Unit-3

Engineering Mathematics-II

Laplace Transform and its Applications

Introduction.

The Laplace transform is a powerful tool to solve differential equations. It transforms an Initial Value Problem in Ordinary Differential Equation to algebraic equations.

Laplace Transform is an integral transform named after its inventor Pierre simon laplace.It Transforms a

Function of a real variable t to a function of a complex variable s.The transform has many application in science and engineering. One important feature of the Laplace Transform is that it can transform analytic problems to algebraic.

The concept of Laplace Transforms are applied in the area of science and technology such as Electric circuit analysis ,Communication engineering control engineering and nuclear physics etc.

Laplace Transform of a Function

Let f(t) be a function of 't' defined for all posiive values of 't'. Then ,the Laplace transform of f(t),represented by L[f(t)].

L[f(t)] = $\int_0^\infty e^{-st} f(t)dt$,where 's' is a parameter which may be a real or complex number.

Formulae

[1] $L(1) = \frac{1}{s}, s > 0$

[2] $L(t^n) = \frac{n!}{s^{n+1}}$

[3] $L(e^{at}) = \frac{1}{s-a},$

[4] L(e^{-at}) $= \frac{1}{s+a}$

[5] $L[\sin at] = \frac{a}{s^2+a^2}$

[6] $L[\cos at] = \frac{s}{s^2+a^2}$

[7] $L(\sinh at) = \frac{a}{s^2-a^2}$

[8] $L(\cosh at) = \frac{s}{s^2-a^2}$

Theorem.1 If $L\{F(t)\} = f(s)\ then\ L(1) = \frac{1}{s}$

Proof: We know that

$$L\{F(t)\} = \int_0^\infty e^{-st} F(t)dt$$

$$L(1) \qquad = \int_0^\infty e^{-st}.1dt \ = [\frac{e^{-st}}{-s}]_0^\infty = \frac{1}{s}$$

Hence proved

Exaple.1 *Find the Laplace Transforms of the following functions* :

$(i)\, t^{-3/2} \quad (ii) \sin 5t \quad (iii) \cos 7t \qquad (iv) \cos^2 3t$

$sol.\ (i)\ L(t^{-3/2}) = \frac{\overline{)-3/2+1}}{s^{-3/2+1}}$ $\qquad L(t^n) = \frac{\overline{)n+1}}{s^{n+1}}$

$$= \frac{\overline{)-1/2}}{s^{-1/2}} = \frac{-2\sqrt{\pi}}{s^{-1/2}}\ Ans.$$

$(ii)\ L(\sin 5t) = \frac{5}{s^2+25}$ $\qquad L(\sin at) = \frac{a}{s^2+a^2}$

$(iii)\ L(\cos 7t) = \frac{s}{s^2+49}$ $\qquad L(\cos at) = \frac{s}{s^2+a^2}$

$(iv)\ L(\cos^2 3t) = L\{\frac{1}{2}(1+\cos 6t)\}$

$$= \frac{1}{2}\{L(1) + L(\cos 6t)\}$$

$$= \frac{1}{2}\left(\frac{1}{s} + \frac{s}{s^2+36}\right)$$

$$Examlpe.2\ \ Find\ the\ \ Laplace\ transform\ of\ \ f(t)=\begin{Bmatrix} 0,\ 0<t<1 \\ t,\ 1<t<2 \\ 0,\ t>2 \end{Bmatrix}$$

$$Sol.\quad L\{f(t)\}=\int_0^{\infty} e^{-st} f(t)dt$$

$$=\int_0^1 0.e^{-st}dt+\int_1^2 te^{-st}dt+\int_2^{\infty} 0e^{-st}dt$$

$$=0+[\frac{te^{-st}}{-s}-\frac{1.e^{-st}}{s^2}]_1^2+0=e^{-s}\left(\frac{1}{s}+\frac{1}{s^2}\right)-e^{-2s}\left(\frac{2}{s}-\frac{1}{s^2}\right)\quad Ans.$$

Prop.-1 Laplace transform of derivative of order n

$$L\{f^n(t)\}=s^n f(s)-s^{n-1}F(0)-s^{n-2}F^{'}(0)-s^{n-3}F''(0)-....-F(0).$$

Prop.-2 Laplace transform of integral of the function f(t)

$$If\ \ L\{f(t)\}=f(s)\ \ then\ \ L\{\int_0^t f(t)dt\}=\frac{1}{s}f(s)$$

Prop.-3 Laplace Transform of $t^n f(t)$ (Multiplication by t)

$$If\ L\{f(t)\}=f(s)\quad then\ L\{t^n.f(t)\}=(-1)^n\frac{d^n}{ds^n}\{f(s)\}, where\ \ n=1,2,3......$$

Prop.-4 Laplace Transform of $\frac{f(t)}{t}$

$$If\ \ L\{f(t)\}=f(s)\ \ then\ \ L\{\frac{f(t)}{t}\}=\int_s^{\infty} f(s)ds$$

Prop.-5 Linearity Property:

$$L[af(t)+bg(t)]=aL[f(t)]+bL[g(t)]$$

Prop.-6 First shifting theorem $If\ L[f(t)]=f(s)\ then\ L[e^{at}f(t)]=f(s-a)$

Existence of Laplace transform:

The Laplace transform of a function $f(t), t \geq 0$ can be found when $\int_0^{\infty} e^{-st} f(t)dt$ exists and this exists if the integral $\int_0^{\lambda} e^{-st} f(t)dt$ can actually be evaluated and its limit as $\lambda \to \infty$ *exists.*

Example.3

Find The Laplace transform of $(t\cos 2t)$

$$Sol.\ Since\ L(\cos 2t) = \frac{s}{s^2+4} \qquad = f(s)$$

$$\Pr operty\ of\ \ multiplication\ by\ t, L\{t^n . f(t)\} = (-1)^n \frac{d^n}{ds^n}\{f(s)\}$$

$$we\ have\ L(t\cos 2t) = (-1)^1 \frac{d}{ds}\frac{s}{s^2+4} \quad = -\frac{s^2-4}{(s^2+4)^2} \qquad Ans.$$

Example.4

Find The Laplace transform of $\left(t^2 e^{=2t}\right)$

$$L(e^{-2t}) = \frac{1}{s+2} = f(s)$$

$$\Pr operty\ of\ \ multiplication\ by\ t, L\{t^n . f(t)\} = (-1)^n \frac{d^n}{ds^n}\{f(s)\}$$

$$= (-1)^2 \frac{d^2}{ds^2}\left(\frac{1}{s+2}\right) = \frac{2}{(s+2)^3} \qquad \text{Ans.}$$

Example.5

Find The Laplace transform of $\left(\frac{\sin 3t}{t}\right)$

$$Sol.\ L(\sin 3t) = \frac{3}{s^2+9}$$

$$By\ division\ property,\quad L\{\frac{f(t)}{t}\} = \int_s^\infty f(s)ds$$

$$we\ have\ L\left(\frac{\sin 3t}{t}\right) = \int_s^\infty \frac{3}{s^2+9}ds = \left\{\tan^{-1}\left(\frac{s}{3}\right)\right\}_s^\infty$$

$$= \frac{\pi}{2} - \tan^{-1}\frac{s}{3}$$

Example.6.

Evaluate $\int_0^\infty te^{-2t}\sin 2t dt$

$$Sol.\quad L(t\sin 2t) = (-1)^1\frac{d}{ds}\left(\frac{2}{s^2+4}\right) = \frac{4s}{(s^2+4)^2}$$

$$\int_0^\infty te^{-st}\sin 2t dt = \frac{4s}{(s^2+4)^2}$$

$put\ s = 2, then$

$$\int_0^\infty te^{-2t}\sin 2t dt = \frac{4*2}{(2^2+4)^2} = \frac{1}{8}\quad Ans.$$

Assignment Sheet-1

Q.1 Evaluate the following:

[i] $L(\frac{e^{-at}-e^{-bt}}{t})$

[ii] $L(\frac{\cos at-\cos bt}{t})$

[iii] $L\left(t^2 e^t \sin 3t\right)$

[iv] $\int_0^\infty te^{-3t}\sin t dt$

[v] $L[\int_0^t e^{-t}\cos t dt]$

[vi] $If\ \ f(t)=\begin{Bmatrix} t^2, 0<t<2 \\ t-1, 2<t<3 \\ 7, t>3 \end{Bmatrix}, then\ L\{f(t)\}$

Inverse Laplace Transform

The following formulae as

$[i]\ L^{-1}\left[\frac{1}{s}\right]=1$ $\qquad$ $[ii]L^{-1}\left[\frac{1}{s-a}\right]=e^{at}$

$[iii]\ L^{-1}\left[\frac{1}{s^n}\right]=\frac{t^{n-1}}{(n-1)!}$ $\qquad$ $[iv]L^{-1}\left[\frac{1}{s^2+a^2}\right]=\frac{1}{a}\sin at$

$[v]L^{-1}\left[\frac{s}{s^2+a^2}\right]=\cos at$

Convolution theorem:

$If\ \ L^{-1}[f(s)]=F(t)\quad and\quad L^{-1}[g(s)]=G(t)\ ,then$

$$L^{-1}[f(s)g(s)].=\int_0^t F(u)G(t-u)du$$

$$Example: 7.\quad L^{-1}\left[\frac{(s+4)}{s^2+4s+13}\right]$$

$$=L^{-1}\left[\frac{(s+2)}{(s+2)^2+9}\right]+L^{-1}\left[\frac{2}{(s+2)^2+9}\right]$$

$$=e^{-2t}L^{-1}\left[\frac{s}{s^2+9}\right]+e^{-2t}L^{-1}\left[\frac{2}{s^2+9}\right]$$

$$=e^{-2t}\cos 3t+e^{-2t}\,\frac{2\sin 3t}{3}\qquad Ans.$$