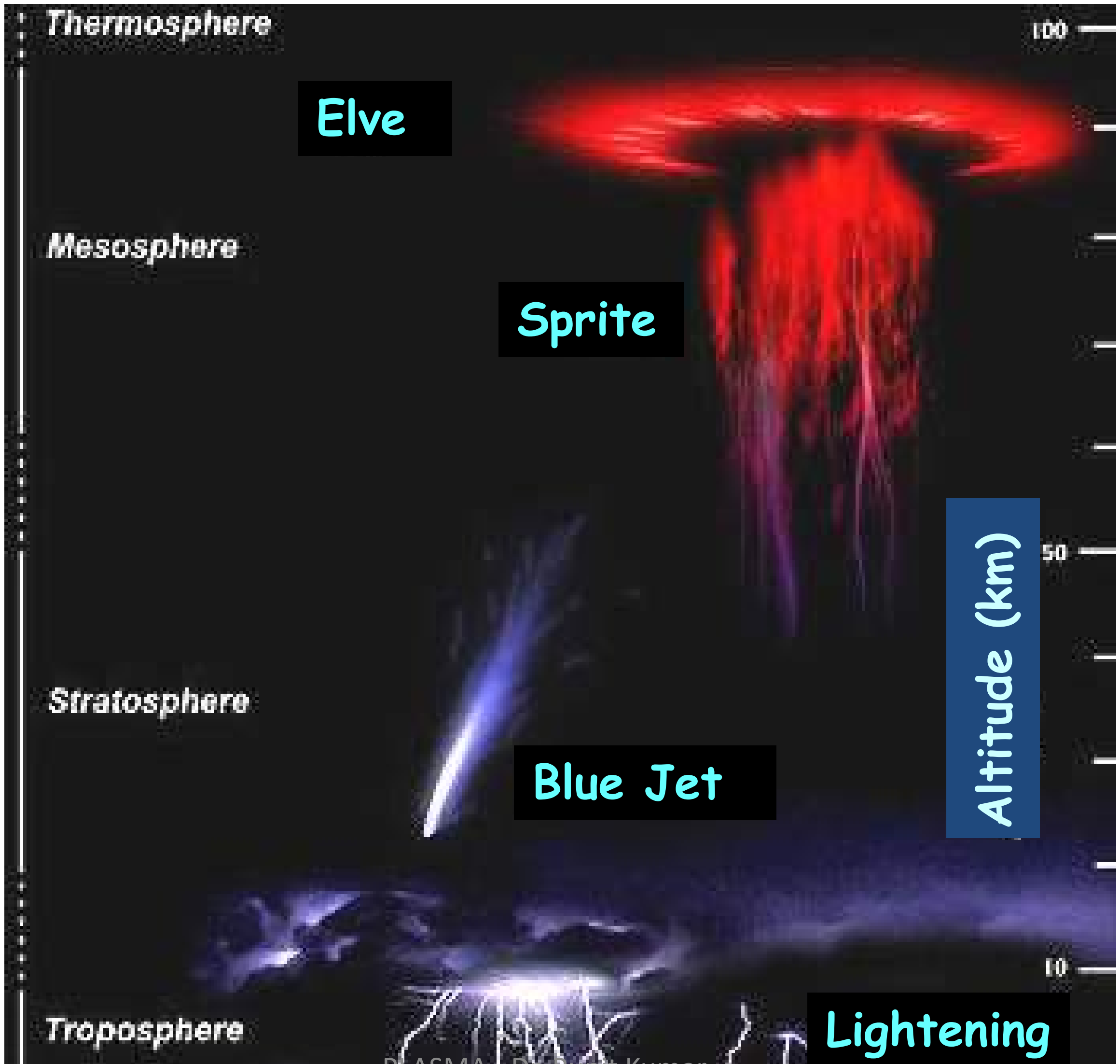


Plasma near Earth (Contd.)

b) Lightening

Plasma near Earth (Contd.)

c) Sprites, Jets, Elves



Plasma near Earth (Contd.)

d) Auroras

- Auroras, are also called the northern and southern lights

- They are natural light displays in the sky, usually observed at night, particularly in the polar regions (north as well as south).

- In northern latitudes - aurora borealis (Roman "goddess of dawn" : Aurora, and the Greek name for "north wind" : Boreas)

- Its southern counterpart - aurora australis is visible from high southern latitudes in Antarctica, South America, or Australasia. *Australis* is the Latin word for "South."

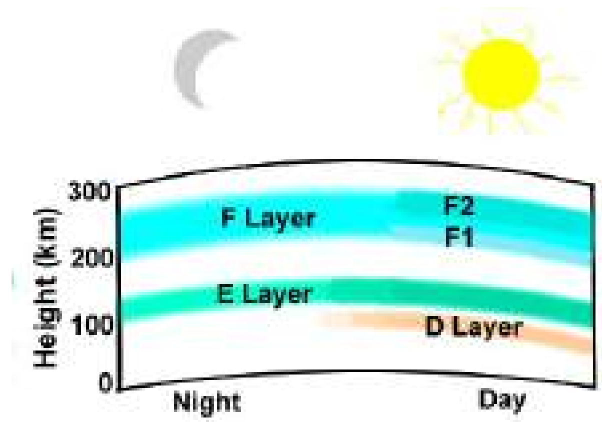
Plasma near Earth (Contd.)

e) Ionization Belts

- The *ionosphere* is a shell of ionized gas that surrounds the Earth, stretching from a height of about 50 km to more than 1000 km.

- At this height, the atmosphere is so thin that the gas ionized by the sun-light remains ionized (as plasma). These layers (E : Heaviside, F : Appleton) appear as "Ionization Belts" around the Earth.

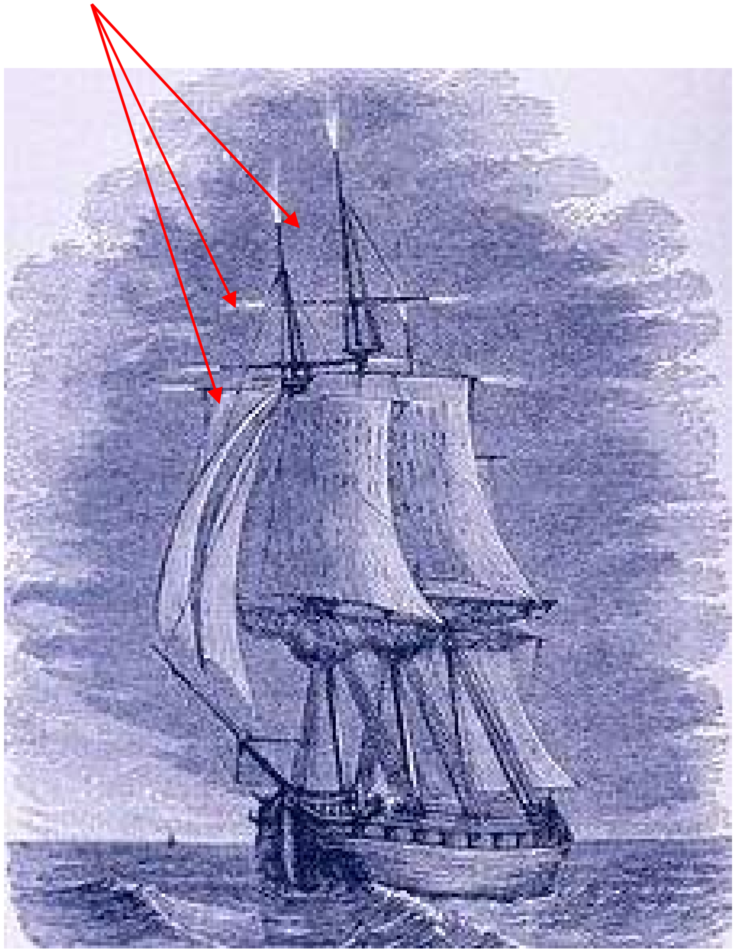
- The radio communication around the globe has been possible mainly due to reflection from these belts.



Plasma near Earth (contd.)

f) St. Elmo's fire

- St. Elmo's fire is a bright blue or violet glow, appearing like fire in some circumstances, from tall, sharply pointed structures such as lightning rods, ship masts, church spires and chimneys, and also on aircraft wing tips.



- Although referred to as "fire", it is in fact, plasma.

- The strong electric field around a pointed object causes ionization of the air molecules, producing the plasma glow visible in low-light conditions.

Plasma on Earth (Artificial)

- **Fluorescent lamp**

  Mercury Lamps

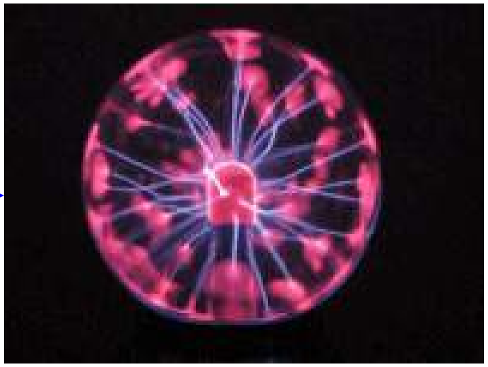
  Sodium vapour lamps

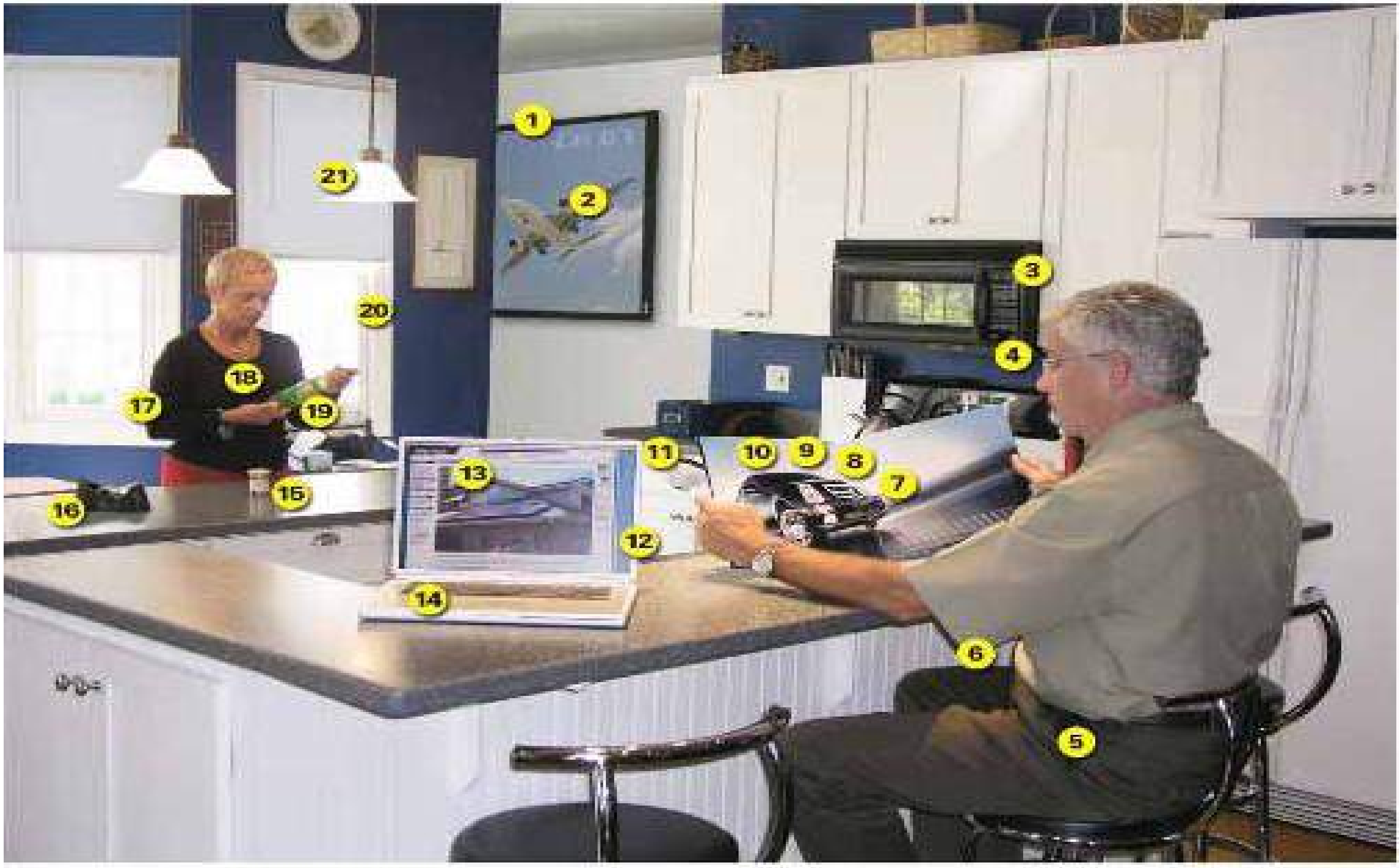
  Neon Signs

  These are all examples of plasma based light sources

- **"Plasma TV" is based on emission of light from plasma**

- **"Plasma Lamp" is another example of plasma light.**



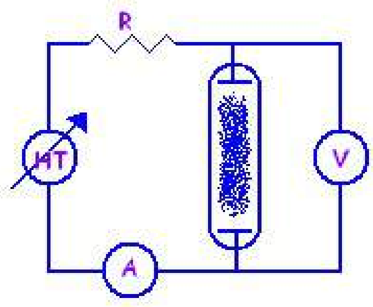


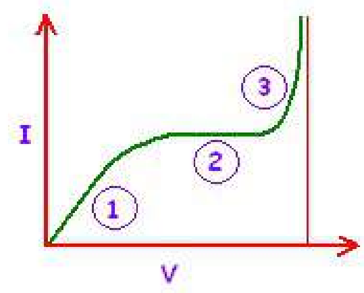
01—Plasma TV
02—Plasma-coated jet turbine blades
03—Plasma-manufactured LEDs in panel
04—Diamondlike plasma CVD eyeglass coating
05—Plasma ion-implanted artificial hip
06—Plasma laser-cut cloth
07—Plasma HID headlamps
08—Plasma-produced H_2 in fuel cell
09—Plasma-aided combustion
10—Plasma muffler
11—Plasma ozone water purification
12—Plasma-deposited LCD screen
13—Plasma-deposited silicon for solar cells
14—Plasma-processed microelectronics
15—Plasma-sterilization in pharmaceutical production
16—Plasma-treated polymers
17—Plasma-treated textiles
18—Plasma-treated heart stent
19—Plasma-deposited diffusion barriers for containers
20—Plasma-sputtered window glazing
21—Compact fluorescent plasma lamp

Plasma Science: Advancing Knowledge in the National Interest (2007), National Research Council, http://books.nap.edu/openbook.php?record_id=11960&page=9

Laboratory sources of plasma

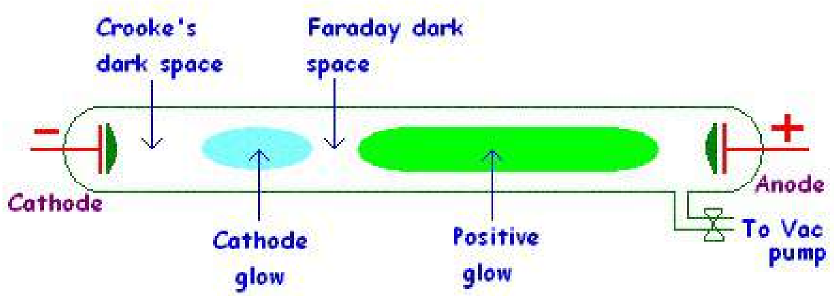
➢ **Arc Discharge (Electrical discharge in gas at high pressure)**





❖ **Examples : Flash in a camera**

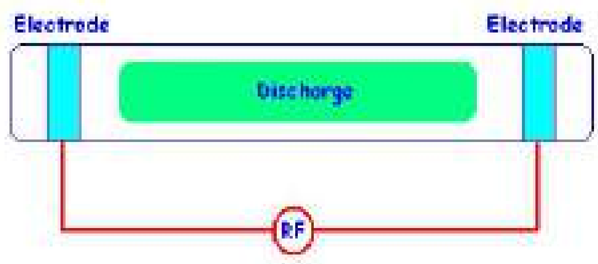
➢ **Glow discharge (Electrical discharge in gases at low pressure)**

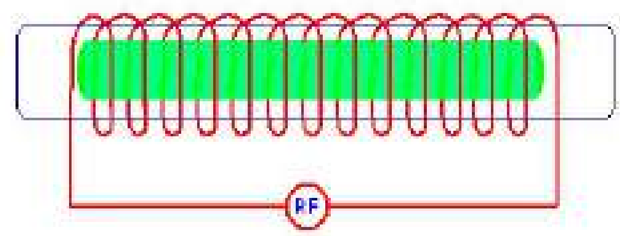


❖ **Example : Fluorescent light, Neon lights**

Laboratory sources of plasma (contd.)

➢ **Plasma production by RF discharge (in low pressure gases)**

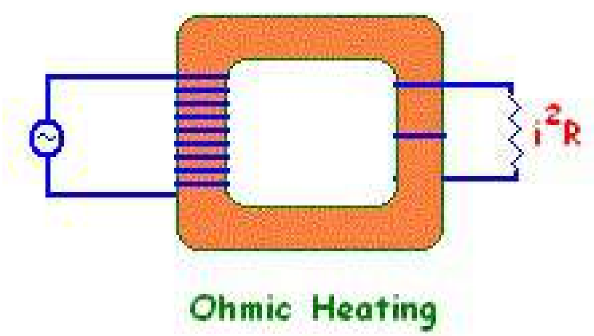


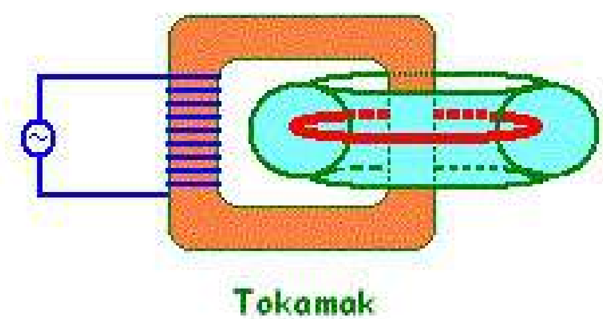


Capacitively coupled

Inductively coupled

➢ **Ohmic heating plasma ["Tokamak" plasma]**





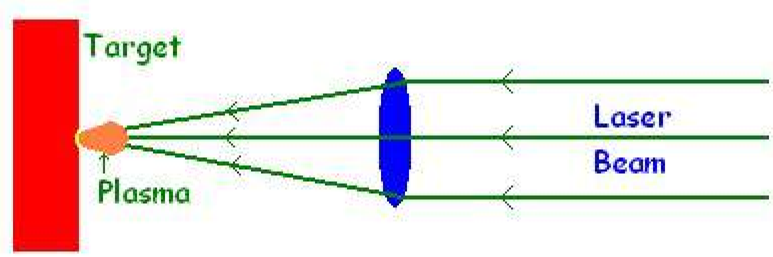
❑ **Typical plasma parameters are : $n_e \sim 10^{14}$ cm^{-3}, $T_e \sim 100$ eV to keV**

Plasma production using light (Lasers)

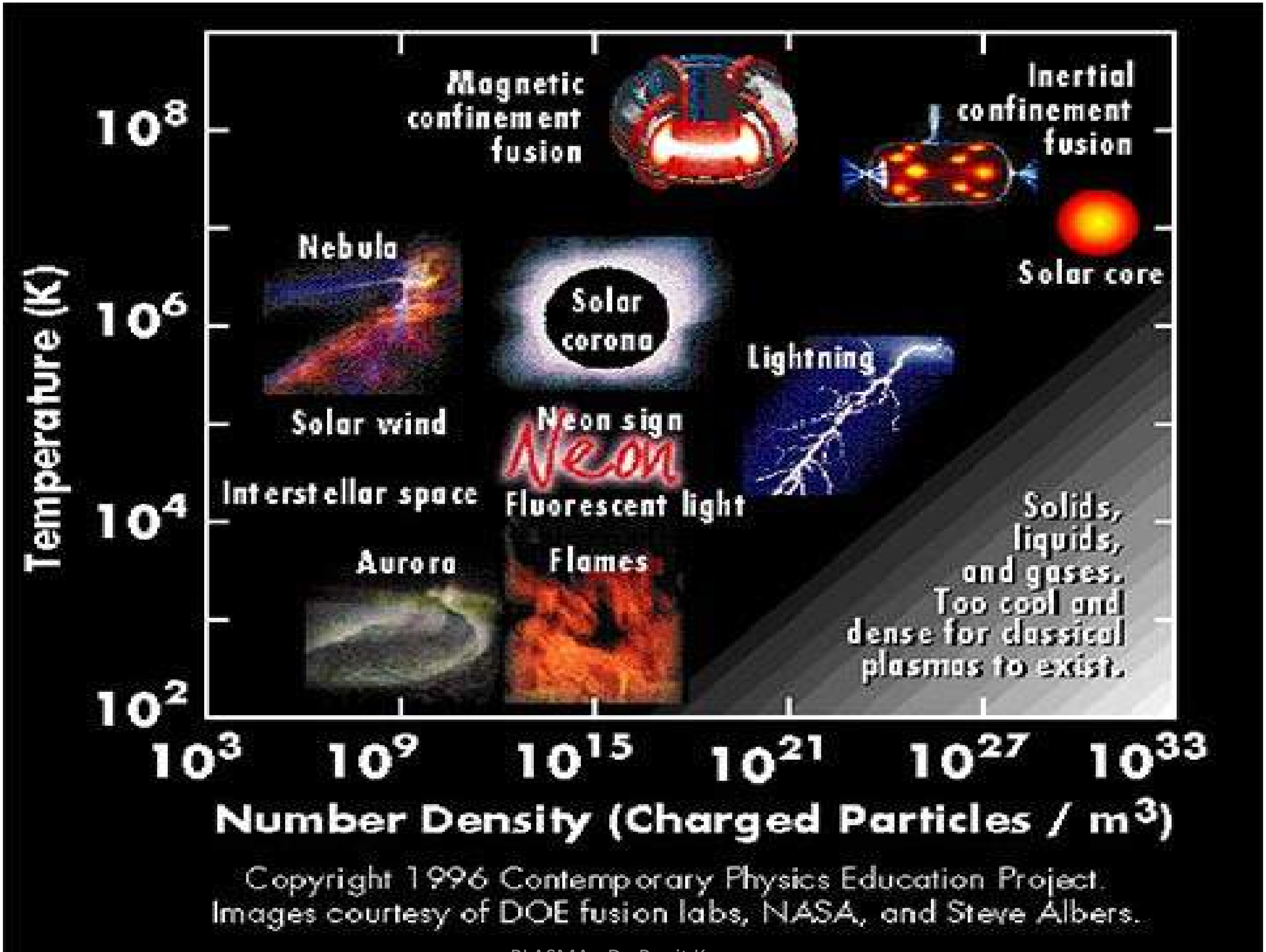
- When laser light is focused on any material, depending on the intensity of the focussed light, different processes take place.
- At very low intensities, the irradiated material (Target) simply gets heated.
- As the intensity is increased, once the temperature reaches the melting point of the target material, melting of the irradiated zone starts.

  This intensity regime is used for laser welding.
- At higher intensities (typically 10^6 W/cm^2), when the boiling point is crossed, the molten material evaporates, leading to creation of a hole in the material.

  This intensity region is used for drilling applications.
- At even higher intensities, when the intensity exceeds 10^9 W/cm^2, the evaporated material gets ionized, leading to formation of plasma.
- Plasma formation by means of focussed laser light is referred to as Laser Produced Plasma.



- Typical plasma parameters are : $n_e \sim 10^{22}$ cm^{-3}, $T_e \sim 100$ eV to 100 keV


Magnetic confinement fusion
Inertial confinement fusion
Nebula
Solar corona
Solar core
Lightning
Solar wind
Neon sign
Neon
Interstellar space
Fluorescent light
Aurora
Flames
Solids, liquids, and gases. Too cool and dense for classical plasmas to exist.
Temperature (K)
10^8
10^6
10^4
10^2
10^3
10^9
10^15
10^21
10^27
10^33
Number Density (Charged Particles / m^3)
Copyright 1996 Contemporary Physics Education Project.
Images courtesy of DOE fusion labs, NASA, and Steve Albers.

Nuclear Fusion

- Fusion reactions are much "cleaner" (i.e. NO radioactive particles of long life are emitted, unlike in Fission reactions)
- Also, we have an inexhaustible supply of deuterium, as ocean water contains 0.015% deuterium by weight.
- The thermo-nuclear neutrons carry the energy released as kinetic energy, which is used to drive a turbine to produce electricity.
- Lawson criterion: For a net energy gain (i.e. energy released ≥ energy used in producing plasma), the interacting particles (density : n) have to be held close to each other for long enough time (τ) for sufficient fusion reactions to take place.
- Mathematically, Laswson Criterion puts the condition :

  $n\,\tau \geq 10^{14}$ cm^{-3}-s for D-T reactions at 10 keV temperature

  $n\,\tau \geq 10^{15}$ cm^{-3}-s for D-D reactions at 10 keV temperature

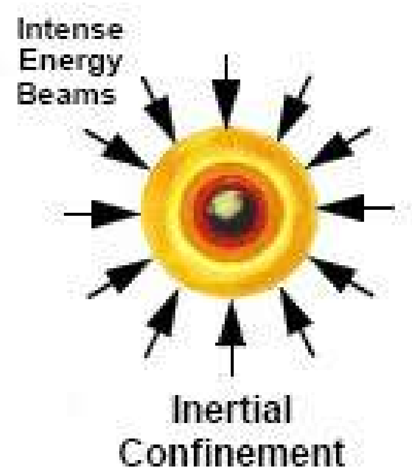
  ➔ Confinement is necessary.

Plasma confinement

- There are three ways a plasma can be confined :

1) Gravitational confinement

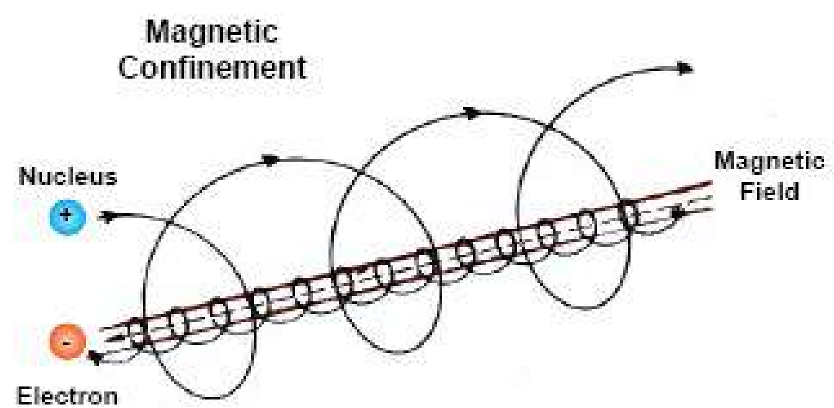
2) Inertial confinement



Lawson Criterion

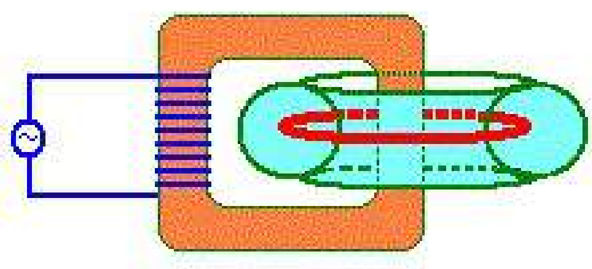
$n\,\tau \geq 10^{14}\ \mathrm{cm^{-3}\text{-}s}$

3) Magnetic confinement

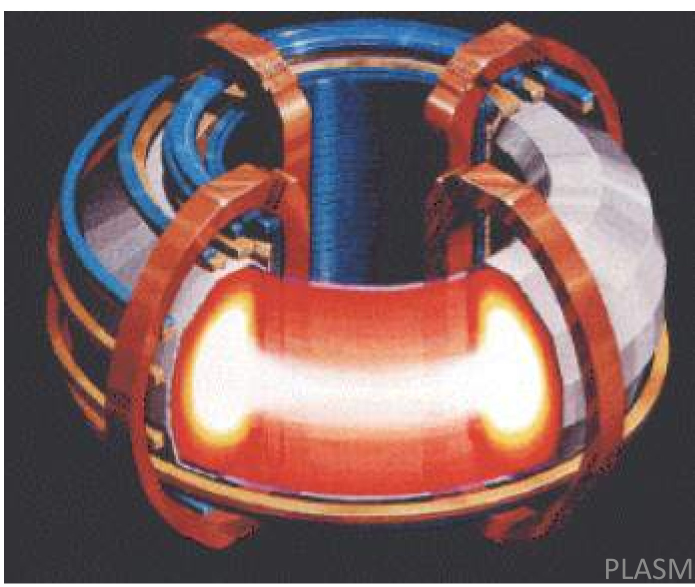


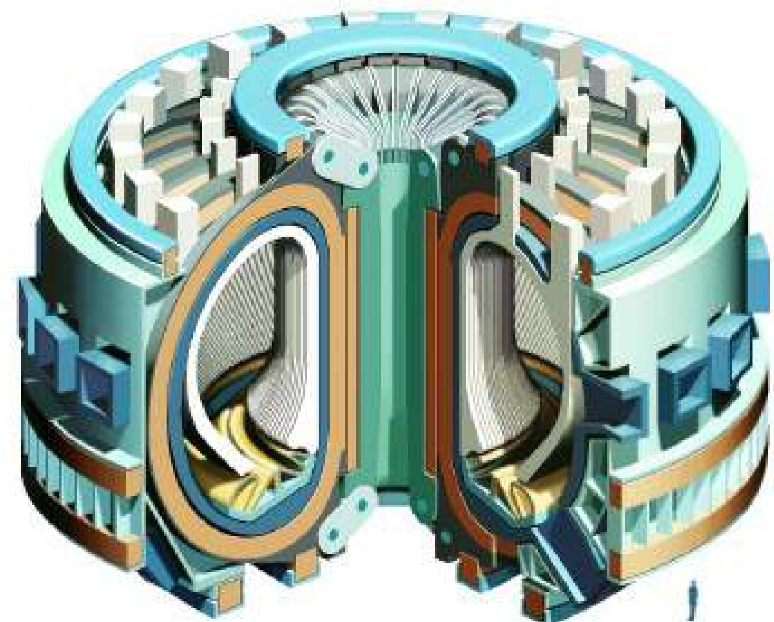
Magnetic Confinement of Plasma

- The **Lorentz force** ($q\,\underline{v}\times\underline{B}$) makes the charged particles go around in a helical path around the lines of force.
- This inhibits plasma expansion perpendicular to the magnetic field i.e. Confinement.



Tokamak





30 m dia, 30 m high

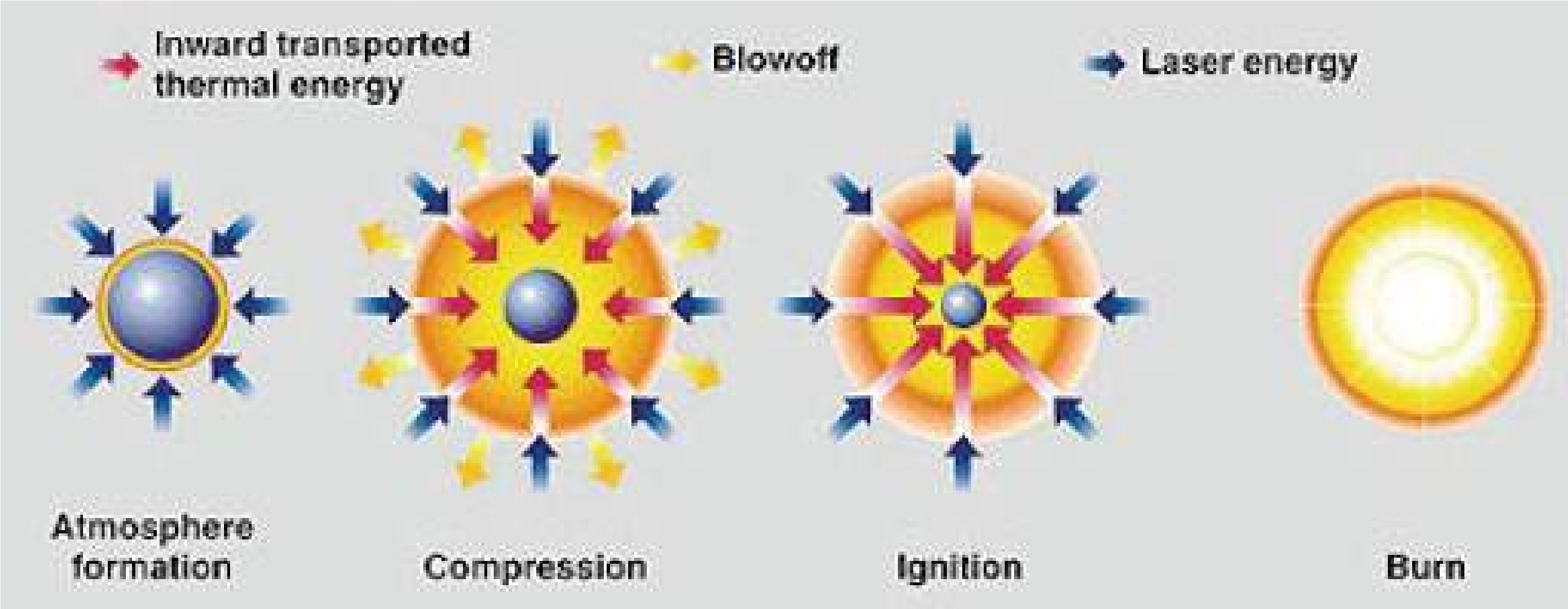
International Thermonuclear Experimental Reactor : ITER

Indian contribution :

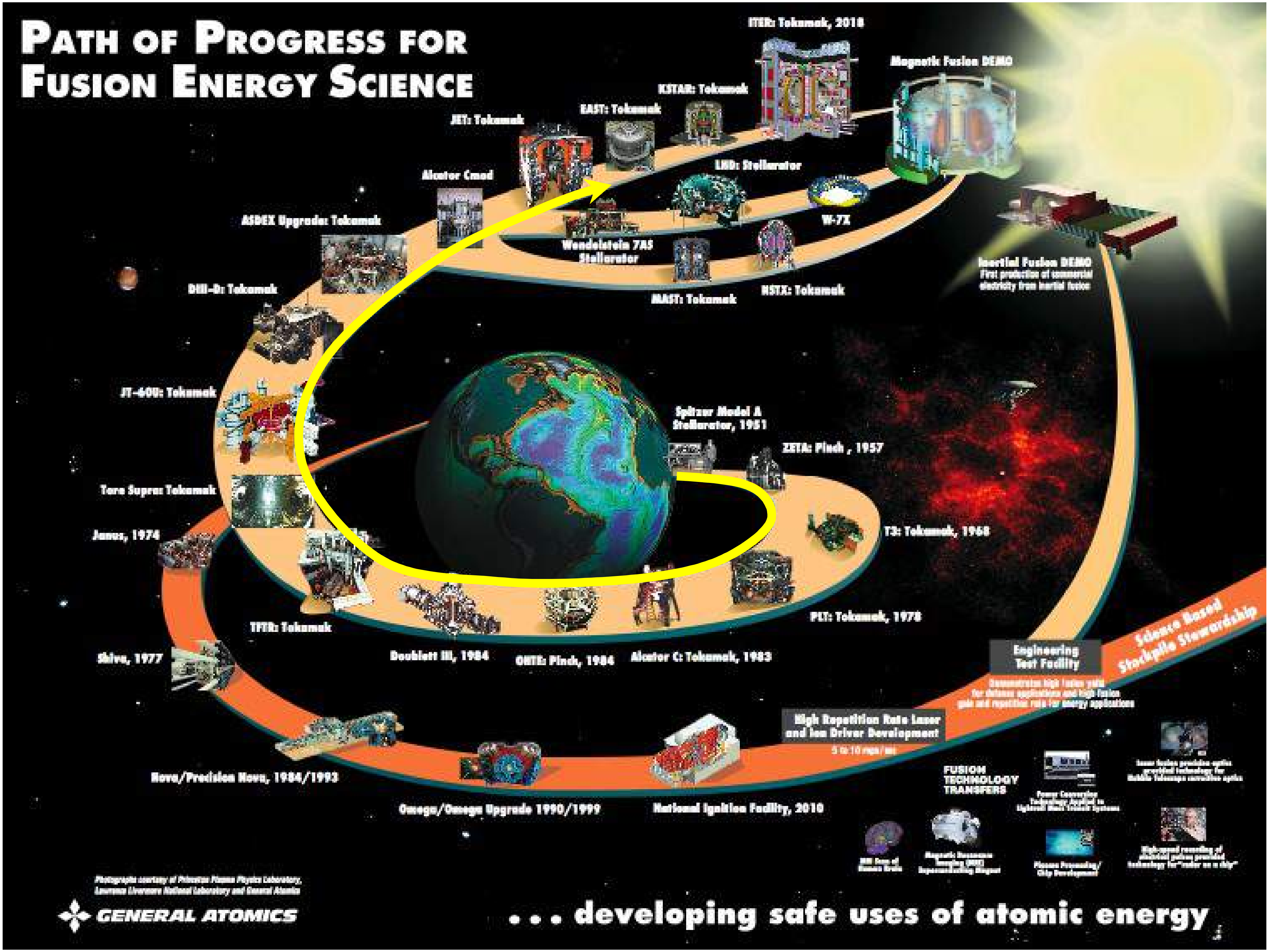
Rs. 2500 crores in 10 years

Inertial Confinement Fusion

- The heating is within few nanoseconds, so that the heated fuel cannot expand much during that time : i.e. It remains **confined by its inertia**.
- As the density is high (solid density), within this time, the Lawson criterion is satisfied.
- This mode of fusion is called **Inertial Confinement Fusion**.



- **Problem :** Needs lasers with **megajoule** level energy.
- Such laser systems have been built in U.S.A. **(National Ignition Facility)**, and France.


PATH OF PROGRESS FOR
FUSION ENERGY SCIENCE
ITER: Tokamak, 2018
Magnetic Fusion DEMO
KSTAR: Tokamak
EAST: Tokamak
JET: Tokamak
LHD: Stellarator
Alcator Cmod
W-7X
ASDEX Upgrade: Tokamak
Wendelstein 7AS
Stellarator
MAST: Tokamak
NSTX: Tokamak
Inertial Fusion DEMO
DIII-D: Tokamak
JT-60U: Tokamak
Spitzer Model A
Stellarator, 1951
ZETA: Pinch , 1957
Tore Supra: Tokamak
T3: Tokamak, 1968
Janus, 1974
TFTR: Tokamak
PLT: Tokamak, 1978
Shiva, 1977
Doublett III, 1984
OHTE: Pinch, 1984
Alcator C: Tokamak, 1983
Engineering
Test Facility
Science Based
Stockpile Stewardship
High Repetition Rate Laser
and Ion Driver Development
Nova/Precision Nova, 1984/1993
Omega/Omega Upgrade 1990/1999
National Ignition Facility, 2010
FUSION
TECHNOLOGY
TRANSFERS
GENERAL ATOMICS
. . . developing safe uses of atomic energy

Properties

Property	Gas	Plasma
Electrical Conductivity	**Very low** Air is an excellent insulator until it breaks down into plasma at electric field strengths above 30 kilovolts per centimeter.	**Usually very high** For many purposes, the conductivity of a plasma may be treated as infinite.
Independently acting species	**One** All gas particles behave in a similar way, influenced by gravity and by collisions	**Two or three** Electrons, ions and neutrons (allowing phenomena-types of waves and instabilities)
Velocity distribution	**Maxwellian** Collisions usually lead to a Maxwellian velocity distribution	**Often non-Maxwellian** Collisional interactions are often weak in hot plasmas
Interactions	**Binary** Two-particle collisions are the rule	**Collective** Waves, or organized motion of plasma

Plasma properties

Type	Electron density n_e (cm^{-3})	Temperature T_e (eV*)
Stars	10^{26}	2×10^3
Laser fusion	10^{25}	3×10^3
Magnetic fusion	10^{15}	10^3
Laser-produced	$10^{18} - 10^{24}$	$10^2 - 10^3$
Discharges	10^{12}	1-10
Ionosphere	10^6	0.1
ISM	1	10^{-2}

Table 1: Densities and temperatures of various plasma types

* $1\mathrm{eV} \equiv 11600\mathrm{K}$

Criteria for the definition of a Plasma

Macroscopic Neutrality :

- In absence of external disturbances a plasma is macroscopically neutral.
- Under equilibrium with no external forces present – volume of plasma sufficiently large to contain large number of particles and yet sufficiently small compared to characteristic lengths for variation of macroscopic parameters such as density and temperature, the net resulting charge is zero.

Quasi-neutrality: number densities of electrons, n_e, and ions, n_i, with charge state Z are *locally balanced*:

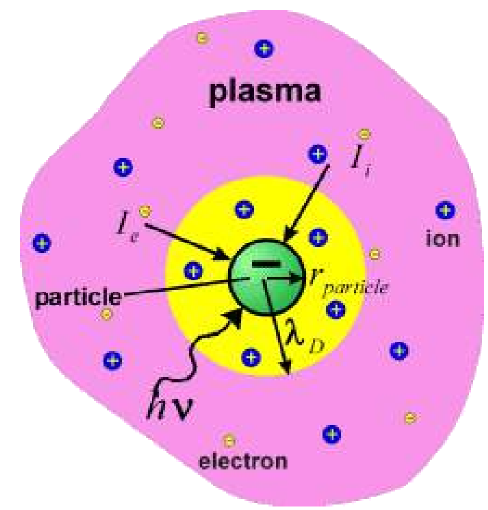
$$n_e \simeq Zn_i. \tag{1}$$

Collective behaviour: long range of Coulomb potential $(1/r)$ leads to nonlocal influence of disturbances in equilibrium.

Macroscopic fields usually dominate over microscopic fluctuations, e.g.:

$$\rho = e(Zn_i - n_e) \Rightarrow \nabla . \boldsymbol{E} = \rho / \varepsilon_0$$

Debye Shielding :



- Provides a measure of the distance over which the influence of the electric field on an individual charged particle is felt by the other charged particles.
- Debye length

$$\lambda_D = \left(\frac{\varepsilon_o kT}{n_e e^2}\right)^{1/2}$$

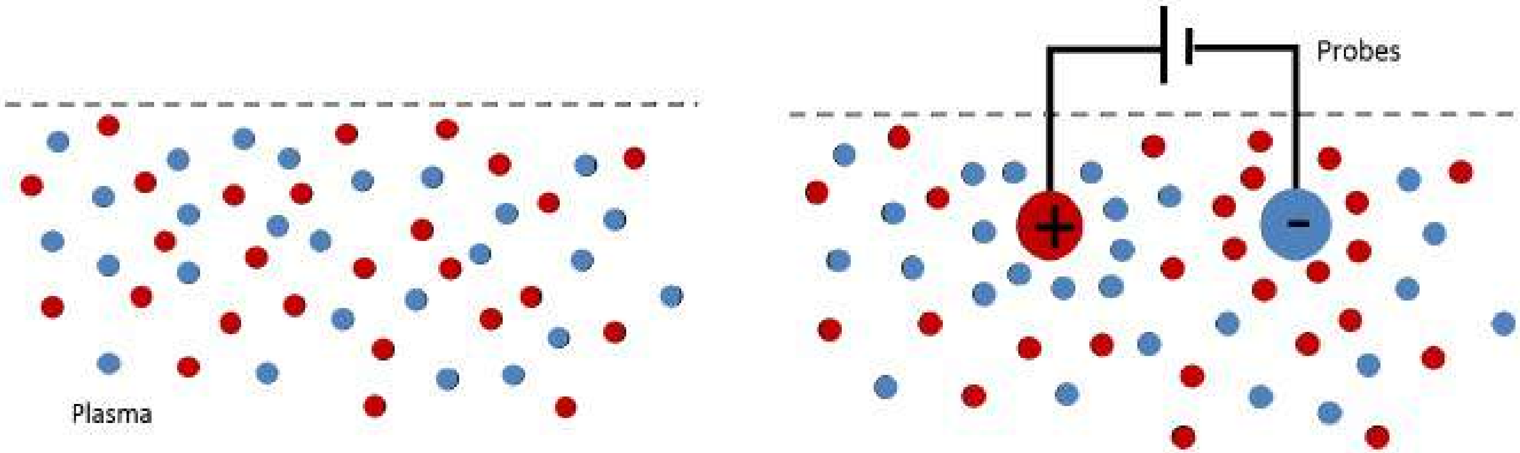
- Physical dimensions of the system should be large compared to Debye length.

$$L >> \lambda_D$$

- No. of electrons in a Debye sphere to be large

$$n_e \lambda_D^3 >> 1$$

Debye shielding



What is the potential $\phi(r)$ of an ion (or positively charged sphere) immersed in a plasma?

Debye shielding (2): ions vs electrons

For equal ion and electron temperatures ($T_e = T_i$), we have:

$$\frac{1}{2} m_e v_e^2 = \frac{1}{2} m_i v_i^2 = \frac{3}{2} k_B T_e \tag{2}$$

Therefore,

$$\frac{v_i}{v_e} = \left(\frac{m_e}{m_i}\right)^{1/2} = \left(\frac{m_e}{A m_p}\right)^{1/2} = \frac{1}{43} \qquad \text{(hydrogen, Z=A=1)}$$

Ions are almost stationary on electron timescale!
To a good approximation, we can often write:

$$n_i \simeq n_0,$$

where the material (eg gas) number density, $n_0 = N_A \rho_m / A$.

Debye shielding (3)

In thermal equilibrium, the electron density follows a Boltzmann distribution*:

$$n_e = n_i \exp(e\phi / k_B T_e) \tag{3}$$

where n_i is the ion density and k_B is the Boltzmann constant.

From Gauss' law (Poisson's equation):

$$\nabla^2 \phi = -\frac{\rho}{\varepsilon_0} = -\frac{e}{\varepsilon_0}(n_i - n_e) \tag{4}$$

Debye shielding (4)

Combining (4) with (3) in spherical geometry[a] and requiring $\phi \to 0$ at $r = \infty$, get solution:

Exercise

$$\phi_D = \frac{1}{4\pi\varepsilon_0}\frac{e^{-r/\lambda_D}}{r}. \qquad (5)$$

with

Debye length

$$\lambda_D = \left(\frac{\varepsilon_0 k_B T_e}{e^2 n_e}\right)^{1/2} = 743\left(\frac{T_e}{\text{eV}}\right)^{1/2}\left(\frac{n_e}{\text{cm}^{-3}}\right)^{-1/2} \text{cm} \qquad (6)$$

---

[a] $\nabla^2 \to \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\phi}{dr}\right)$

Debye sphere

An *ideal* plasma has many particles per Debye sphere:

$$N_D \equiv n_e \frac{4\pi}{3} \lambda_D^3 \gg 1. \tag{7}$$

$\Rightarrow$ Prerequisite for collective behaviour.

Alternatively, can define *plasma parameter*:

$$g \equiv \frac{1}{n_e \lambda_D^3}$$

Classical plasma theory based on assumption that $g \ll 1$, which also implies dominance of collective effects over collisions between particles.

Plasma Frequency :

- By some external means small charge separation is produced
- Oscillations are produced when the disturbance is removed instantaneously
- Result in fast collective oscillations of electrons about massive ions

$$\omega_{pe} = \left(\frac{n_e e^2}{m_e \varepsilon_o}\right)^{1/2}$$

- Collisions between electrons and neutral particles damp these oscillations
- For less damping (so that electrons can behave independently)

$$\nu_{pe} > \nu_{en}$$

Plasma Oscillations (1)

$$\nabla.\vec{E} = 4\pi e(n_0 - n_e)$$

Poisson's equation

$$\frac{\partial}{\partial t} n_e + \nabla \cdot n_e \vec{\mathrm{v}} = 0$$

Particle conservation

$$\frac{\partial}{\partial t} n_e m_e \vec{\mathrm{v}} + \nabla \cdot \left(n_e m_e \vec{\mathrm{v}}\vec{\mathrm{v}} \right) = -n_e e\vec{E}$$

Momentum conservation

Linearize

$$\boxed{\begin{aligned} n_e &= n_0 + \tilde{n}(\vec{r},t) \\ \vec{\mathrm{v}} &= \vec{\mathrm{v}}_0 + \tilde{\vec{\mathrm{v}}}(\vec{r},t) \\ \vec{E} &= \vec{E}_0 + \tilde{\vec{E}}(\vec{r},t) \end{aligned}}$$

Assume

$$\boxed{\begin{aligned} n_i &= n_0 \\ \vec{\mathrm{v}}_0 &= 0 \\ \vec{E}_0 &= 0 \end{aligned}}$$

$$\Rightarrow \quad \nabla.\left(\vec{E}_0 + \tilde{\vec{E}} \right) = \nabla.\tilde{\vec{E}} = -4\pi e\tilde{n}$$

Linearized Poisson's equation

Plasma Oscillations (2)

Particle conservation

$$\frac{\partial}{\partial t}\left(\cancelto{0}{n_0} + \tilde{n}(\vec{r},t)\right) = -\nabla\cdot\left[\left(n_0 + \tilde{n}\right)\left(\vec{\mathrm{v}}_0 + \tilde{\vec{\mathrm{v}}}\right)\right]$$

$$= \cancelto{0}{-\nabla\cdot\left(n_0\vec{\mathrm{v}}_0\right)} \cancelto{0}{-\nabla\cdot\left(\tilde{n}\vec{\mathrm{v}}_0\right)} - \nabla\cdot\left(n_0\tilde{\vec{\mathrm{v}}}\right) \cancelto{0}{-\nabla\cdot\left(\tilde{n}\tilde{\vec{\mathrm{v}}}\right)}$$

$$= -n_0\nabla\cdot\tilde{\vec{\mathrm{v}}}$$

$$\Rightarrow \quad \frac{\partial\tilde{n}}{\partial t} = -n_0\nabla\cdot\tilde{\vec{\mathrm{v}}}$$

Linearized Particle Conservation Equation

Plasma Oscillations (3)

$$\frac{\partial}{\partial t} n_e m\vec{v} + \nabla \cdot \left(n_e m \vec{\mathrm{v}}.\vec{\mathrm{v}}\right) = -n_e e\vec{E}$$

Momentum conservation

$$m\frac{\partial}{\partial t}\left(n_0 + \tilde{n}\right)\left(\vec{\mathrm{v}}_0 + \tilde{\vec{\mathrm{v}}}\right) + \nabla \cdot \left(\cancelto{0}{\left(n_0 + \tilde{n}\right) m \tilde{\vec{\mathrm{v}}}.\tilde{\vec{\mathrm{v}}}}\right) = -(n_0 + \tilde{n})e\tilde{\vec{E}}$$

$$\Rightarrow \quad mn_0 \frac{\partial \tilde{\vec{\mathrm{v}}}}{\partial t} = -n_0 e\tilde{\vec{E}}$$

Linearized momentum equation

Derivation of Cold Plasma Oscillations

$$\frac{\partial \tilde{n}}{\partial t} = -n_0 \nabla \cdot \tilde{\vec{\mathrm{v}}}$$

using $$m \frac{\partial \tilde{\vec{\mathrm{v}}}}{\partial t} = -e\tilde{\vec{E}}$$

or

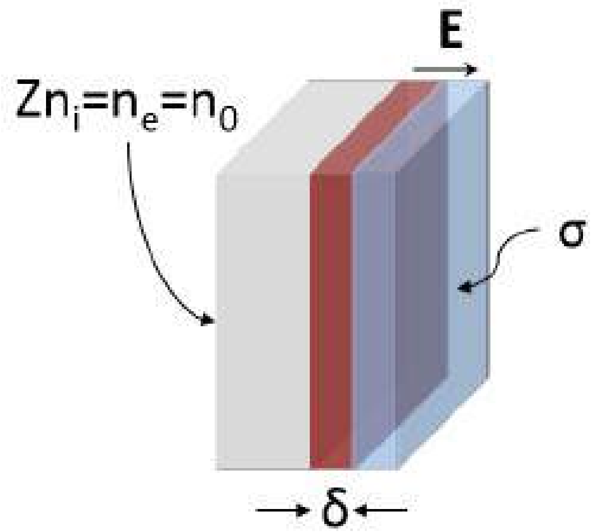
$$\frac{\partial^2 \tilde{n}}{\partial t^2} = -n_0 \nabla \cdot \frac{\partial \tilde{\vec{\mathrm{v}}}}{\partial t} = n_0 \nabla \cdot \frac{e}{m} \tilde{E}$$

using $\nabla . \tilde{E} = -4\pi e \tilde{n}$

$$= -\omega_p^2 \tilde{n}$$

Plasma frequency $$\omega_p^2 = \frac{4\pi n_0 e^2}{m_e}$$

Plasma oscillations: capacitor model



Consider electron layer displaced from plasma slab by length δ. This creates two 'capacitor' plates with surface charge $\sigma = \pm e n_e \delta$, resulting in an electric field:

$$\boldsymbol{E} = \frac{\sigma}{\varepsilon_0} = \frac{e n_e \delta}{\varepsilon_0}$$

Capacitor model (2)

The electron layer is accelerated back towards the slab by this restoring force according to:

$$m_e \frac{dv}{dt} = -m_e \frac{d^2\delta}{dt^2} = -eE = \frac{e^2 n_e \delta}{\varepsilon_0}$$

Or:

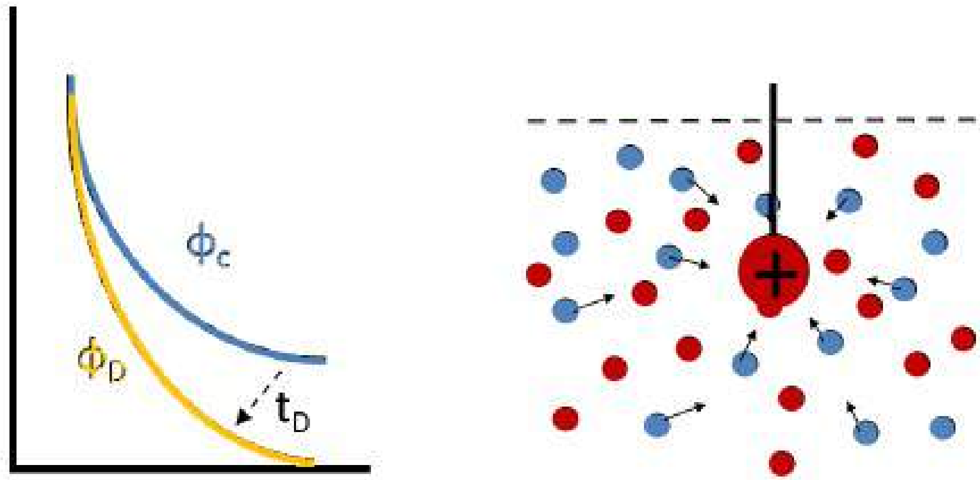
$$\frac{d^2\delta}{dt^2} + \omega_p^2 \delta = 0,$$

where

Electron plasma frequency

$$\omega_p \equiv \left(\frac{e^2 n_e}{\varepsilon_0 m_e}\right)^{1/2} \simeq 5.6 \times 10^4 \left(\frac{n_e}{\mathrm{cm}^{-3}}\right)^{1/2} \mathrm{s}^{-1}. \qquad (8)$$

Response time to create Debye sheath



For a plasma with temperature T_e (and thermal velocity $v_{te} \equiv \sqrt{k_B T_e / m_e}$), one can also define a characteristic *reponse time* to recover quasi-neutrality:

$$t_D \simeq \frac{\lambda_D}{v_{te}} = \left(\frac{\varepsilon_0 k_B T_e}{e^2 n_e} \cdot \frac{m}{k_B T_e} \right)^{1/2} = \omega_p^{-1}.$$

External fields: underdense vs. overdense

If the plasma response time is shorter than the period of a external electromagnetic field (such as a laser), then this radiation will be *shielded out*.

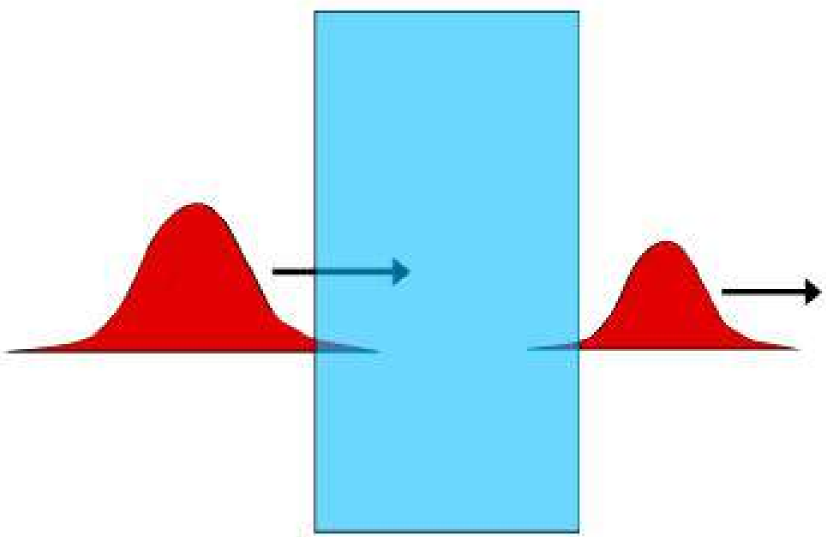


Figure 1: Underdense, $\omega > \omega_p$: plasma acts as nonlinear refractive medium

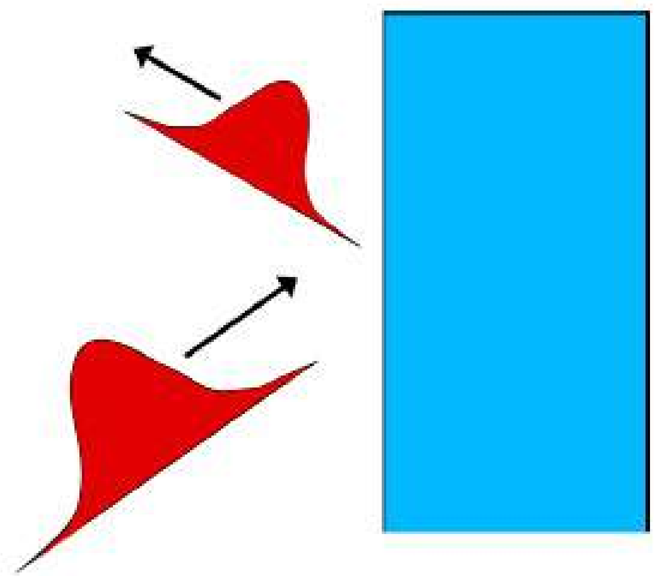


Figure 2: Overdense, $\omega < \omega_p$: plasma acts like mirror

Plasma creation: field ionization

At the Bohr radius

$$a_B = \frac{\hbar^2}{me^2} = 5.3 \times 10^{-9} \text{ cm},$$

the electric field strength is:

$$\begin{aligned} E_a &= \frac{e}{4\pi\varepsilon_0 a_B^2} \\ &\simeq 5.1 \times 10^9 \text{ Vm}^{-1}. \end{aligned} \tag{10}$$

This leads to the *atomic intensity*:

$$\begin{aligned} I_a &= \frac{\varepsilon_0 c E_a^2}{2} \\ &\simeq 3.51 \times 10^{16} \text{ Wcm}^{-2}. \end{aligned} \tag{11}$$

A laser intensity of $I_L > I_a$ will *guarantee ionization* for any target material, though in fact this can occur well below this threshold value (eg: $\sim 10^{14}$ Wcm^{-2} for hydrogen) via *multiphoton* effects .

Degree of ionization

- The degree of ionization - proportion of atoms that have lost or gained electrons.
- Ionized gas with 1% of ionization can have the characteristics of a plasma (i.e., response to magnetic fields and high electrical conductivity).
- The degree of ionization, α is defined as

$$\alpha = n_i/(n_i + n_a)$$

n_i is the number density of ions
n_a is the number density of neutral atoms.

Temperature

- Very high temperatures are usually needed to sustain ionization, which is a defining feature of a plasma.

- The degree of plasma ionization is determined by the "electron temperature" relative to the ionization energy, (and more weakly by the density), in a relationship called the Saha equation.

- At low temperatures, ions and electrons tend to recombine into bound states—atoms, and the plasma will eventually become a gas.

Thermal vs. non-thermal plasmas

- Thermal plasmas have electrons and the heavy particles at the same temperature, i.e., they are in thermal equilibrium with each other.

- Non-thermal plasmas on the other hand have the ions and neutrals at a much lower temperature (normally room temperature), whereas electrons are much "hotter".

Hot and Cold Plasmas

- “Hot" if it is nearly fully ionized
- “Cold" if only a small fraction (for example 1%) of the gas molecules are ionized
- Even in a "cold" plasma, the electron temperature is still typically several thousand degrees Celsius.

Magnetization

- Plasma with a magnetic field strong enough to influence the motion of the charged particles is said to be magnetized.

- Particle on average completes at least one gyration around the magnetic field before making a collision.

- It is often the case that the electrons are magnetized while the ions are not.

- Magnetized plasmas are *anisotropic*, meaning that their properties in the direction parallel to the magnetic field are different from those perpendicular to it.

Theoretical description of Plasma Phenomena

- Dynamic behavior – governed by interaction between the plasma particles and the internal fields produced by the particle themselves and the externally applied fields.

- Interaction of charged particles with e.m. fields is governed by Lorentz force

$$F = \frac{dp}{dt} = q(\vec{E} + \vec{v} \times \vec{B})$$

- In principle – describe the dynamics of a plasma by solving the equations of motion for each particle in the plasma under the combined influence of the externally applied fields and the internal fields generated by all other plasma particles.

- E. M. fields obey Maxwell's eqs.

$$\nabla . E = \frac{\rho}{\varepsilon_o}$$

$$\nabla . B = 0$$

$$\nabla \times E = -\frac{\partial B}{\partial t}$$

$$\nabla \times B = \mu_o (J + \varepsilon_0 \frac{\partial E}{\partial t})$$

- Plasma charge and current densities

$$\rho_p = \frac{1}{\delta V} \sum_i q_i$$

$$J_p = \frac{1}{\delta V} \sum_i q_i V_i$$

Logical framework of Plasma Physics

Plasma Dynamics – self-consistent interaction between e.m. fields and statistically large number of charged particles.

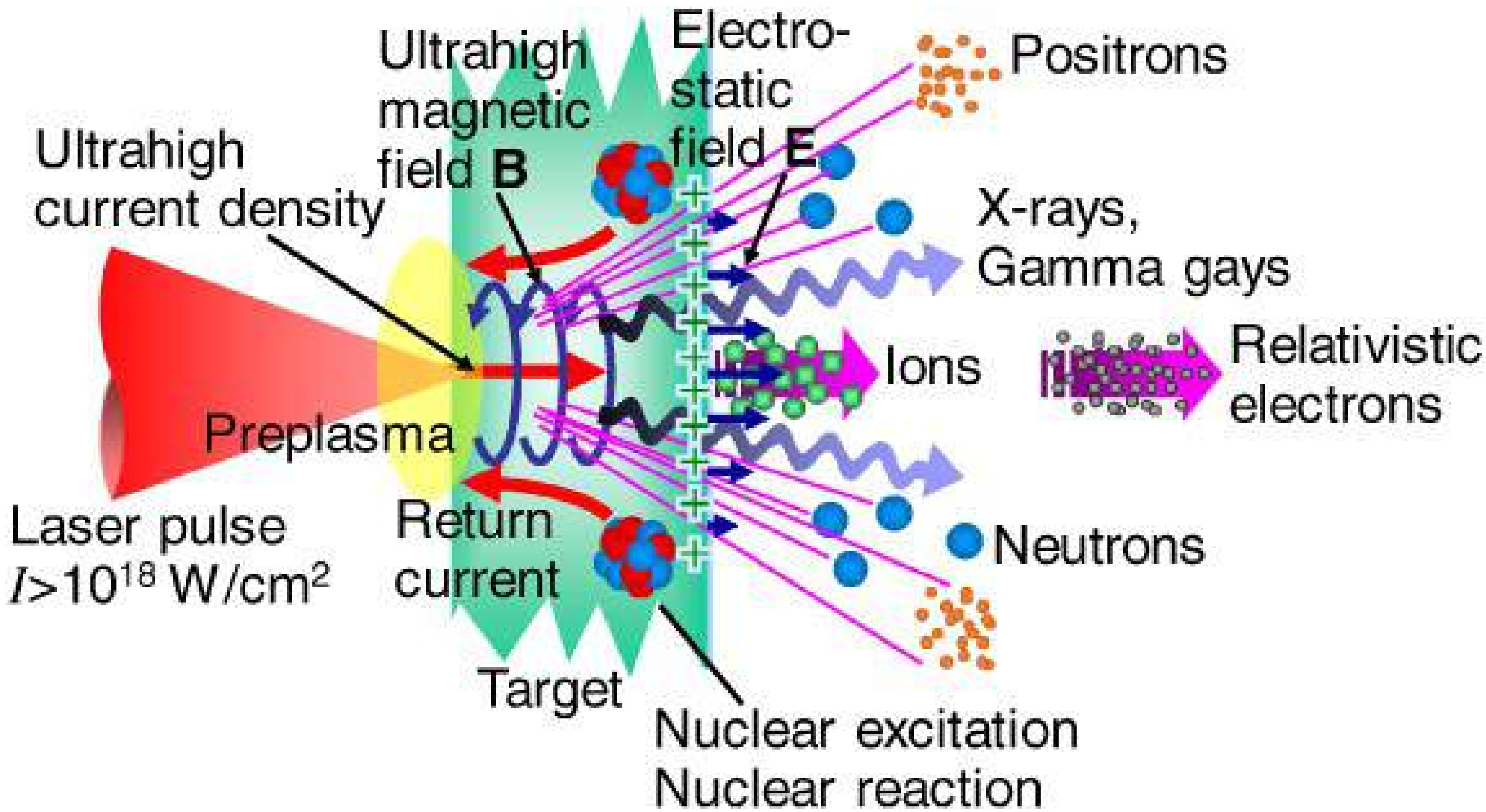
Lorentz equation

(gives x_j, v_j for each particle from knowledge of E(x,t), B(x,t))

$\updownarrow$

Maxwell's equations

(gives E(x,t), B(x,t) from x_j , v_j)


Ultrahigh
magnetic
field B
Electro-
static
field E
Positrons
Ultrahigh
current density
X-rays,
Gamma gays
Ions
Relativistic
electrons
Preplasma
Laser pulse
I>10[18] W/cm[2]
Return
current
Neutrons
Target
Nuclear excitation
Nuclear reaction

Laser Plasma Interaction

The propagation of laser light through pre-ionized plasma is governed by the dispersion relation $c^2 k^2 = \omega^2 - \omega_p^2$, where ω is laser frequency and k is propagation constant. For studying the interaction process three categories have been defined:

Underdense : $\omega > \omega_p$
Critically dense: $\omega = \omega_p$
Overdense: $\omega < \omega_p$

The dimension less amplitude $a_0 \left(= eE_0 / mc\omega\right)$ serves as a parameter which determines the strength of interaction. Depending on the value of a_0 the laser plasma interaction may be:

Non-relativistic: $a_0 << 1$
Mildly relativistic: $a_0 < 1$
Ultra relativistic: $a_0 >> 1$

- **Interaction of intense lasers with plasma involves a number of interesting nonlinear physical phenomenon including self-focusing, wakefield generation and quasi-static magnetic field generation.**

- **Experiments report that quasi-static magnetic fields (both axial and azimuthal) of the order of MG are generated when intense laser beams interact with underdense plasma.**

- **These fields affect the propagation characteristics of the laser pulses and hence play vital role in fast ignition schemes of inertial confinement fusion, charged particle acceleration, harmonic generation and other nonlinear effects.**

Wave Propagation in Plasma :

The approaches are :

(i) First principles, $N -$ body molecular dynamics

(ii) Phase-space methods - the Vlasov-Boltzmann equation

(iii) Two - fluid equations

(iv) Magnetohydrodynamics (single magnetized fluid)

- Start with the two-fluid equations for a plasma with
  Finite temperature $(T_e > 0)$
  Collisionless $(\nu_{ie} \approx 0)$
  and non-relativistic, so that $u << c$.

- The equations governing the plasma dynamics under these conditions are

$$n_s m_s \frac{du}{dt} = n_s q_s (E + u_s \times B) - \nabla P_s,$$

$$\frac{\partial n_s}{\partial t} + \nabla .(n_s u_s) = 0,$$

$$\frac{d(P_s n_s^{-\gamma_s})}{dt} = 0,$$

where P_s is the thermal pressure of species S and γ_s is the specific heat ratio.

- In absence of fields, and assuming strict quasi-neutrality $(n_e = Zn_i = n; u_e = u_i = u)$, we recover the more familiar Navier-Stokes equations

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho u) = 0,$$

$$\frac{\partial u}{\partial t} + (u.\nabla)u = \frac{1}{\rho}\nabla P.$$

- By contrast, in the plasma accelerator context we usually deal with time–scales over which the ions can be assumed to be motionless, i.e. $u_i = 0,$ and also unmagnetized plasma, so that the momentum equation reads

$$n_e m_e \frac{du_e}{dt} = -en_e E - \nabla P_e.$$

Longitudinal (Langmuir) waves :

- Simplify the dynamical equations by setting $u_i = 0$, restricting the electron motion to one dimension (x) and taking $\frac{\partial}{\partial y} = \frac{\partial}{\partial z} = 0$;

$$\frac{\partial n_e}{\partial t} + \frac{\partial (n_e u_e)}{\partial x} = 0,$$

$$n_e\left(\frac{\partial u_e}{\partial t} + u_e \frac{\partial u_e}{\partial x}\right) = \frac{-e}{m} n_e E - \frac{1}{m}\frac{\partial P_e}{\partial x},$$

$$\frac{d}{dt}\left(\frac{P_e}{n_e^{\gamma_e}}\right) = 0.$$

- The above equations has three equations and four unknowns. We need an expression for the electric field, which can be found from Gauss's law with $Zn_i = n_0$,

$$\frac{\partial E}{\partial x} = \frac{e}{\varepsilon_0}(n_0 - n_e).$$

- The above system of equations is nonlinear and, apart from a few special cases, cannot be solved exactly. A common technique for analysing waves in plasmas is to linearize the equations, which involves assuming that the perturbed amplitudes are small compared to the equilibrium values, i.e.

$$n_e = n_0 + n_1$$

$$u_e = u_1$$

$$P_e = P_0 + P_1,$$

$$E = E_1$$

where, $n_1 << n_0$ and $P_1 << P_0$.

- Substituting these expressions and neglecting all products of perturbations such as $n_1 \partial_t u_1$ and $u_1 \partial_x u_1$,we get a set of linear equations for the perturbed quantities

$$\frac{\partial n_1}{\partial t} + n_0 \frac{\partial u_1}{\partial x} = 0,$$

$$n_0 \frac{\partial u_1}{\partial x} = -\frac{e}{m} n_0 E_1 - \frac{1}{m} \frac{\partial P_1}{\partial x},$$

$$\frac{\partial E_1}{\partial x} = -\frac{e}{\varepsilon_0} n_1,$$

$$P_1 = 3 k_B T_e n_1.$$

- The expression for P_1 results from the specific heat ratio γ_e and from assuming isothermal back-ground electrons, $P_0 = k_B T_e n_0$ (ideal gas).

- We can now eliminate E_1, P_1 and u_1 to get

$$\left(\frac{\partial^2}{\partial t^2} - 3 v_{te}^2 \frac{\partial^2}{\partial x^2} + \omega_p^2 \right) n_1 = 0,$$

with $v_{te}^2 = k_B T_e / .m_e$ Finally, we look for plane-wave solutions of the form $A = A_0 \exp\{i(\omega t - kx)\}$, so that our derivative operators are transformed as follows:

$$\frac{\partial}{\partial x} \to -ik; \quad \frac{\partial}{\partial t} \to i\omega$$

- Substitution of above yields the Bohm–Gross dispersion rRelation

$$\omega^2 = \omega_p^2 + 3k^2 v_{te}.$$

Transverse waves :

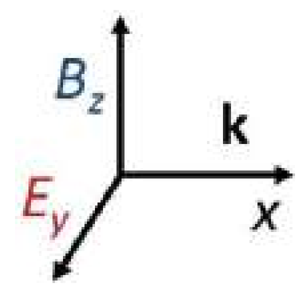
- Linearize 3rd and 4th Maxwell's equations and again apply the harmonic approximation $\frac{\partial}{\partial t} \rightarrow i\omega$ to get

$$\nabla \times E_1 = -i\omega B_1,$$

$$\nabla \times B_1 = -\mu_0 J_1 + i\varepsilon_0 \mu_0 \omega E_1,$$

where the transverse current density is given by

$$J_1 = -n_0 e u_1$$



- This time we look for EM plane wave solutions with E_1 perpendicular to k and also assume that the group and phase velocities are large enough, $v_p, v_g >> v_{te}$ so that we have a cold plasma with $P_e = n_0 k_B T_e \approx 0$

- The linearized fluid velocity and corresponding current are then

$$u_1 = -\frac{e}{i\omega m_e} E_1,$$

$$J_1 = \frac{n_0 e^2}{i\omega m_e} E_1 \equiv \sigma E_1,$$

where σ is the AC electrical conductivity.

- By analogy with dielectric media, in which Ampere's law is usually written as $\nabla\times B_1 = \mu_0 \partial_t D_1$, one can show that

$$D_1 = \varepsilon_0 \varepsilon E_1,$$

with

$$\varepsilon = 1 + \frac{\sigma}{i\omega\varepsilon_0} = 1 - \frac{\omega_p^2}{\omega^2}.$$

- From above, it follows that

$$\eta \equiv \sqrt{\varepsilon} = \frac{ck}{\omega} = \left(1 - \frac{\omega_p^2}{\omega^2}\right)^{1/2}$$

with

$$\omega^2 = \omega_p^2 + c^2 k^2$$

- For under dense plasma ($n_e << n_c$),

Phase velocity $$v_p = \frac{\omega}{k} \approx c\left(1 + \frac{\omega_p^2}{2\omega^2}\right) > c;$$

Group velocity $$v_g = \frac{\partial\omega}{\partial k} \approx c\left(1 - \frac{\omega_p^2}{2\omega^2}\right) < c.$$

- In the opposite case of an overdense plasma , where $n_e > n_c$, the refractive index η becomes imaginary and the wave can no longer propagate.

Nonlinear wave propagation :

- Two assumptions:

  (i) the ions are initially singly charged $(Z = 1)$ and are treated as an immobile $(v_i = 0)$, homogeneous background with $(n_0 = Zn_i)$;

  (ii) thermal motion can be neglected, since the temperature remains low compared to the typical oscillation energy in the laser field.

- The starting equations are then as follows:

$$\frac{\partial P}{\partial t} + (v.\nabla)P = -e(E + v \times B),$$

$$\nabla \cdot E = \frac{e}{\varepsilon_0}(n_0 - n_e),$$

$$\nabla \times E = -\frac{\partial B}{\partial t},$$

$$c^2 \nabla \times B = -\frac{e}{\varepsilon_0} n_e v + \frac{\partial E}{\partial t}$$

- Assume a plane-wave geometry with the transverse electromagnetic fields given by $B_L = (0,0,B_z)$ and $E_L = (0,E_y,0)$

- Transverse electron momentum is $P_y = eA_y$, where $E_y = \partial A_y / \partial t$.

This relation expresses conservation of canonical momentum.

- Substituting $E = -\nabla\phi - \partial A / \partial t$ and $B = \nabla \times A$ into Ampere's equation yields

$$c^2 \nabla \times (\nabla \times A) + \frac{\partial^2 A}{\partial t^2} = \frac{J}{\varepsilon_0} - \nabla \frac{\partial \phi}{\partial t},$$

- The current is given by $J = -en_e v.$

- Splitting the current into rotational (solenoidal) and irrotational (longitudinal) parts,

$$J = J_{\perp} + J_{\parallel}$$

$$J_{\parallel} - \frac{1}{c^2} \nabla \frac{\partial \phi}{\partial t} = 0$$

- Applying the coulomb gauge $\nabla \cdot A = 0$ and $v_y = eA_y / \gamma$

$$\frac{\partial^2 A_y}{\partial t^2} - c^2 \nabla^2 A_y = \mu_0 J_y = -\frac{e^2 n_e}{\varepsilon_0 m_e \gamma} A_y .$$

- The nonlinear source term on the right-hand side contains two important bits of physics : $n_e = n_0 + \delta n$, which couples the EM wave to plasma waves, and $\gamma = \sqrt{1 + P^2 / m_e^2 c^2}$ which introduces relativistic effects through the increased electron inertia.

- Longitudinal component of the momentum equation

$$\frac{d}{dt}\left(\gamma m_e v_x\right) = -eE_x - \frac{e^2}{2m_e\gamma}\frac{\partial A_y^2}{\partial x}$$

- Eliminate v_x using the x component of Ampère’s law (21):

$$0 = -\frac{e}{\varepsilon_0} n_e v_x + \frac{\partial E_x}{\partial t}.$$

- The electron density can be determined via Poisson’s equation

$$n_e = n_0 - \frac{\varepsilon_0}{e}\frac{\partial E_x}{\partial x}.$$

- The above (closed) set of equations can in principle be solved numerically for arbitrary pump strengths. For the moment, we simplify things by linearizing the plasma fluid quantities. Let

$$n_e \approx n_0 + n_1 + \ldots\ldots\ldots\ \ldots.,$$

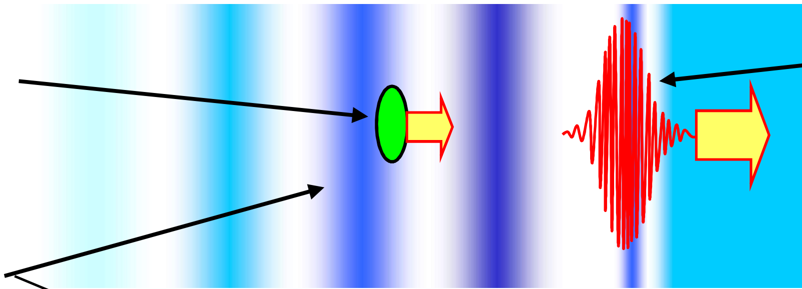
$$v_x \approx v_1 + v_2 + \ldots\ldots\ldots\ \ldots.,$$

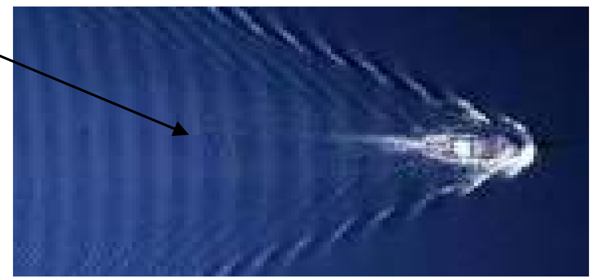
and neglect products of perturbations such as $n_1 v_1$. This leads to

$$\left(\frac{\partial^2}{\partial t^2} + \frac{\omega_p^2}{\gamma_0}\right) E_x = -\frac{\omega_p^2 e}{2 m_e \gamma_0^2} \frac{\partial}{\partial x} A_y^2.$$

- The driving force on the right hand side is the *relativistic ponderomotive force.*

Plasma based electron acceleration

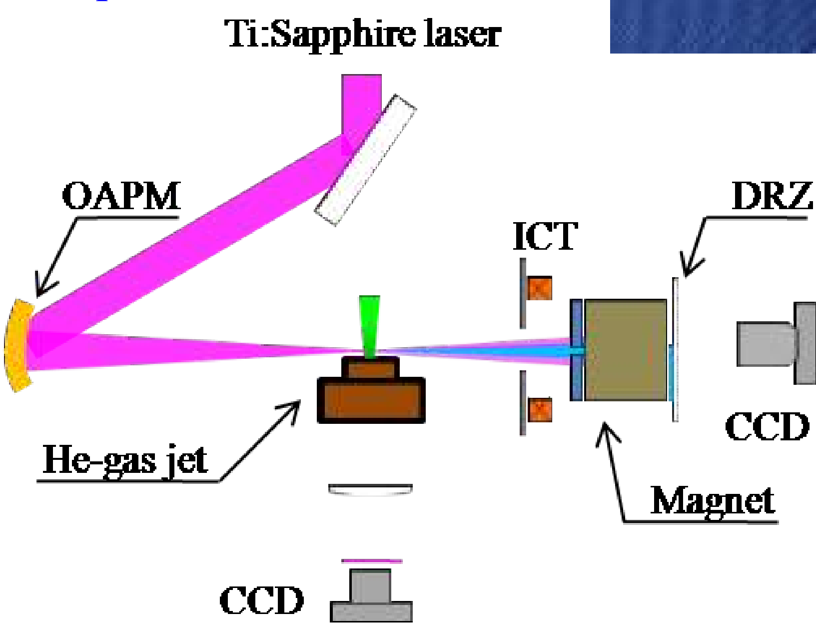


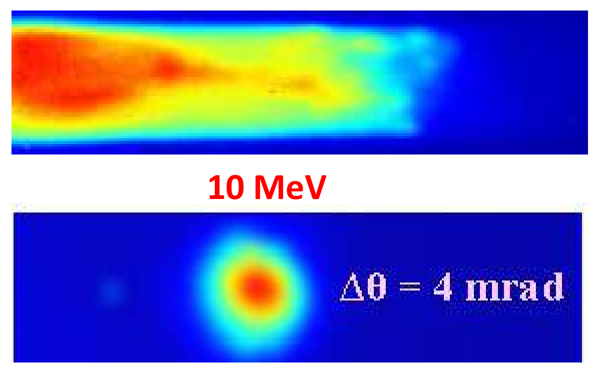


Electrons have been already accelerated to GeV energy using this technique.

Electric fields in plasmas ~ 1000 times higher than those possible in normal accelerators

➔ Compact accelerator





Future compact electron accelerators will be Laser Plasma based !

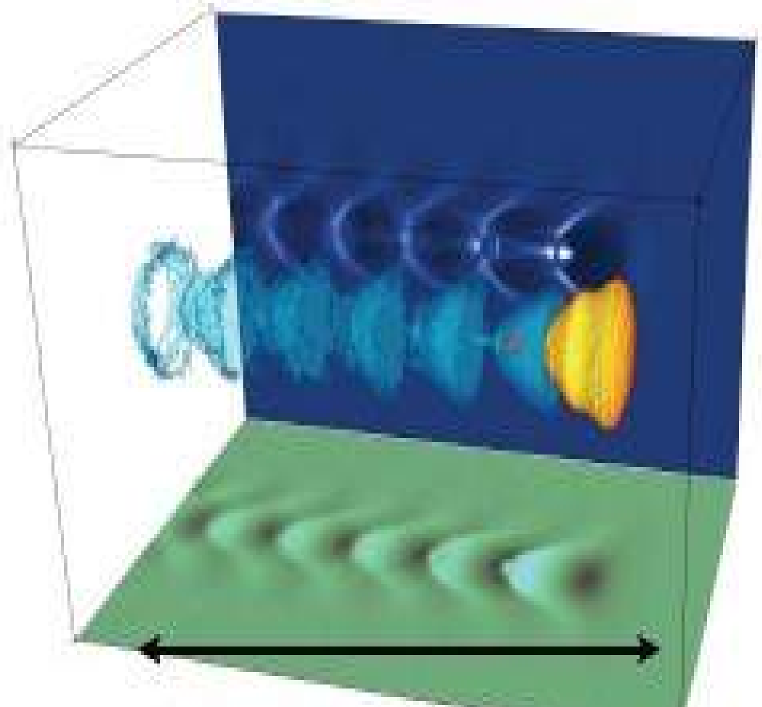
Compactness of Laser Plasma Accelerators

RF Cavity

1 m => 50 MeV Gain

Electric field < 100 MV/m

Plasma Cavity



1mm => 100 MeV

Electric field > 100 GV/m

1979 Relativistic plasma waves with Laser pulse

VOLUME 43, NUMBER 4 PHYSICAL REVIEW LETTERS 23 JULY 1979

Laser Electron Accelerator

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Department of Physics, University of California, Los Angeles, California 90024
(Received 9 March 1979)

An intense electromagnetic pulse can create a weak of plasma oscillations through the action of the nonlinear ponderomotive force. Electrons trapped in the wake can be accelerated to high energy. Existing glass lasers of power density 10^{18}W/cm^2 shone on plasmas of densities 10^{18} cm^{-3} can yield gigaelectronvolts of electron energy per centimeter of acceleration distance. This acceleration mechanism is demonstrated through computer simulation. Applications to accelerators and pulsers are examined.

Such a wake is most effectively generated if the length of the electromagnetic wave packet is half the wavelength of the plasma waves in the wake:

$$L_t = \lambda_w/2 = \pi c/\omega_p . \quad (2)$$

=> Laser wakefield

An alternative way of exciting the plasmon is to inject two laser beams with slightly different frequencies (with frequency difference $\Delta\omega \sim \omega_p$) so that the beat distance of the packet becomes $2\pi c/\omega_p$. The mechanism for generating the wakes

=> Laser beatwave

2004 The Dream Beam

Monoenergetic beams of relativistic electrons from intense laser–plasma interactions

High-quality electron beams from a laser wakefield accelerator using plasma-channel guiding

A laser–plasma accelerator producing monoenergetic electron beams

Plasma based Particle Accelerators

- What makes the system potentially useful is the possibility of generating waves of very high charge separation that travel through the plasma.

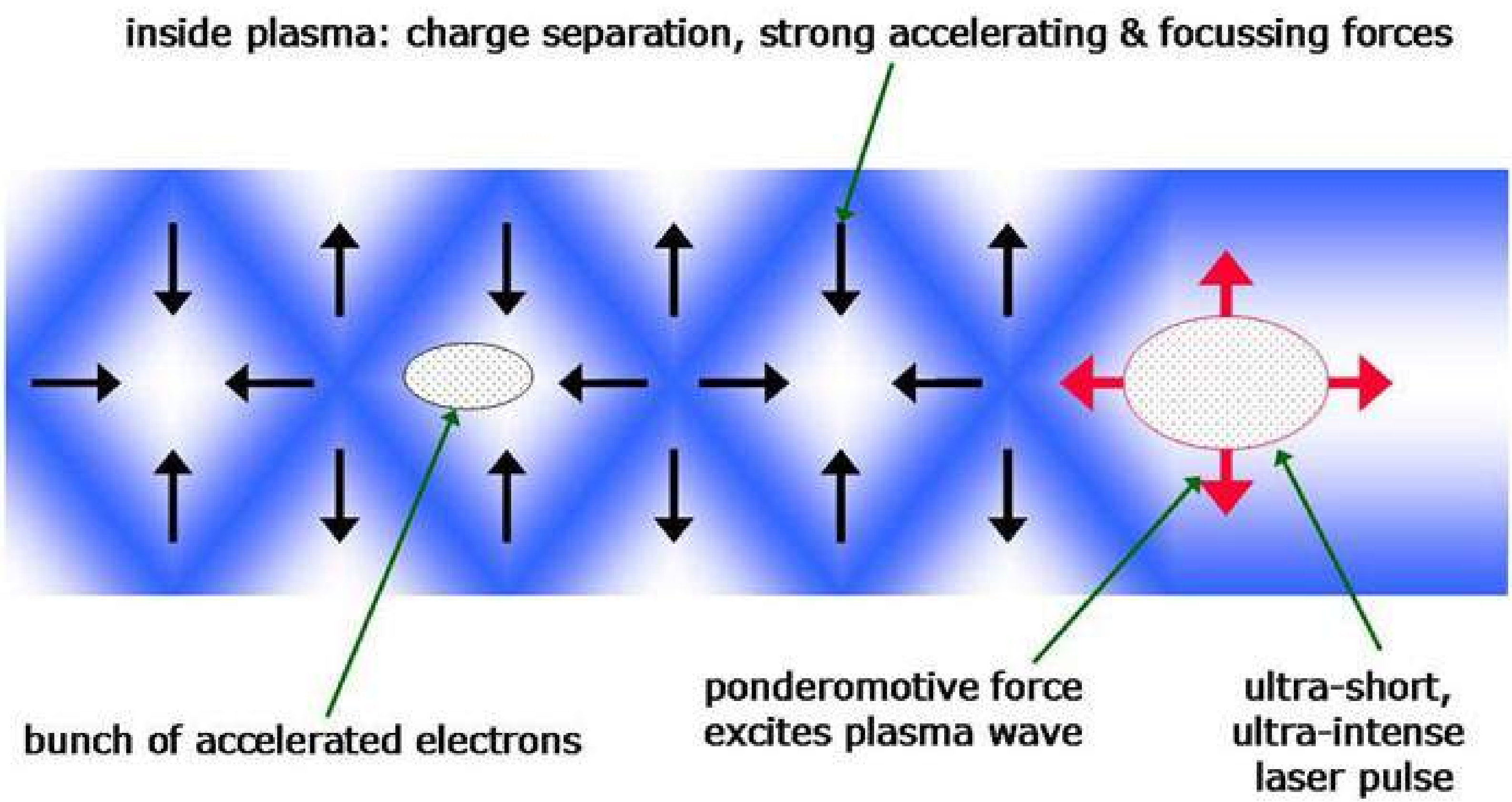
- When an intense, short, laser pulse is fired into a plasma, a plasma wave can be launched – the duration of the laser pulse must be approximately equal to the period of the plasma wave.

- As the pulse rises & falls the plasma electron density oscillates..

- The force doing this is the "ponderomotive force" working along the direction of the pulse and results in the generation of a "plasma wave" in the "wake" of the laser pulse..

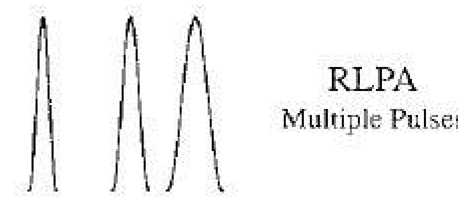
- This leads to a small area of very strong potential gradient following the laser pulse. It is this "wakefield" that is used for particle acceleration.

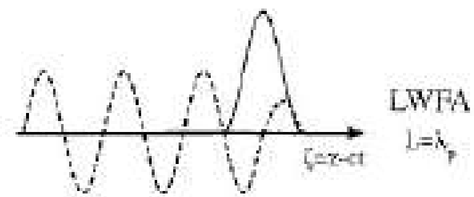
Ponderomotive Acceleration

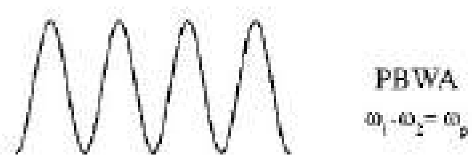


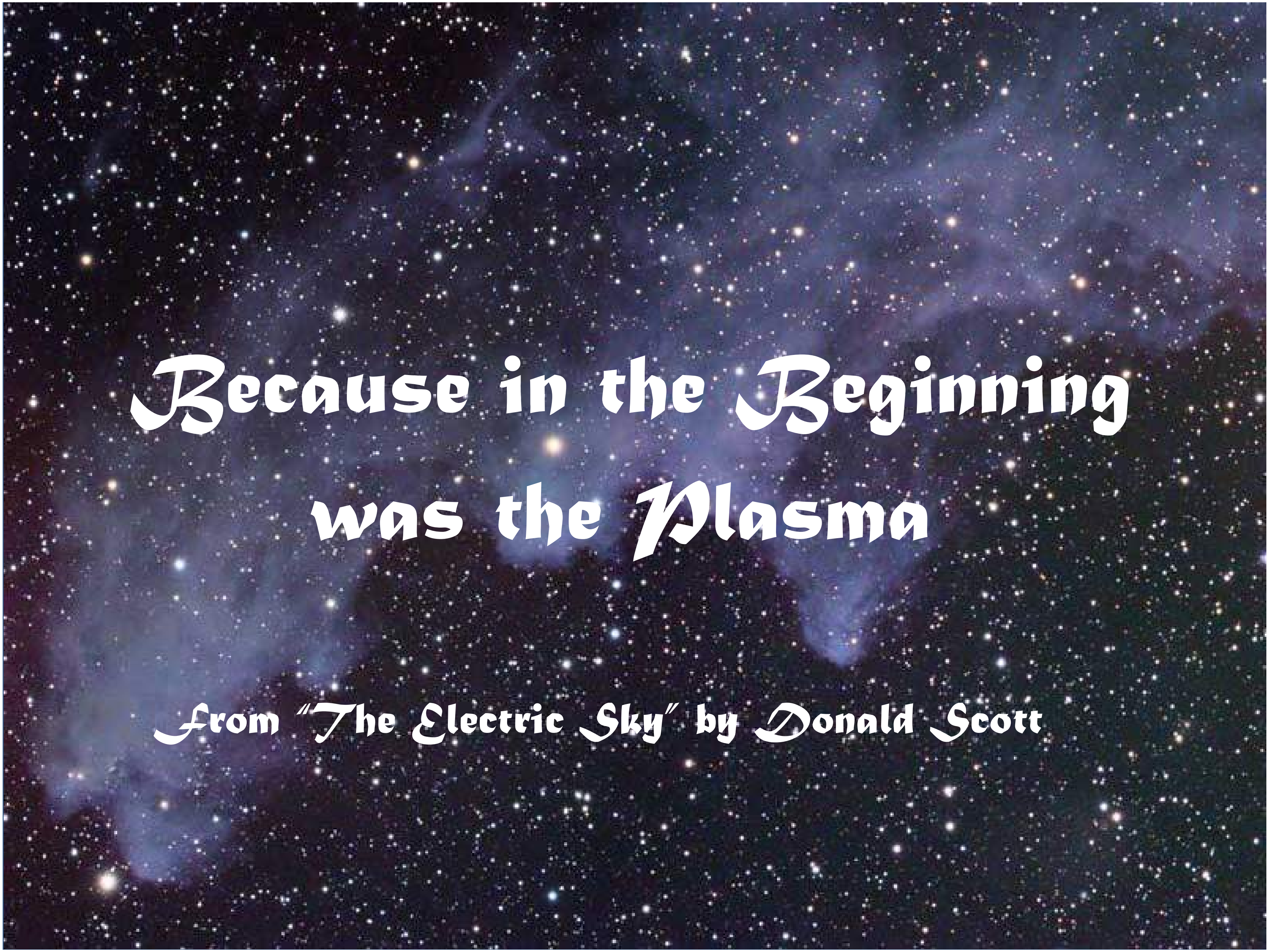
Plasma Wakefield Accelerators

- Plasma Wake Field Accelerator(PWFA)
- Laser Wake Field Accelerator(LWFA)
- Plasma Beat Wave Accelerator(PBWA)
- Self Modulated Laser Wake Field Accelerator(SMLWFA)








Because in the Beginning
was the Plasma
From "The Electric Sky" by Donald Scott

Department of Physics
M.Sc. (Physics), Semester-IV
Elective Paper
PHYE – 401
Physics at LHC and Beyond
Unit – III

Compiled by : Dr. Punit Kumar

<u>Acknowledgement</u> : These notes have been compiled from the lectures delivered in the 'Plasma Accelerator School' at CERN.

<u>Syllabus</u>

Need of the hour: Plasma-based accelerators. Definition and characteristics of plasma, collective behaviour, plasma oscillations, plasma frequency. Propagation of electromagnetic waves in plasma, plasma electron quiver velocity, current density, linear dispersion relation, phase and group velocity, refractive index. Under dense, critically dense and over dense plasmas. Interaction of plasma with intense laser radiation fields, nonlinear interaction- relativistic quiver velocity, dispersion, ponderomotive forces.

1 Plasma types and definitions

Plasmas are often described as the fourth state of matter, alongside gases, liquids and solids, a definition which does little to illuminate their main physical attributes. In fact, a plasma can exhibit behaviour characteristic of all three of the more familiar states, depending on its density and temperature, so we obviously need to look for other distinguishing features. A simple textbook definition of a plasma [1, 2] would be: a *quasi-neutral* gas of charged particles showing *collective* behaviour. This may seem precise enough, but the rather fuzzy-sounding terms of 'quasi-neutrality' and 'collectivity' require further explanation. The first of these, 'quasi-neutrality', is actually just a mathematical way of saying that even though the particles making up a plasma consist of free electrons and ions, their overall charge densities cancel each other in equilibrium. So if n_e and n_i are, respectively, the number densities of electrons and ions with charge state Z, then these are *locally balanced*, i.e.

$$n_e \simeq Z n_i. \tag{1}$$

The second property, 'collective' behaviour, arises because of the long-range nature of the $1/r$ Coulomb potential, which means that local disturbances in equilibrium can have a strong influence on remote regions of the plasma. In other words, macroscopic fields usually dominate over short-lived microscopic fluctuations, and a net charge imbalance $\rho = e(Zn_i - n_e)$ will immediately give rise to an electrostatic field according to Gauss's law,

$$\nabla \quad \mathbf{E} = \rho/\varepsilon_0.$$

Likewise, the same set of charges moving with velocities v_e and v_i will give rise to a *current* density $J = e(Zn_i v_i - n_e v_e)$. This in turn induces a magnetic field according to Ampère's law,

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}.$$

It is these internally driven electric and magnetic fields that largely determine the dynamics of the plasma, including its response to externally applied fields through particle or laser beams—as, for example, in the case of plasma-based accelerator schemes.

Now that we have established what plasmas are, it is natural to ask where we can find them. In fact they are rather ubiquitous: in the cosmos, 99% of the visible universe—including stars, the interstellar medium and jets of material from various astrophysical objects—is in a plasma state. Closer to home, the ionosphere, extending from around 50 km (equivalent to 10 Earth radii) to 1000 km, provides vital protection from solar radiation to life on Earth. Terrestrial plasmas can be found in fusion devices (machines designed to confine, ignite and ultimately extract energy from deuterium–tritium fuel), street lighting, industrial plasma torches and etching processes, and lightning discharges. Needless to say, plasmas play a central role in the topic of the present school, supplying the medium to support very large travelling-wave field structures for the purpose of accelerating particles to high energies. Table 1 gives a brief overview of these various plasma types and their properties.

Table 1: Densities and temperatures of various plasma types

Type	**Electron density** n_e **(cm^{-3})**	**Temperature** T_e **(eV[a])**
Stars	10^{26}	2×10^3
Laser fusion	10^{25}	3×10^3
Magnetic fusion	10^{15}	10^3
Laser-produced	10^{18}–10^{24}	10^2–10^3
Discharges	10^{12}	1–10
Ionosphere	10^6	0.1
Interstellar medium	1	10^{-2}

[a] 1 eV ≡ 11 600 K.

1.1 Debye shielding

In most types of plasma, quasi-neutrality is not just an ideal equilibrium state; it is a state that the plasma actively tries to achieve by readjusting the local charge distribution in response to a disturbance. Consider a hypothetical experiment in which a positively charged ball is immersed in a plasma; see Fig. 1. After some time, the ions in the ball's vicinity will be repelled and the electrons will be attracted, leading to an altered average charge density in this region. It turns out that we can calculate the potential $\phi(r)$ of this ball after such a readjustment has taken place.

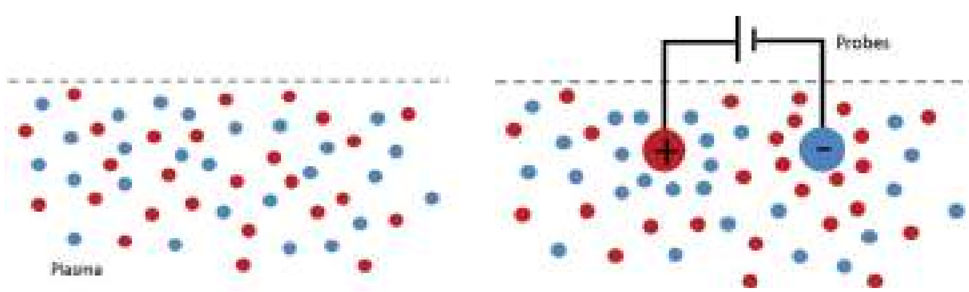


Fig. 1: Debye shielding of charged spheres immersed in a plasma

First of all, we need to know how fast the electrons and ions actually move. For equal ion and electron temperatures ($T_e = T_i$), we have

$$\frac{1}{2}m_e\overline{v}_e^2 = \frac{1}{2}m_i\overline{v}_i^2 = \frac{3}{2}k_B T_e. \tag{2}$$

Therefore, for a hydrogen plasma, where $Z = A = 1$,

$$\frac{\overline{v}_{\mathrm{i}}}{\overline{v}_{\mathrm{e}}} = \left(\frac{m_{\mathrm{e}}}{m_{\mathrm{i}}}\right)^{1/2} = \left(\frac{m_{\mathrm{e}}}{Am_{\mathrm{p}}}\right)^{1/2} = \frac{1}{43}.$$

In other words, the ions are almost stationary on the electron time-scale. To a good approximation, we often write

$$n_{\mathrm{i}} \simeq n_0, \tag{3}$$

where $n_0 = N_{\mathrm{A}}\rho_{\mathrm{m}}/A$ is the material (e.g. gas) number density, with ρ_{m} being the usual mass density and N_{A} the Avogadro constant. In thermal equilibrium, the electron density follows a Boltzmann distribution [1],

$$n_{\mathrm{e}} = n_{\mathrm{i}} \exp(e\phi/k_{\mathrm{B}}T_{\mathrm{e}}), \tag{4}$$

where n_{i} is the ion density, k_{B} is the Boltzmann constant, and $\phi(r)$ is the potential created by the external disturbance. From Gauss's law (Poisson's equation), we can also write

$$\nabla^2\phi = -\frac{\rho}{\varepsilon_0} = -\frac{e}{\varepsilon_0}(n_{\mathrm{i}} - n_{\mathrm{e}}). \tag{5}$$

So now we can combine (5) with (4) and (3) in spherical geometry[1] to eliminate n_{e} and arrive at a physically meaningful solution:

$$\phi_{\mathrm{D}} = \frac{1}{4\pi\varepsilon_0}\frac{\exp(-r/\lambda_{\mathrm{D}})}{r}. \tag{6}$$

This condition supposes that $\phi \to 0$ at $r = \infty$. The characteristic length-scale λ_{D} inside the exponential factor is known as the *Debye length*, and is given by

$$\lambda_{\mathrm{D}} = \left(\frac{\varepsilon_0 k_{\mathrm{B}} T_{\mathrm{e}}}{e^2 n_{\mathrm{e}}}\right)^{1/2} = 743\left(\frac{T_{\mathrm{e}}}{\mathrm{eV}}\right)^{1/2}\left(\frac{n_{\mathrm{e}}}{\mathrm{cm}^{-3}}\right)^{-1/2}\mathrm{cm}. \tag{7}$$

The Debye length is a fundamental property of nearly all plasmas of interest, and depends equally on the plasma's temperature and density. An *ideal* plasma has many particles per Debye sphere, i.e.

$$N_{\mathrm{D}} \equiv n_{\mathrm{e}}\frac{4\pi}{3}\lambda_{\mathrm{D}}^3 \gg 1, \tag{8}$$

which is a prerequisite for the collective behaviour discussed earlier. An alternative way of expressing this condition is via the so-called *plasma parameter*,

$$g \equiv \frac{1}{n_{\mathrm{e}}\lambda_{\mathrm{D}}^3}, \tag{9}$$

which is essentially the reciprocal of N_{D}. Classical plasma theory is based on the assumption that $g \ll 1$, which implies dominance of collective effects over collisions between particles. Therefore, before we refine our plasma classification, it is worth taking a quick look at the nature of collisions between plasma particles.

1.2 Collisions in plasmas

Where $N_{\mathrm{D}} \leq 1$, screening effects are reduced and collisions will dominate the particle dynamics. In intermediate regimes, collisionality is usually measured via the *electron–ion collision rate*, given by

$$\nu_{\mathrm{ei}} = \frac{\pi^{3/2} n_{\mathrm{e}} Z e^4 \ln\Lambda}{2^{1/2}(4\pi\varepsilon_0)^2 m_{\mathrm{e}}^2 v_{\mathrm{te}}^3}\ \mathrm{s}^{-1}, \tag{10}$$

---

[1] $\nabla^2 \to \frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\phi}{\mathrm{d}r}\right)$.

where $v_{te} \equiv \sqrt{k_B T_e/m_e}$ is the electron thermal velocity and $\ln\Lambda$ is a slowly varying term, called the Coulomb logarithm, which typically takes a numerical value of order 10–20. The numerical coefficient in expression (10) may vary between different texts depending on the definition used. Our definition is consistent with that in Refs. [5] and [3], which define the collision rate according to the average time taken for a thermal electron to be deflected by 90° via multiple scatterings from fixed ions. The collision frequency can also be written as

$$\frac{\nu_{ei}}{\omega_p} \simeq \frac{Z \ln\Lambda}{10 N_D} \quad \text{with } \ln\Lambda \simeq 9N_D/Z.$$

where ω_p is the electron plasma frequency defined below in Eq. (11).

1.3 Plasma classification

Armed with our definition of plasma ideality, Eq. (8), we can proceed to make a classification of plasma types in density–temperature space. This is illustrated for a few examples in Fig. 2; the 'accelerator' plasmas of interest in the present school are found in the middle of this chart, having densities corresponding to roughly atmospheric pressure and temperatures of a few eV (10^4 K) as a result of field ionization; see Section 1.5.

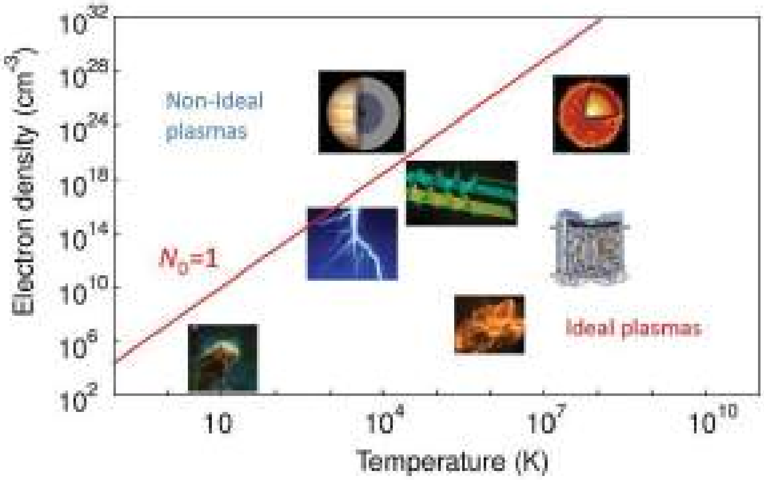


Fig. 2: Examples of plasma types in the density–temperature plane

1.4 Plasma oscillations

So far we have considered characteristics, such as density and temperature, of a plasma in equilibrium. We can also ask how fast the plasma will respond to an external disturbance, which could be due to electromagnetic waves (e.g. a laser pulse) or particle beams. Consider a quasi-neutral plasma slab in which an electron layer is displaced from its initial position by a distance δ, as illustrated in Fig. 3. This creates two 'capacitor' plates with surface charge $\sigma = \pm e n_e \delta$, resulting in an electric field

$$\mathrm{E} = \frac{\sigma}{\varepsilon_0} = \frac{e n_e \delta}{\varepsilon_0}.$$

The electron layer is accelerated back towards the slab by this restoring force according to

$$m_e \frac{\mathrm{d}v}{\mathrm{d}t} = -m_e \frac{\mathrm{d}^2\delta}{\mathrm{d}t^2} = -eE = \frac{e^2 n_e \delta}{\varepsilon_0},$$

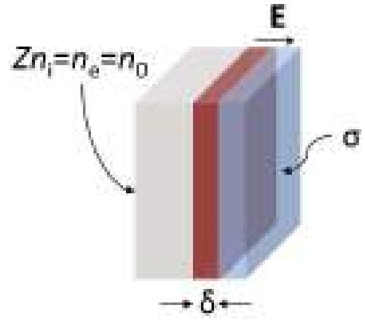


Fig. 3: Slab or capacitor model of an oscillating electron layer

or

$$\frac{\mathrm{d}^2\delta}{\mathrm{d}t^2} + \omega_\mathrm{p}^2\,\delta = 0$$

where

$$\omega_\mathrm{p} \equiv \left(\frac{e^2 n_\mathrm{e}}{\varepsilon_0 m_\mathrm{e}}\right)^{1/2} \simeq 5.6 \times 10^4 \left(\frac{n_\mathrm{e}}{\mathrm{cm}^{-3}}\right)^{1/2} \mathrm{s}^{-1} \tag{11}$$

is the *electron plasma frequency*.

This quantity can be obtained via another route by returning to the Debye sheath problem of Section 1.1 and asking how quickly it would take the plasma to adjust to the insertion of the foreign charge. For a plasma of temperature T_e, the response time to recover quasi-neutrality is just the ratio of the Debye length to the thermal velocity $v_\mathrm{te} \equiv \sqrt{k_\mathrm{B} T_\mathrm{e}/m_\mathrm{e}}$; that is,

$$t_\mathrm{D} \simeq \frac{\lambda_\mathrm{D}}{v_\mathrm{te}} = \left(\frac{\varepsilon_0 k_\mathrm{B} T_\mathrm{e}}{e^2 n_\mathrm{e}} \;\; \frac{m}{k_\mathrm{B} T_\mathrm{e}}\right)^{1/2} = \omega_\mathrm{p}^{-1}.$$

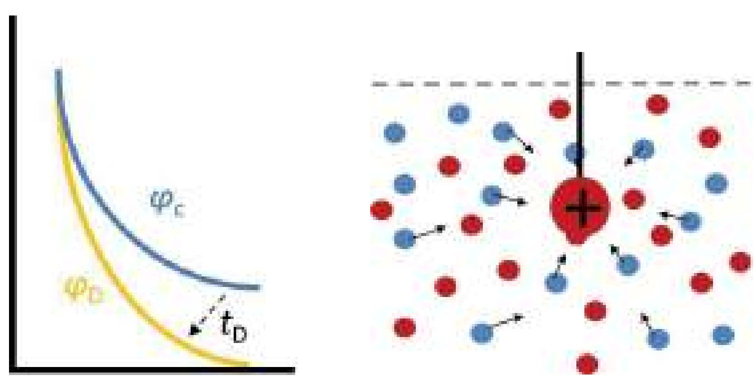


Fig. 4: Response time to form a Debye sheath

If the plasma response time is shorter than the period of a external electromagnetic field (such as a laser), then this radiation will be *shielded out*. To make this statement more quantitative, consider the ratio

$$\frac{\omega_\mathrm{p}^2}{\omega^2} = \frac{e^2 n_\mathrm{e}}{\varepsilon_0 m_\mathrm{e}} \cdot \frac{\lambda^2}{4\pi^2 c^2}.$$

Setting this to unity defines the wavelength λ_μ for which $n_e = n_c$, or

$$n_c \simeq 10^{21} \lambda_\mu^{-2} \text{ cm}^{-3}. \tag{12}$$

Radiation with wavelength $\lambda > \lambda_\mu$ will be reflected. In the pre-satellite/cable era, this property was exploited to good effect in the transmission of long-wave radio signals, which utilizes reflection from the ionosphere to extend the range of reception.

Typical gas jets have $P \sim 1$ bar and $n_e = 10^{18}$–10^{19} cm^{-3}, and the critical density for a glass laser is $n_c(1\mu) = 10^{21}$ cm^{-3}. Gas-jet plasmas are therefore *underdense*, since $\omega^2/\omega_p^2 = n_e/n_c \ll 1$. In this case, *collective effects* are important if $\omega_p \tau_{int} > 1$, where τ_{int} is some characteristic interaction time, such as the duration of a laser pulse or particle beam entering the plasma. For example, if $\tau_{int} = 100$ fs and $n_e = 10^{17}$ cm^{-3}, then $\omega_p \tau_{int} = 1.8$ and we will need to consider the plasma response on the interaction time-scale. Generally this is the situation we seek to exploit in all kinds of plasma applications, including short-wavelength radiation, nonlinear refractive properties, generation of high electric/magnetic fields and, of course, particle acceleration.

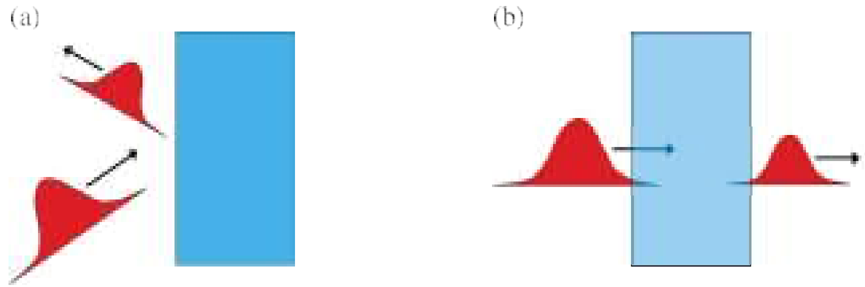


Fig. 5: (a) Overdense plasma, with $\omega < \omega_p$, showing mirror-like behaviour. (b) Underdense plasma, with $\omega > \omega_p$, which behaves like a nonlinear refractive medium.

1.5 Plasma creation

Plasmas are created via ionization, which can occur in several ways: through collisions of fast particles with atoms; through photoionization by electromagnetic radiation; or via electrical breakdown in strong electric fields. The latter two are examples of *field ionization*, which is the mechanism most relevant to the plasma accelerator context. To get some idea of when field ionization occurs, we need to know the typical field strength required to strip electrons away from an atom. At the Bohr radius

$$a_B = \frac{\hbar^2}{me^2} = 5.3 \times 10^{-9} \text{ cm},$$

the electric field strength is

$$E_a = \frac{e}{4\pi\varepsilon_0 a_B^2} \simeq 5.1 \times 10^9 \text{ V m}^{-1}. \tag{13}$$

This threshold can be expressed as the so-called *atomic intensity*,

$$I_a = \frac{\varepsilon_0 c E_a^2}{2} \simeq 3.51 \times 10^{16} \text{ W cm}^{-2}. \tag{14}$$

A laser intensity of $I_L > I_a$ will therefore guarantee ionization for any target material, though in fact ionization can occur well below this threshold (e.g. around 10^{14} W cm^{-2} for hydrogen) due to *multiphoton* effects. Simultaneous field ionization of many atoms produces a plasma with electron density n_e and temperature $T_e \sim 1$–10 eV.

1.6 Relativistic threshold

Before we discuss wave propagation in plasmas, it is useful to have some idea of the strength of the external fields used to excite them. To do this, we consider the classical equation of motion for an electron exposed to a linearly polarized laser field $\mathbf{E} = \hat{y}E_0 \sin\omega t$:

$$\frac{dv}{dt} \simeq \frac{-eE_0}{m_e} \sin\omega t.$$

This implies that the electron will acquire a velocity

$$v = \frac{eE_0}{m_e\omega} \cos\omega t = v_{osc} \cos\omega t, \tag{15}$$

which is usually expressed in terms of a dimensionless oscillation amplitude

$$a_0 \equiv \frac{v_{osc}}{c} \equiv \frac{p_{osc}}{m_e c} \equiv \frac{eE_0}{m_e\omega c}. \tag{16}$$

In many articles and books a_0 is referred to as the 'quiver' velocity or momentum; it can exceed unity, in which case the normalized momentum (third expression) is more appropriate, since the real particle velocity is just pinned to the speed of light. The laser intensity I_L and wavelength λ_L are related to E_0 and ω through

$$I_L = \frac{1}{2}\varepsilon_0 c E_0^2, \qquad \lambda_L = \frac{2\pi c}{\omega}.$$

By substituting these into (16) one can show that

$$a_0 \simeq 0.85(I_{18}\lambda_\mu^2)^{1/2}, \tag{17}$$

where

$$I_{18} = \frac{I_L}{10^{18}\ \mathrm{W\,cm^{-2}}}, \qquad \lambda_\mu = \frac{\lambda_L}{\mu\mathrm{m}}.$$

From this expression it can be seen that we will have relativistic electron velocities, or $a_0 \sim 1$, for intensities $I_L \geq 10^{18}$ W cm^{-2}, at wavelengths $\lambda_L \simeq 1\ \mu$m.

2 Wave propagation in plasmas

The theory of wave propagation is an important subject in its own right, and has inspired a vast body of literature and a number of textbooks [4, 5, 8]. There are a great many possible ways in which plasmas can support waves, depending on the local conditions, the presence of external electric and magnetic fields, and so on. Here we will concentrate on two main wave forms: longitudinal oscillations of the kind we have encountered already, and electromagnetic waves. To derive and analyse wave phenomena, there are several possible theoretical approaches, with the suitability of each depending on the length- and time-scales of interest, which in laboratory plasmas can range from nanometres to metres and from femtoseconds to seconds. These approaches are:

(i) first-principles N-body molecular dynamics;
(ii) phase-space methods—the Vlasov–Boltzmann equation;
(iii) two-fluid equations;
(iv) magnetohydrodynamics (single magnetized fluid).

The first is rather costly and limited to much smaller regions of plasma than usually needed to describe the common types of wave. Indeed, the number of particles needed for first-principles modelling of a tokamak would be around 10^{21}; a laser-heated gas requires 10^{20} particles, still way out of reach of even the most powerful computers available. Clearly a more tractable model is needed, and in fact many plasma phenomena can be analysed by assuming that each charged particle component of density n_s and velocity $\mathbf{u}_s$ behaves in a fluid-like manner, interacting with other species (s) via the electric and magnetic fields; this is the idea behind approach (iii). The rigorous way to derive the governing equations in this approximation is via *kinetic theory*, starting from method (ii) [2, 5], which is beyond the scope of this paper. Finally, slow wave phenomena on more macroscopic, ion time-scales can be handled with approach (iv) [2].

For the present purposes, we therefore start from the two-fluid equations for a plasma with finite temperature ($T_e > 0$) that is assumed to be collisionless ($\nu_{ie} \simeq 0$) and non-relativistic, so that the fluid velocities are such that $u \ll c$. The equations governing the plasma dynamics under these conditions are

$$\frac{\partial n_s}{\partial t} + \nabla \cdot (n_s \mathbf{u}_s) = 0, \tag{18}$$

$$n_s m_s \frac{\mathrm{d}\mathbf{u}_s}{\mathrm{d}t} = n_s q_s (\mathbf{E} + \mathbf{u}_s \times \mathbf{B}) - \nabla P_s, \tag{19}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(P_s n_s^{-\gamma_s}) = 0, \tag{20}$$

where P_s is the thermal pressure of species s and γ_s the specific heat ratio, or $(2+N)/N$ with N the number of degrees of freedom.

The continuity equation (18) tells us that (in the absence of ionization or recombination) the number of particles *of each species* is conserved. Noting that the charge and current densities can be written as $\rho_s = q_s n_s$ and $\mathbf{J}_s = q_s n_s \mathbf{u}_s$, respectively, Eq. (18) can be rewritten as

$$\frac{\partial \rho_s}{\partial t} + \nabla \cdot \mathbf{J}_s = 0, \tag{21}$$

which expresses the conservation of *charge*.

Equation (19) governs the motion of a fluid element of species s in the presence of electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$. In the absence of fields, and assuming strict quasi-neutrality ($n_e = Zn_i = n$; $\mathbf{u}_e = \mathbf{u}_i = \mathbf{u}$), we recover the more familiar *Navier–Stokes* equations

$$\begin{aligned} &\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \\ &\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = \frac{1}{\rho}\nabla P. \end{aligned} \tag{22}$$

By contrast, in the plasma accelerator context we usually deal with time-scales over which the ions can be assumed to be motionless, i.e. $\mathbf{u}_i = 0$, and also unmagnetized plasmas, so that the momentum equation reads

$$n_e m_e \frac{\mathrm{d}\mathbf{u}_e}{\mathrm{d}t} = -e\, n_e \mathbf{E} - \nabla P_e. \tag{23}$$

Note that $\mathbf{E}$ can include both external and internal field components (via charge separation).

2.1 Longitudinal (Langmuir) waves

A characteristic property of plasmas is their ability to transfer momentum and energy via collective motion. One of the most important examples of this is the oscillation of electrons against a stationary ion background, or *Langmuir waves*. Returning to the two-fluid model, we can simplify (18)–(20) by setting $\mathbf{u}_i = 0$, restricting the electron motion to one dimension (x) and taking $\frac{\partial}{\partial y} = \frac{\partial}{\partial z} = 0$:

$$\frac{\partial n_e}{\partial t} + \frac{\partial}{\partial x}(n_e u_e) = 0,$$

$$n_e\left(\frac{\partial u_e}{\partial t}+u_e\frac{\partial u_e}{\partial x}\right)=-\frac{e}{m}n_eE-\frac{1}{m}\frac{\partial P_e}{\partial x}, \tag{24}$$
$$\frac{d}{dt}\left(\frac{P_e}{n_e^3}\right)=0.$$

The system (24) has three equations and four unknowns. To close it, we need an expression for the electric field, which, since $\mathbf{B}=0$, can be found from Gauss's law (Poisson's equation) with $Zn_i=n_0$:

$$\frac{\partial E}{\partial x}=\frac{e}{\varepsilon_0}(n_0-n_e). \tag{25}$$

The system of equations (24)–(25) is nonlinear and, apart from a few special cases, cannot be solved exactly. A common technique for analysing waves in plasmas is to *linearize* the equations, which involves assuming that the perturbed amplitudes are small compared to the equilibrium values, i.e.

$$\begin{aligned} n_e &= n_0+n_1,\\ u_e &= u_1,\\ P_e &= P_0+P_1,\\ E &= E_1, \end{aligned}$$

where $n_1 \ll n_0$ and $P_1 \ll P_0$. Upon substituting these expressions into (24)–(25) and neglecting all products of perturbations such as $n_1\partial_t u_1$ and $u_1\partial_x u_1$, we get a set of linear equations for the perturbed quantities:

$$\begin{aligned} \frac{\partial n_1}{\partial t}+n_0\frac{\partial u_1}{\partial x} &= 0,\\ n_0\frac{\partial u_1}{\partial t} &= -\frac{e}{m}n_0E_1-\frac{1}{m}\frac{\partial P_1}{\partial x},\\ \frac{\partial E_1}{\partial x} &= -\frac{e}{\varepsilon_0}n_1, \end{aligned} \tag{26}$$

$$P_1=3k_BT_e\,n_1.$$

The expression for P_1 results from the specific heat ratio $\gamma_e=3$ and from assuming isothermal background electrons, $P_0=k_BT_en_0$ (ideal gas); see Kruer's book [5]. We can now eliminate E_1, P_1 and u_1 from (26) to get

$$\left(\frac{\partial^2}{\partial t^2}-3v_{te}^2\frac{\partial^2}{\partial x^2}+\omega_p^2\right)n_1=0, \tag{27}$$

with $v_{te}^2=k_BT_e/m_e$ and ω_p given by (11) as before. Finally, we look for plane-wave solutions of the form $A=A_0\exp\{i(\omega t-kx)\}$, so that our derivative operators are transformed as follows: $\frac{\partial}{\partial t}\to i\omega$ and $\frac{\partial}{\partial x}\to -ik$. Substitution into (27) yields the Bohm–Gross dispersion relation

$$\omega^2=\omega_p^2+3k^2v_{te}^2. \tag{28}$$

This and other dispersion relations are often depicted graphically on a chart such as that in Fig. 6, which gives an overview of which propagation modes are permitted for low- and high-wavelength limits.

2.2 Transverse waves

To describe *transverse* electromagnetic (EM) waves, we need two additional Maxwell's equations, Faraday's law and Ampère's law, which we will introduce properly later; see Eqs. (38) and (39). For the time being, it is helpful to simplify things by making use of our previous analysis of small-amplitude longitudinal waves. Therefore, we linearize and again apply the harmonic approximation $\frac{\partial}{\partial t}\to i\omega$ to get

$$\nabla\times\mathbf{E}_1=-i\omega\mathbf{B}_1, \tag{29}$$

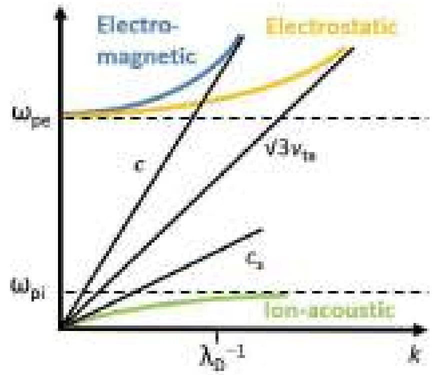


Fig. 6: Schematic illustration of dispersion relations for Langmuir, electromagnetic and ion-acoustic waves

$$\nabla \times \mathbf{B}_1 = \mu_0 \mathbf{J}_1 + \mathrm{i}\varepsilon_0\mu_0\omega \mathbf{E}_1, \tag{30}$$

where the transverse current density is given by

$$\mathbf{J}_1 = -n_0 e \mathbf{u}_1. \tag{31}$$

This time we look for pure EM plane-wave solutions with $\mathbf{E}_1 \perp \mathbf{k}$ (see Fig. 7) and also assume that the group and phase velocities are large enough, $v_\mathrm{p}, v_g \gg v_\mathrm{te}$, so that we have a *cold* plasma with $P_e = n_0 k_\mathrm{B} T_e \simeq 0$.

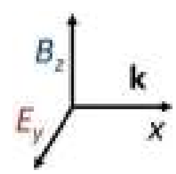


Fig. 7: Geometry for electromagnetic plane-wave analysis

The linearized electron fluid velocity and corresponding current are then

$$\begin{aligned} \mathbf{u}_1 &= -\frac{e}{\mathrm{i}\omega m_e}\mathbf{E}_1, \\ \mathbf{J}_1 &= \frac{n_0 e^2}{\mathrm{i}\omega m_e}\mathbf{E}_1 \equiv \sigma \mathbf{E}_1, \end{aligned} \tag{32}$$

where σ is the *AC electrical conductivity*. By analogy with dielectric media (see, e.g., Ref. [7]), in which Ampère's law is usually written as $\nabla \times \mathbf{B}_1 = \mu_0 \partial_t \mathbf{D}_1$, by substituting (32) into (39) one can show that

$$\mathbf{D}_1 = \varepsilon_0 \varepsilon \mathbf{E}_1,$$

with

$$\varepsilon = 1 + \frac{\sigma}{\mathrm{i}\omega\varepsilon_0} = 1 - \frac{\omega_\mathrm{p}^2}{\omega^2}. \tag{33}$$

From (33) it follows immediately that

$$\eta \equiv \sqrt{\varepsilon} = \frac{ck}{\omega} = \left(1 - \frac{\omega_\mathrm{p}^2}{\omega^2}\right)^{1/2}, \tag{34}$$

with

$$\omega^2 = \omega_p^2 + c^2 k^2. \tag{35}$$

The above expression can also be found directly by elimination of $\mathbf{J}_1$ and $\mathbf{B}_1$ from Eqs. (29)–(32). From the dispersion relation (35), also depicted in Fig. 6, a number of important features of EM wave propagation in plasmas can be deduced. For *underdense* plasmas ($n_e \ll n_c$),

$$\text{phase velocity} \quad v_p = \frac{\omega}{k} \simeq c\left(1 + \frac{\omega_p^2}{2\omega^2}\right) > c\,;$$

$$\text{group velocity} \quad v_g = \frac{\partial \omega}{\partial k} \simeq c\left(1 - \frac{\omega_p^2}{2\omega^2}\right) < c\,.$$

In the opposite case of an *overdense* plasma, where $n_e > n_c$, the refractive index η becomes imaginary and the wave can no longer propagate, becoming evanescent instead, with a decay length determined by the *collisionless skin depth* c/ω_p; see Fig. 8.

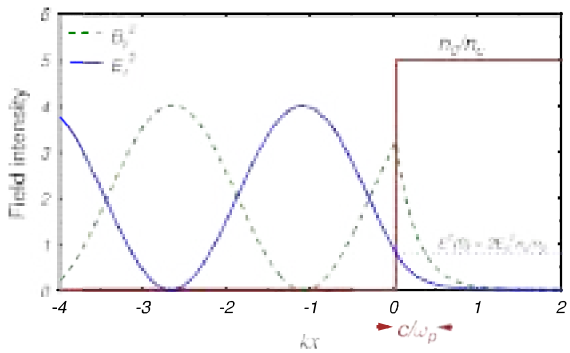


Fig. 8: Electromagnetic fields resulting from reflection of an incoming wave by an overdense plasma slab

2.3 Nonlinear wave propagation

So far we have considered purely longitudinal or transverse waves; linearizing the wave equations ensures that any nonlinearities or coupling between these two modes is excluded. While this is a reasonable approximation for low-amplitude waves, it is inadequate for describing strongly driven waves in the relativistic regime of interest in plasma accelerator schemes. The starting point of most analyses of nonlinear wave propagation phenomena is the Lorentz equation of motion for the electrons in a *cold* ($T_e = 0$) unmagnetized plasma, together with Maxwell's equations [5,6]. We make two further assumptions: (i) that the ions are initially singly charged ($Z = 1$) and are treated as an immobile ($v_i = 0$), homogeneous background with $n_0 = Zn_i$; (ii) that thermal motion can be neglected, since the temperature remains low compared to the typical oscillation energy in the laser field ($v_{osc} \gg v_{te}$). The starting equations (in SI units) are then as follows:

$$\frac{\partial \mathbf{p}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{p} = -e(\mathbf{E} + \mathbf{v} \times \mathbf{B}), \tag{36}$$

$$\nabla \cdot \mathbf{E} = \frac{e}{\varepsilon_0}(n_0 - n_e), \tag{37}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{38}$$

$$c^2 \nabla \times \mathbf{B} = -\frac{e}{\varepsilon_0} n_e \mathbf{v} + \frac{\partial \mathbf{E}}{\partial t}, \tag{39}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{40}$$

where $\mathbf{p} = \gamma m_e \mathbf{v}$ and $\gamma = (1 + p^2/m_e^2 c^2)^{1/2}$.

To simplify matters, we first assume a plane-wave geometry like that in Fig. 7, with the transverse electromagnetic fields given by $\mathbf{E}_L = (0, E_y, 0)$ and $\mathbf{B}_L = (0, 0, B_z)$. From Eq. (36), the transverse electron momentum is then simply

$$p_y = eA_y, \tag{41}$$

where $E_y = -\partial A_y/\partial t$. This relation expresses conservation of canonical momentum. Substituting $\mathbf{E} = -\nabla\phi - \partial\mathbf{A}/\partial t$ and $\mathbf{B} = \nabla \times \mathbf{A}$ into Ampère's equation (39) yields

$$c^2\nabla \times (\nabla \times \mathbf{A}) + \frac{\partial^2 \mathbf{A}}{\partial t^2} = \frac{\mathbf{J}}{\varepsilon_0} - \nabla\frac{\partial\phi}{\partial t},$$

where the current is given by $\mathbf{J} = -en_e\mathbf{v}$. Now we use a bit of vectorial wizardry, splitting the current into rotational (solenoidal) and irrotational (longitudinal) parts,

$$\mathbf{J} = \mathbf{J}_\perp + \mathbf{J}_\parallel = \nabla \times \mathbf{\Pi} + \nabla\Psi,$$

from which we can deduce (see Jackson's book [7]) that

$$\mathbf{J}_\parallel - \frac{1}{c^2}\nabla\frac{\partial\phi}{\partial t} = 0.$$

Finally, by applying the Coulomb gauge $\nabla \cdot \mathbf{A} = 0$ and $v_y = eA_y/\gamma$ from (41), we obtain

$$\frac{\partial^2 A_y}{\partial t^2} - c^2\nabla^2 A_y = \mu_0 J_y = -\frac{e^2 n_e}{\varepsilon_0 m_e \gamma} A_y. \tag{42}$$

The nonlinear source term on the right-hand side contains two important bits of physics: $n_e = n_0 + \delta n$, which couples the EM wave to plasma waves, and $\gamma = \sqrt{1 + \mathbf{p}^2/m_e^2 c^2}$, which introduces relativistic effects through the increased electron inertia. Taking the *longitudinal* component of the momentum equation (36) gives

$$\frac{\mathrm{d}}{\mathrm{d}t}(\gamma m_e v_x) = -eE_x - \frac{e^2}{2m_e\gamma}\frac{\partial A_y^2}{\partial x}.$$

We can eliminate v_x using the x component of Ampère's law (39):

$$0 = -\frac{e}{\varepsilon_0} n_e v_x + \frac{\partial E_x}{\partial t}.$$

And the electron density can be determined via Poisson's equation (37):

$$n_e = n_0 - \frac{\varepsilon_0}{e}\frac{\partial E_x}{\partial x}.$$

The above (closed) set of equations can in principle be solved numerically for arbitrary pump strengths. For the moment, we simplify things by linearizing the *plasma* fluid quantities. Let

$$n_e \simeq n_0 + n_1 + \cdots,$$
$$v_x \simeq v_1 + v_2 + \cdots,$$

and neglect products of perturbations such as $n_1 v_1$. This leads to

$$\left(\frac{\partial^2}{\partial t^2} + \frac{\omega_p^2}{\gamma_0}\right) E_x = -\frac{\omega_p^2 e}{2m_e\gamma_0^2}\frac{\partial}{\partial x} A_y^2. \tag{43}$$

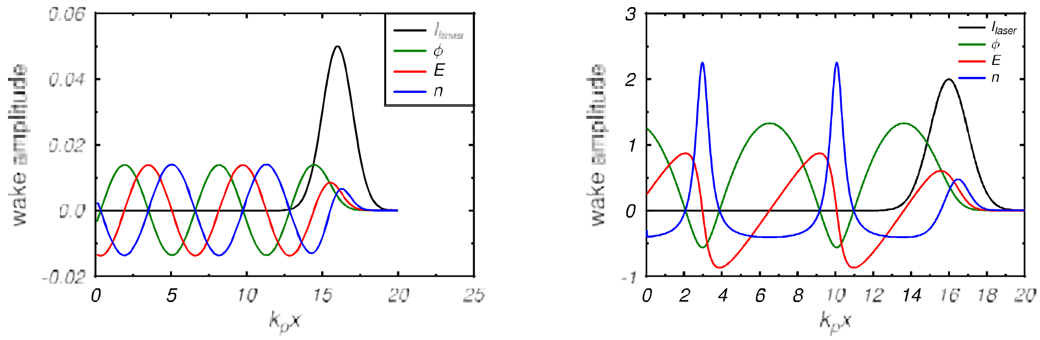


Fig. 9: Wakefield excitation by a short-pulse laser propagating in the positive x direction in the linear regime (left) and nonlinear regime (right).

The driving term on the right-hand side is the *relativistic ponderomotive force*, with $\gamma_0 = (1 + a_0^2/2)^{1/2}$. Some solutions of Eq. (43) are shown in Fig. 9, for low- and high-intensity laser pulses. The properties of the wakes will be discussed in detail in other lectures, but we can already see some obvious qualitative differences between the linear and nonlinear wave forms; the latter are typically characterized by a spiked density profile, a sawtooth electric field, and a longer wavelength.

The coupled fluid equations (42) and (43) and their fully nonlinear counterparts describe a wide range of nonlinear laser–plasma interaction phenomena, many of which are treated in the later lectures of this school, including plasma wake generation, blow-out regime laser self-focusing and channelling, parametric instabilities, and harmonic generation. Plasma-accelerated particle *beams*, on the other hand, cannot be treated using fluid theory and require a more sophisticated kinetic approach, usually assisted by numerical models solved with the aid of powerful supercomputers.

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Appendices

A Useful constants and formulae

Table A.1: Commonly used physical constants

Name	Symbol	Value (SI)	Value (cgs)
Boltzmann constant	k_B	1.38×10^{-23} J K^{-1}	1.38×10^{-16} erg K^{-1}
Electron charge	e	1.6×10^{-19} C	4.8×10^{-10} statcoul
Electron mass	m_e	9.1×10^{-31} kg	9.1×10^{-28} g
Proton mass	m_p	1.67×10^{-27} kg	1.67×10^{-24} g
Planck constant	h	6.63×10^{-34} J s	6.63×10^{-27} erg-s
Speed of light	c	3×10^{8} m s^{-1}	3×10^{10} cm s^{-1}
Dielectric constant	ε_0	8.85×10^{-12} F m^{-1}	—
Permeability constant	μ_0	$4\pi \times 10^{-7}$	—
Proton/electron mass ratio	m_p/m_e	1836	1836
Temperature = 1eV	e/k_B	11 604 K	11 604 K
Avogadro number	N_A	6.02×10^{23} mol^{-1}	6.02×10^{23} mol^{-1}
Atmospheric pressure	1 atm	1.013×10^{5} Pa	1.013×10^{6} dyne cm^{-2}

Table A.2: Formulae in SI and cgs units

Name	Symbol	Formula (SI)	Formula (cgs)
Debye length	λ_D	$\left(\frac{\varepsilon_0 k_B T_e}{e^2 n_e}\right)^{1/2}$ m	$\left(\frac{k_B T_e}{4\pi e^2 n_e}\right)^{1/2}$ cm
Particles in Debye sphere	N_D	$\frac{4\pi}{3}\lambda_D^3$	$\frac{4\pi}{3}\lambda_D^3$
Plasma frequency (electrons)	ω_{pe}	$\left(\frac{e^2 n_e}{\varepsilon_0 m_e}\right)^{1/2}$ s^{-1}	$\left(\frac{4\pi e^2 n_e}{m_e}\right)^{1/2}$ s^{-1}
Plasma frequency (ions)	ω_{pi}	$\left(\frac{Z^2 e^2 n_i}{\varepsilon_0 m_i}\right)^{1/2}$ s^{-1}	$\left(\frac{4\pi Z^2 e^2 n_i}{m_i}\right)^{1/2}$ s^{-1}
Thermal velocity	$v_{te} = \omega_{pe}\lambda_D$	$\left(\frac{k_B T_e}{m_e}\right)^{1/2}$ m s^{-1}	$\left(\frac{k_B T_e}{m_e}\right)^{1/2}$ cm s^{-1}
Electron gyrofrequency	ω_c	eB/m_e s^{-1}	eB/m_e s^{-1}
Electron–ion collision frequency	ν_{ei}	$\frac{\pi^{3/2} n_e Z e^4 \ln\Lambda}{2^{1/2}(4\pi\varepsilon_0)^2 m_e^2 v_{te}^3}$ s^{-1}	$\frac{4(2\pi)^{1/2} n_e Z e^4 \ln\Lambda}{3 m_e^2 v_{te}^3}$ s^{-1}
Coulomb logarithm	$\ln\Lambda$	$\ln\frac{9N_D}{Z}$	$\ln\frac{9N_D}{Z}$

Table A.3: Useful formulae, with T_e in eV, n_e and n_i in cm^{-3}, and wavelength λ_L in μm

Plasma frequency	$\omega_{pe} = 5.64 \times 10^4 n_e^{1/2}\ \mathrm{s}^{-1}$
Critical density	$n_c = 10^{21} \lambda_L^{-2}\ \mathrm{cm}^{-3}$
Debye length	$\lambda_D = 743\, T_e^{1/2} n_e^{-1/2}\ \mathrm{cm}$
Skin depth	$\delta = c/\omega_p = 5.31 \times 10^5 n_e^{-1/2}\ \mathrm{cm}$
Electron–ion collision frequency	$\nu_{ei} = 2.9 \times 10^{-6} n_e T_e^{-3/2} \ln\Lambda\ \mathrm{s}^{-1}$
Ion–ion collision frequency	$\nu_{ii} = 4.8 \times 10^{-8} Z^4 \left(\frac{m_p}{m_i}\right)^{1/2} n_i T_i^{-3/2} \ln\Lambda\ \mathrm{s}^{-1}$
Quiver amplitude	$a_0 = \frac{p_{osc}}{m_e c} = \left(\frac{I\lambda_L^2}{1.37 \times 10^{18}\ \mathrm{W\,cm^{-2}\,\mu m^2}}\right)^{1/2}$
Relativistic focusing threshold	$P_c = 17 \left(\frac{n_c}{n_e}\right)\ \mathrm{GW}$