Steady flow in Pipes :

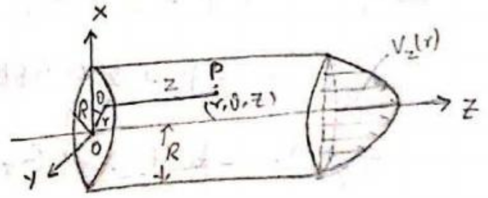
2006

(a) Flow through a pipe (The Hagen - poiseuille flow)

Let us Consider Laminar flow of InComp. fluid without body forces through a ~~pipe~~ long straight pipe of Circular Cross Section with axial Symmetry. let z be the direction of flow along the axis of the pipe and r denote the radial direction measured outward from the z-axis. The velocity Components V_r and V_θ in radial and tangential directions, respectively, are Zero.

under ~~these~~ these Conditions Eqn of Continuity in cylindrical Polar Coordinates, with $V_r = V_\theta = 0$. also $\frac{\partial}{\partial \theta} = 0$

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(\rho r . V_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho V_\theta) + \frac{\partial}{\partial z}(\rho V_z) = 0$$



reduces to Eqn

$$\frac{\partial V_z}{\partial z} = 0.$$

i.e $V_z = V_z(r, \theta)$

but due to axial symmetry

$$V_z = V_z(r) \quad - (1)$$

Navier - stokes Eqn for Viscous, In Comp. fluid (when viscosity is Constant) in Cylindrical polar Coordinate System are

$$\rho\left(\frac{DV_r}{Dt} - \frac{V_\theta^2}{r}\right) = \rho X_r - \frac{\partial p}{\partial r} + \mu\left[\nabla^2 V_r - \frac{V_r}{r^2} - \frac{2}{r^2}\frac{\partial V_\theta}{\partial \theta}\right]$$

$$\rho\left(\frac{DV_\theta}{Dt} - \frac{V_r . V_\theta}{r}\right) = \rho X_\theta - \frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\left[\nabla^2 V_\theta + \frac{2}{r^2}\frac{\partial V_r}{\partial V_\theta} - \frac{V_\theta}{r}\right]$$

$$\rho \frac{DV_z}{Dt} = \rho X_z - \frac{\partial p}{\partial z} + \mu \nabla^2 V_z$$

Since $V_r = V_\theta = 0$, $\frac{\partial}{\partial \theta} = 0$ and $X_r = X_\theta = X_z = 0$

So above Eqn reduces to

$$\frac{\partial p}{\partial r} = 0 \quad -(2)$$

$$\frac{\partial p}{\partial \theta} = 0 \quad -(3)$$

and $$\rho \, V_z \frac{\partial V_z}{\partial z} = -\frac{\partial p}{\partial z} + \mu \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \cancelto{0}{\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}} + \frac{\partial^2}{\partial z^2} \right) V_z \quad -(4)$$

Eqn (2) and (3) shows that $p = p(z)$. Since V_z is a fn of r only therefore Eqn (4) reduces into

$$0 = -\frac{dp}{dz} + \mu \left(\frac{d^2 V_z}{dr^2} + \frac{1}{r}\frac{dV_z}{dr} \right)$$

i.e $$\mu \left(\frac{d^2 V_z}{dr^2} + \frac{1}{r}\frac{dV_z}{dr} \right) = \frac{dp}{dz} \quad -(5)$$

This Eqn is the balance of the shearing force and pressure force so that there is no resultant force in the direction of z.

~~Consider~~ Eqn (5) ~~again~~ shows that $\frac{dp}{dz}$ is Constant.
Under this Condition Soln of Eqn (5) is

$$r \cdot \frac{d^2 V_z}{dr^2} + \frac{dV_z}{dr} = \frac{1}{\mu}\frac{dp}{dz} \cdot r$$

$$\Rightarrow \frac{d}{dr}\left(r \frac{dV_z}{dr} \right) = \frac{1}{\mu} \cdot \frac{dp}{dz} \cdot r$$

$$\Rightarrow r \frac{dV_z}{dr} = \frac{1}{\mu}\frac{dp}{dz}\frac{r^2}{2} + A$$

$$\Rightarrow \frac{dV_z}{dr} = \frac{1}{\mu}\frac{dp}{dz} \cdot \frac{r}{2} + \frac{A}{r}$$

$$\Rightarrow V_z = \frac{1}{\mu}\frac{dp}{dz}\frac{r^2}{4} + A \log r + B$$

$$V_z = \left(\frac{1}{4\mu}\frac{dp}{dz}\right)r^2 + A.\log r + B \quad \text{—} (6)$$

A, B are the Constants of Integration which are determined by the ~~B~~ boundary Conditions.

The first boundary ~~Conside~~ Condition is found from the Symmetry of the flow which requires

$$\text{when } r = 0 \ , \ \frac{dV_z}{dr} = 0$$

The Second Condition is the no slip Condition at the wall

$$\text{when } r = R \ , \ V_z = 0$$

using these Condition we get

$$A = 0$$

$$B = -\frac{1}{4\mu}\frac{dp}{dz} \cdot R^2$$

Substituting these values in (6) we get velocity distribution of the Hagen- Poiseuille flow through a pipe.

$$V_z = \frac{-R^2}{4\mu}\frac{dp}{dz} \cdot \left[1 - \left(\frac{r}{R}\right)^2\right] \quad \text{—} (7)$$

which has the form of Paraboloid of revolution. the velocity distribution given by (7) Can only occur when the flow reaches a fully developed state and ~~lam~~ laminar. The flow reaches a fully developed state when it travells almost 100 pipe- diameters length from the entrance.

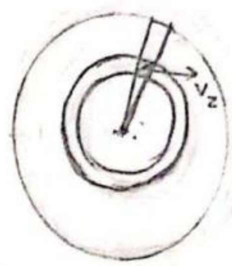
The maximum and Average velocities:

It is clearly from Eqn (7) that maxi. velocity occurs at the centre of the pipe where $r = 0$.

$$(V_z)_{max} = \frac{-R^2}{4\mu}\left(\frac{dp}{dz}\right)$$

where $\frac{dp}{dz} < 0$.

The average velocity $V_{z(av)}$ can be obtained from



$$V_{z(av)} = \frac{1}{\pi R^2}\int_0^{2\pi}\int_0^R V_z \, r dr d\theta$$

$$= \frac{1}{\pi R^2} 2\pi \int_0^R V_z \cdot r dr$$

$$= \frac{2}{R^2}\left(-\frac{R^2}{4\mu}\frac{dp}{dz}\right)\int_0^R r\cdot\left\{1 - \frac{r^2}{R^2}\right\} dr$$

$$= -\frac{1}{2\mu}\frac{dp}{dz}\left(\frac{r^2}{2} - \frac{r^4}{4R^2}\right)_0^R$$

$$= -\frac{1}{2\mu}\frac{dp}{dz}\cdot\left(\frac{R^2}{2} - \frac{R^2}{4}\right)$$

$$V_{z(av)} = -\frac{R^2}{8\mu}\frac{dp}{dz}$$

$$V_{z(av)} = \frac{(V_z)_{max}}{2}$$

the Volumetric flow

Q = flow of volume in unit time through Cross-section

$$= \int_0^{2\pi}\int_0^R V_z \cdot r\,d\theta\,dr$$

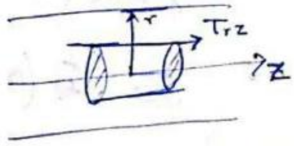
$$= \pi R^2 \cdot V_{(avg)}$$

$$Q = \frac{\pi}{8\mu} R^4 \left(-\frac{dp}{dz}\right)$$

mass flux per unit time passing any cross-section $M = \rho \frac{\pi R^2}{8\mu}\left(-\frac{dp}{dz}\right)$

This result is Poiseuille's law. we can calculate μ.

(iii) The shearing stress at the wall:



Shearing stress at any Point is $\tau_{rz} = +\mu\left(\frac{\partial V_r}{\partial z} + \frac{\partial V_z}{\partial r}\right)$

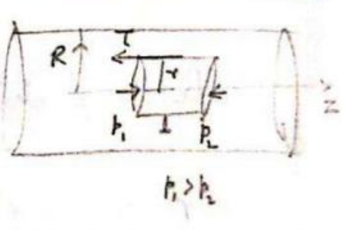
$$= +\mu \frac{\partial V_z}{\partial r}$$

$$\tau_{rz} = +\mu \frac{dV_z}{dr} = \left(+\frac{dp}{dz}\right)\frac{r}{2} \quad \text{using } (7)$$

at the wall Shearing stress $= -(\tau_{rz})_{r=R} = \frac{R}{2} \frac{dp}{dz}$

$$(\tau_{rz})_{r=R} = -\frac{R}{2}\frac{dp}{dz}$$

$$= +4\mu \frac{V_{av}}{R}$$



$p_1 > p_2$

$(p_1 - p_2)\pi r^2 = \tau\, 2\pi r l$