Department of Physics
B.Sc. (Semester – II)
Paper – I (Optics)
Unit – IV

by

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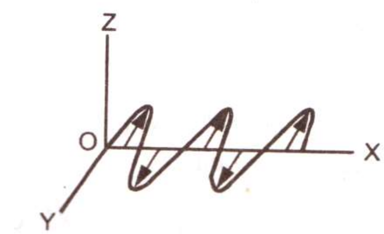
<u>Topics Covered</u>

Jones matrix, matrix representation of plane polarized waves, matrices for polarizers, retardation plates and rotators

- Method developed by R. Clark Jones

- Alight disturbance propagating along x – axis

$$y = \hat{j} a_y e^{i(kx - \omega t)}$$



- The horizontally polarized light disturbance

$$y = a_y e^{i\phi_x}$$

where $\phi_x = kx - \omega t$ and $z = 0$

• Jones – polarised light can be represented by a column vector whose first element represents the amplitude and phase of Y – component and the second element represents the Z-component.

$$J_1 = \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} a_y e^{i\phi_x} \\ 0 \end{bmatrix}$$

• Intensity is normalised and phase indicating factor is omitted

$$J_1 = \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

• Horizontally polarised light is represented by Jone's vector J_1 as

$$J_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \leftrightarrow \text{horizontally polarised light}$$

- The disturbance in vertical plane (Y - Z plane)

$$z = \hat{k} a_z e^{i(kx - \omega t + \phi)}$$

ϕ is the phase difference with respect to horizontal disturbance.

- The vertically polarized light disturbance

$$\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ a_z e^{i\phi_x} e^{i\phi} \end{bmatrix}$$

- Omitting the phase factor and considering that intensity is normalised, the Jones vector J_2 for vertically polarised light can be

$$J_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

for vertically polarised light $(\updownarrow)$

- If the disturbance is polarized at 45^o

$$J_3 = \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} ae^{i\phi_x} \cos\pi/4 \\ ae^{i\phi_x} \sin\pi/4 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- The linearly polarised light may be obtained by adding a horizontally polarised light and a vertically polarised light

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

So, a normailsed linearly polarised light at 45^o

$$J_3 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- If right and left handed circularly polarised light (of equal amplitude) are combined

$$\begin{bmatrix}1\\-i\end{bmatrix}+\begin{bmatrix}1\\i\end{bmatrix}=\begin{bmatrix}2\\0\end{bmatrix}=2\begin{bmatrix}1\\0\end{bmatrix}$$

State of polarization of a light beam	*Jones vector*
Linearly polarized with the vibration direction along the x-axis (horizontal).	$\begin{bmatrix}1\\0\end{bmatrix}$
Linearly polarized with the vibration direction along the y-axis (vertical).	$\begin{bmatrix}0\\1\end{bmatrix}$
Linearly polarized with the vibration direction at 45° relative to the x-axis.	$\begin{bmatrix}1\\1\end{bmatrix}$
Right circular polarization.	$\begin{bmatrix}1\\-i\end{bmatrix}$
Left circular polarization.	$\begin{bmatrix}1\\i\end{bmatrix}$

There is no Jones representation of unpolarized light.

Jone's matrices for polarizers and retardation plates

Jones vectors are useful in computing the effect of inserting linear optical elements into the paths of beams of light of given polarization.

A polarizer (such as a Nicol prism), a quarter-wave plate and a half-wave plate are examples of 'linear optical elements'. We can associate with each one of them a 2×2 matrix called a Jones matrix.

We have seen that the state of polarization of a beam can be represented by a column vector $\begin{bmatrix} A \\ B \end{bmatrix}$. Let $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ represent the Jones matrix of a linear optical element. When the beam passes through the element, the emergent beam will, in general, have a different state of polarization represented by, say $\begin{bmatrix} A' \\ B' \end{bmatrix}$. Jones matrices can be used to find the state of polarisation of the emergent beam. The Jones matrix is a transformation matrix which transforms the vector $\begin{bmatrix} A \\ B \end{bmatrix}$ into the vector $\begin{bmatrix} A' \\ B' \end{bmatrix}$ as follows :

$$\begin{bmatrix} A' \\ B' \end{bmatrix} = T \begin{bmatrix} A \\ B \end{bmatrix}$$

Here, $T = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is the transformation matrix.

Let us express the general transformation by the following matrix operation:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \begin{bmatrix} A \\ B \end{bmatrix} \qquad = \qquad \begin{bmatrix} A' \\ B' \end{bmatrix}$$

Transformation matrix	Incident light matrix	Emergent light matrix

Let us denote the incident light by the following matrix.

$$\begin{bmatrix} ae^{i\phi_1} \\ be^{i\phi_2} \end{bmatrix}$$

Consider the following transformation matrix.

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

By the operation of transformation matrix on incident light matrix, we get the emergent light as horizontally linearly polarised light.

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} ae^{i\phi_1} \\ be^{i\phi_2} \end{bmatrix} = \begin{bmatrix} ae^{i\phi_1} \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

The matrix $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ represents the linear polariser with axis horizontal.

Similarly, the matrix $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ represents the linear polariser with axis vertical as shown below :

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} ae^{i\phi_1} \\ be^{i\phi_2} \end{bmatrix} = \begin{bmatrix} 0 \\ be^{i\phi_2} \end{bmatrix} \rightarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

If $\phi_1 = \phi_2 = 0$ and $a = b = 1$, then

$$\frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Here matrix $\frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ represents a linear polariser with axis at + 45°.

Similarly, matrix $\frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ represents a linear polariser with axis at $-45°$.

$$\frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The emergent light has zero intensity *i.e.*, cut off. So the matrix represents a polariser of $-45°$.

There is no Jones vector representation for unpolarized light. Jones matrices are useful only when we commence with light that is initially polarized in some manner.

If light in some state of polarization given by $\begin{bmatrix} A \\ B \end{bmatrix}$ is sent through a train of n optical elements having Jones matrices $T_1, T_2, ...T_n$, the final state of polarization is given by

$$\begin{bmatrix} A' \\ B' \end{bmatrix} = T_n \ldots T_2 T_1 \begin{bmatrix} A \\ B \end{bmatrix}.$$

Linear Polariser	*Jones Matrix*
Linear polarizer with the transmission axis horizontal. $(\leftrightarrow)$	$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$
Linear polarizer with the transmission axis vertical. $(\updownarrow)$	$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

Linear polarizer with the transmission axis at + 45°.	$\frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$
Linear polarizer with the transmission axis at – 45°.	$\frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

Jone’s Matrices Quarter and Half Wave Plates

In Jones’ matrix formalism, a linear retarder, either right handed or left handed can be represented by a matrix

$$[M] = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\delta} \end{bmatrix}.$$

A $\lambda/4$ plate introduces a phase change of $\pi/2$. Hence Jones’ matrix for $\lambda/4$ plate can be represented as

$$[J]_{\lambda/4} = \begin{bmatrix} 1 & 0 \\ 0 & e^{\pm i\pi/2} \end{bmatrix}.$$

The positive sign is used when a $\lambda/4$ plate matrix produces a right circularly polarised light. The negative sign is used when $\lambda/4$ plate matrix produces left circularly polarised light. Thus

$$[J]_{\lambda/4} = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \lambda/4 \text{ plate matrix producing } RCP.$$

$$[J]_{\lambda/4} = \begin{bmatrix} 1 & 0 \\ 0 & -i \end{bmatrix} \lambda/4 \text{ plate matrix producing } LCP.$$

A $\lambda/2$ plate introduces a phase change of π. Hence the Jones’ matrix for half wave plate is

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Optical Element	*Jones Matrix*
Quarter-wave plate with the fast axis horizontal.	$\begin{bmatrix} 1 & 0 \\ 0 & -i \end{bmatrix}$
Quarter-wave plate with the fast axis vertical.	$\begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$
Quarter-wave plate with the fast axis at $\pm$ 45°.	$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & \pm i \\ \mp i & 1 \end{bmatrix}$
Half-wave plate with the fast axis either vertical or horizontal.	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
Half-wave plate with the fast axis at + 45°.	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Example 1. *Consider a linear polariser with its transmission axis horizontal. Describe what would happen when*

(i) Linearly polarised light of horizontal orientation is incident on it.

(ii) The same light but now with vertical orientation is incident on it.

Solution. (*i*) $\begin{bmatrix} A' \\ B' \end{bmatrix} = T \begin{bmatrix} A \\ B \end{bmatrix}$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

The emergent light is linearly horizontally polarised.

(*ii*) Let us denote the emergent light by the matrix $\begin{bmatrix} A' \\ B' \end{bmatrix}$.

Then, $$\begin{bmatrix} A' \\ B' \end{bmatrix} = T \begin{bmatrix} A \\ B \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

There is no emergent light.

Example 2. *What is the polarisation of emergent light when the light first be horizontally polarised, then passed, through a + 45° polariser and finally through a – 45° polariser ?*

Solution. The matrix operation is shown below.

$$\frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Hence no light is transmitted.

Example 3. (*i*) *Determine the polarization of the emerging light when a quarter-wave plate is inserted into a beam of linearly polarized light with the vibration direction making an angle of 45° with the x-axis. Assume the fast axis as horizontal.*

(ii) Work out the same example, for a half-wave plate (instead of the quarter-wave plate) with its fast axis at 45°.

Solution. (*i*) $\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}; T = \begin{bmatrix} 1 & 0 \\ 0 & -i \end{bmatrix}$

$$\therefore \begin{bmatrix} A' \\ B' \end{bmatrix} = T\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -i \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

The emerging light beam is right circularly polarized.

(*ii*) Here $T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$$\therefore \qquad \begin{bmatrix} A' \\ B' \end{bmatrix} = T\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

This is a case where the direction of vibration of the incident light is along a privileged direction of the crystal. The polarized beam is transmitted as it is.

Jone's Matrices for Circularly Polarised Light

Consider a vibration in Y-direction represented as

$$\mathbf{y} = \hat{\mathbf{j}}\, a_1\, e^{i(kx-\omega t)} = \hat{\mathbf{j}}\, a_1\, e^{i\phi_x} \qquad \ldots(1)$$

Here, $\phi_x = (kx - \omega t)$ and $k =$ propagation constant $= 2\pi/\lambda$.

Further consider an orthogonal vibration propagating along X-axis and having displacement along Y-axis. This can be represented as

$$\mathbf{z} = \hat{\mathbf{k}}\, a_2\, e^{i(\phi_x + \phi)} \qquad \ldots(2)$$

Here, ϕ is the phase difference between two linear vibrations.

When $a_1 = a_2$ and $\phi = \pi/2$, then the resultant of two vibrations is $a(\mathbf{j} + \hat{\mathbf{k}}\, e^{i\pi/2})\, e^{i\phi_x}$. Thus the two components fluctuate in different phases. The two components can be represented as

$$y = \text{real part of } ae^{i\phi_x} = a\cos\phi_x \qquad \ldots(3)$$

and

$$z = \text{real part of } ae^{i(\phi_x + \pi/2)}$$

$$= a\cos(\phi_x + \pi/2) = -a\sin\phi_x \qquad \ldots(4)$$

From Eqs. (3) and (4) we get

$$\frac{y^2}{a^2} + \frac{z^2}{a^2} = 1 \qquad \ldots(5)$$

Thus the light is circularly polarised light.

Special Cases

(*a*) **When Z-component leads in phase by $\pi/2$**

(*i*) Let $\phi_x = 0$ at time $t = t_1$.

So $\omega t_1 = kx$ or $t_1 = kx/\omega$.

From Eqs. (3) and (4), we get $y = a$ and $z = 0$.

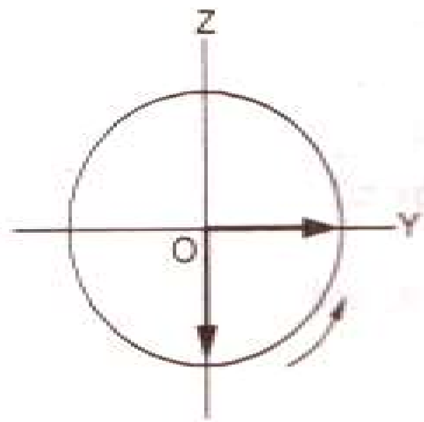


Fig. 16.2.

(*ii*) Let $\phi_x = \pi/2$ at time $t = t_2$.

Then $t_2 = \dfrac{kx}{\omega} - \dfrac{\pi}{2\omega}$ $(t_2 << t_1)$.

From Eqs. (3) and (4), we get $y = 0$ and $z = -a$.

Thus, as seen from Fig. 16.2, as the time passes the resultant displacement vector passes from a configuration $[y = 0, z = -a]$ to a configuration $[y = a, z = 0]$. Thus the resultant displacement vector rotates in a circle in *anti-clockwise sense*. In general, if the Z vibration leads Y vibration by $(2n + 1/2)\pi$, where $n = 0, 1, 2 \ldots$ etc., then the resultant vibration would be rotating anticlockwise in a circle. The light in this case is called right circularly polarised (RCP) light. In Jones' matrix formalism

$$[J]_{RCP} = \begin{bmatrix} a \\ ae^{i\pi/2} \end{bmatrix} e^{i\phi_x} \rightarrow \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ i \end{bmatrix} \quad \ldots(6)$$

with intensity normalised.

(*b*) **When Z-component lags in phase by $\pi/2$**

Let $y = a \cos \phi_x$ and $z = a \cos\left[\phi_x - \frac{\pi}{2}\right] = a \sin \phi_x$

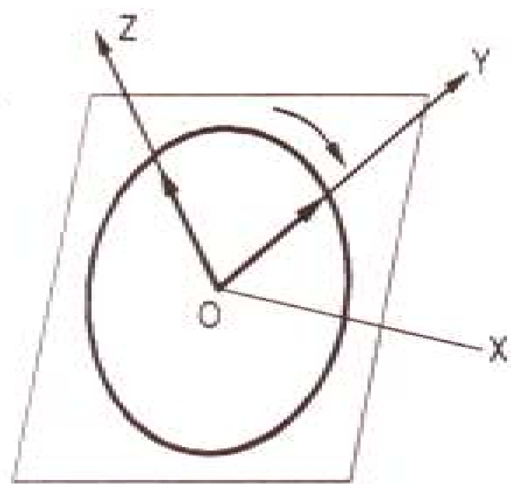


Fig. 16.3.

(*i*) When $\phi_x = 0, t_1 = \frac{kx}{\omega}$.

Now $y = a$ and $z = 0$.

(*ii*) When $\phi_x = \frac{\pi}{2}, t_2 = \frac{kx}{\omega} - \frac{\pi}{2\omega}$.

Now $y = 0$ and $z = a$.

As the time passes, the resultant vector passes from the configuration $[y = 0, z = a]$ to configuration $y = a, z = 0$ as shown in Fig. 16.3. The light is called left circularly polarised (LCP) light.

In Jones' formalism

$$[J]_{LCP} = \begin{bmatrix} a \\ ae^{-i\pi/2} \end{bmatrix} e^{i\phi_x} \rightarrow \frac{a}{\sqrt{2}}\begin{bmatrix} 1 \\ -i \end{bmatrix} \rightarrow \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -i \end{bmatrix}$$

(*c*) **When Y-component leads by $\pi/2$**

Let $y = a\cos\left[\frac{\pi}{2} + \phi_x\right] = -a\sin\phi_x$ and $z = a\cos\phi_x$.

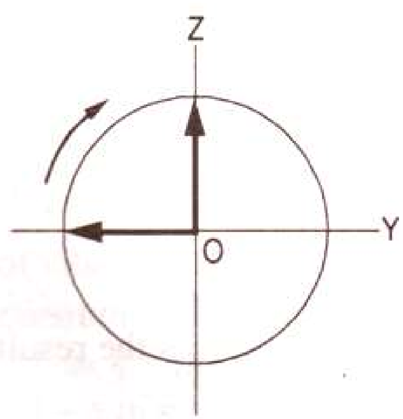


Fig. 16.4.

(*i*) When $\phi_x = 0, t_1 = \frac{kx}{\omega}$. Now $y = 0$ and $z = a$.

(*ii*) When $\phi_x = \frac{\pi}{2} . t_2 = \frac{kx}{\omega} - \frac{\pi}{2\omega}$. Now $y = -a$ and $z = 0$.

Thus as the time passes, the resultant vector passes from configuration [$y = -a$, $z = 0$] to configuration [$y = 0$, $z = a$] as shown in Fig. 16.4. The light is called left circularly polarised light.

In Jones formalism

$$[J]_{LCP} = \frac{1}{\sqrt{2}}\begin{bmatrix} i \\ 1 \end{bmatrix}$$

(d) **When Y component lags by π/2**

Let $y = a \cos\left[\phi_x - \frac{\pi}{2}\right] = a \sin \phi_x$ and $z = a \cos \phi_x$.

(*i*) When $\phi_x = 0, t_1 = \frac{kx}{\omega}$.

Now $y = 0$ and $z = a$.

(*ii*) When $\phi_x = \frac{\pi}{2}, t_2 = \frac{kx}{\omega} - \frac{\pi}{2\omega}$.

Now $y = a$ and $z = 0$.

Now we have right circularly polarised light.

In Jones' formalism $[J]_{RCP} = \frac{1}{\sqrt{2}}\begin{bmatrix} -i \\ 1 \end{bmatrix}$

The following Table lists Jones matrices for circularly polarised light.

Nature of light	*Jones matrix*
RCP when Z-component leads.	$\frac{1}{\sqrt{2}}\begin{bmatrix}1\\ i\end{bmatrix}$
LCP when Z-component lags.	$\frac{1}{\sqrt{2}}\begin{bmatrix}1\\ -i\end{bmatrix}$
LCP when Y-component leads.	$\frac{1}{\sqrt{2}}\begin{bmatrix}i\\ 1\end{bmatrix}$
RCP when Y-component **lags**.	$\frac{1}{\sqrt{2}}\begin{bmatrix}-i\\ 1\end{bmatrix}$

Jone's Matices for Elliptically Polarised Light

Elliptically polarised light is the superposition of horizontally polarised light and vertically polarised light. The two lights are represented as

$$\mathbf{y} = \hat{\mathbf{j}}\, a_1\, e^{i(kx-\omega t)} = \hat{\mathbf{j}}\, a_1\, e^{i\phi_x}$$

and

$$\mathbf{z} = \hat{\mathbf{k}}\, a_2\, e^{i(kx-\omega t+\phi)} = \hat{\mathbf{k}}\, a_2\, e^{i\phi_x}\, e^{i\phi}$$

The resultant of two vibrations is

$$(\hat{\mathbf{j}}\, a_1 + \hat{\mathbf{k}}\, a_2\, e^{i\phi})\, e^{i\phi_x}$$

The y and z components of resultant vibration are

y = real part of $a_1\, e^{i\phi_x} = a_1 \cos \phi_x$

z = real part of $a_2\, e^{i(\phi_x+\phi)} = a_2 \cos(\phi_x + \phi)$.

From these equations, we get

$$\frac{y^2}{a_1^2} + \frac{z^2}{a_2^2} - \frac{2yz}{a_1 a_2} \cos\phi = \sin^2 \phi$$

When $\phi = (2n+1)\,\pi/2$, $n = 0, 1, 2, 3 \ldots$, the equation takes the form

$$\frac{y^2}{a_1^2} + \frac{z^2}{a_2^2} = 1.$$

This represents an ellipse with its axes along y and z directions.
Thus we get elliptically polarised light.

In Jones' formalism, elliptically polarised light is represented as

$$[J]_{REP} = \begin{bmatrix} a_1 \\ a_2 e^{i\phi} \end{bmatrix} \rightarrow \begin{bmatrix} a_1 \\ ia_2 \end{bmatrix}$$

This is right elliptically polarised light.
Similarly, the left elliptically polarised light is represented as

$$[J]_{LEP} = \begin{bmatrix} a_1 \\ -ia_2 \end{bmatrix}$$

The following Table lists Jones matrices for elliptically polarised light :

Nature of light	*Jones matrix*
Right handed elliptically polarised light [REP]	$\begin{bmatrix} a_1 \\ ia_2 \end{bmatrix}$
Left handed elliptically polarised light [LEP]	$\begin{bmatrix} a_1 \\ -ia_2 \end{bmatrix}$