1 STOCHASTIC PROCESSES AND THEIR CLASSIFICATION

1.1 DEFINITION AND EXAMPLES

Definition 1. **Stochastic process** or **random process** is a collection of random variables ordered by an index set.

☛ ***Example 1.*** Random variables $X_0, X_1, X_2, \ldots$ form a stochastic process ordered by the *discrete index set* $\{0, 1, 2, \ldots\}$. Notation: $\{X_n : n = 0, 1, 2, \ldots\}$.

☛ ***Example 2.*** Stochastic process $\{Y_t : t \geq 0\}$. with *continuous index set* $\{t : t \geq 0\}$.

The indices n and t are often referred to as "time", so that X_n is a **descrete-time process** and Y_t is a **continuous-time process**.

Convention: the index set of a stochastic process is always infinite.

The range (possible values) of the random variables in a stochastic process is called the **state space** of the process. We consider both discrete-state and continuous-state processes.

Further examples:

☛ ***Example 3.*** $\{X_n : n = 0, 1, 2, \ldots\}$, where the state space of X_n is $\{0, 1, 2, 3, 4\}$ representing which of four types of transactions a person submits to an on-line database service, and time n corresponds to the number of transactions submitted.

☛ ***Example 4.*** $\{X_n : n = 0, 1, 2, \ldots\}$, where the state space of X_n is $\{1, 2\}$ representing whether an electronic component is acceptable or defective, and time n corresponds to the number of components produced.

☛ ***Example 5.*** $\{Y_t : t \geq 0\}$, where the state space of Y_t is $\{0, 1, 2, \ldots\}$ representing the number of accidents that have occurred at an intersection, and time t corresponds to weeks.

☛ ***Example 6.*** $\{Y_t : t \geq 0\}$, where the state space of Y_t is $\{0, 1, 2, \ldots, s\}$ representing the number of copies of a software product in inventory, and time t corresponds to days.

☛ ***Example 7.*** $\{Y_t : t \geq 0\}$, where the state space of Y_t is $\{0, 1, 2, \ldots\}$ representing the number of cars parked in a parking garage at a shopping mall, and time t corresponds to hours.

Lecture notes for the course *Stochastic Models* at FD ČVUT.
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Stochastic processes and sample paths

The distinction between a stochastic process and a sample path of that process is important. We can derive statements about how a process will gehave from a **stochastic-process model**. A **sample path** is a record of how a process actually did behave in one instance. Sample paths are generated by executing algorithm simulation with specific seeds or streams for the pseudorandom-number generator.

> A **sample path** is a collection of **time-ordered data** describing what happened to a dynamic process in one instance.
> A **stochastic process** is a probability model describing a collection of **time-ordered random variables** that represent the possible sample paths.

Stochastic processes can be classified on the basis of the nature of their parameter space and state space.

1.2 CLASSIFICATION

1. Stochastic Processes with Discrete Parameter and State Spaces

☛ ***Example 8.*** **A Brand-Switching Model for Consumer Behavior**

Before introducing a new brand of coffee, a manufacturer wants to study consumer behavior relative to the brands already available in the market. Suppose there are three brands already available in the market. Suppose there are three brands on sale, say A, B, C. The consumers either buy the same brand for a few months or change their brands every now and then. There is also a strong possibility that when a superior brand is introduced, come of the old brands will be left with only a few customers. Sample surveys are used to gauge consumer behavior.

In such a survey, conducted over a period of time, suppose the estimates obtained for the consumer brand-switching behavior are as follows: Out of those why buy A is one month, during the next months 60% buy A again, 30% switch to brand B and 10% switch to brand C. For brands B and C these figures are, B to A 50%, B to B 30%, B to C 20%, C to A 40%, C to B 40%, C to C 20%.

If we are interested in the number of people who buy a certain brand of coffee, then that number could be represented as a stochastic process. The behavior of the consumer can also be considered a stochastic process that can enter three different states A, B, C. Some of the questions that arise are: What is the expected number of months that a consumer stays wity one specific brand? What are the mean and variance of the number using a particular brand after a certain number of months? Whichis the product preferred most by the customers in the long run?

Suppose, for instance, that consumer preferences are observed on a monthly basis. Then we have a discrete-time, discrete-state stochastic process.

[Bhat; Miller, 2002], p. 1

☛ ***Example 9.*** **A Queueing Problem**

A bus taking students back and forth between the dormitory complex and the student union arrives at the student union several times during the day. The bus has a capacity to seat K persons. If there are K or fewer waiting persons when the bus arrives, it departs with the available number. If there are more than K waiting, the driver allows the first K to board, and the remaining persons must wait for the next bus. The university administrators would like to know at each time period during the day how many students are left behind when a bus departs.

The number waiting at the bus stop is a stochastic process dependent on the arrival process (e.g., with some probability distribution). Some desirable characteristics of the system are: the reduction in the number waiting at any time, minimizing the time a student has to wait for a ride, and minimum operational cost.

Consider the number of students waiting at the time of arrival of a bus when time is counted by the number of times the bus arrives. Then we again have a discrete-time, discrete-state stochastic process.

[Bhat; Miller, 2002], p. 2

2. Stochastic Processes with Continuous Parameter and Discrete State Space

☛ ***Example 10.*** In Example 9 consider the number of students waiting for a bus at any time of day – in this case the parameter space is continuous and we speak about a continuous-time, discrete state stochastic process.

☛ ***Example 11.*** **A Population Growth**

Consider the size of a population at a given time – we have again a continuous-time, discrete state stochastic process (the population is finite).

3. Stochastic Processes with Discrete Parameter and Continuous State Space

☛ ***Example 12.*** **Stock Market**

Consider the values of the Dow-Jones Index at the end of the nth week. Then we have a discrete-time stochastic process with the continuous state space $\langle 0, \infty)$.

The probabilistic analysis of such process is very helpful to the participants to make judicious decisions about buying and selling stocks.

[Kulkarni, 1995], p. 6

☛ ***Example 13.*** In Example 9 consider waiting time of the n-th student arriving at a bus stop – in this case we also have a continuous-state stochastic process.

☛ ***Example 14.*** **A Time-Sharing Computer System**

Jobs of varied length come to a computing center from various sources. The number of jobs arriving, as well as their length, can be said to follow certain distributions. Under these conditions the number of jobs waiting at any time and the time a job has to spend in the system can be represented by stochastic processes. Under a strictly first-come, first-served policy, there is a good chance of a long job delaying a much more important shorter fob over a long period of time. For the efficient operation of the system, in addition to minimizing the number of jobs waiting and the total delay, it may be necessary to adopt a different service policy. A round-robin policy in which the service is performed on a single job only for a certain length of time, say 3 or 5 sec, and those jobs that need more service are put back in the queue, is one of the common practices adopted under these conditions.

Consider accumulated workload observed at specified points in time.

[Bhat; Miller, 2002], p. 3

4. Stochastic Processes with Continuous Parameter and State Spaces

☛ ***Example 15.*** In Example 14 consider waiting time of an arriving job until it gets into service, with the arriving time o the job now the parameter.

1.3 FURTHER READING

Stochastic Processes

In addition to the recommended literature, more on stochastic processes including other examples can be found in following Internet addresses:

- http://math.uc.edu/~brycw/probab/books/smplbook/smplbook.html
- http://asrl.ecn.uiowa.edu/dbricker/Stacks_pdf1/Stochastic_Processes_Intro%20.pdf
- http://asrl.ecn.uiowa.edu/dbricker/ss_notes.html
- http://cpk.auc.dk/dicom/E02/StochasticProcesses.htm

Probability Theory

If you need to remind some knowledge on probability theory, you can look (besides lecture notes and textbook for the obligatory course on probability theory in the fourth semester) at these pages:

- http://asrl.ecn.uiowa.edu/dbricker/Stacks_pdf1/Misc_Prob_Results.pdf
- http://keskus.hut.fi/opetus/s38143/2001/luennot/E_lect01.pdf
- http://asrl.ecn.uiowa.edu/dbricker/Stacks_pdf1/Leibnitz_Rule.pdf
- http://www.dartmouth.edu/~chance/teaching_aids/books_articles/probability_book/
  - .../Chapter1.pdf (Discrete probability distribution)
  - .../Chapter2.pdf (Continuous Probability Densities)
  - .../Chapter3.pdf (Combinatorics)
  - .../Chapter4.pdf (Conditional Probability)
  - .../Chapter5.pdf (Important Distributions and Densities)
  - .../Chapter6.pdf (Expected Value and Variance)
  - .../Chapter7.pdf (Sums of Independent Random Variables)
  - .../Chapter8.pdf (Law of Large Numbers)
  - .../Chapter9.pdf (Central Limit Theorem)