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B.Tech-IV SEM

Discrete Mathematical Structure-AS-404

Propositions

- Proposition A statement that is either true or false but not both is called a proposition.

• Examples of propositions –

“1 + 1 = 2” · · · True

“1 + 1 > 3” · · · False

“Singapore is in Europe.” · · · False

• Examples (which are not propositions)

“1 + 1 > x” · · · ×

“What a great book!” . . . ×

“Is Singapore in Asia?” . . .×

EXAMPLE: We know what sentences are (I hope):

1. John is going to the store.
2. That guy is going to the store.
3. John, go to the store.
4. Did John go to the store?

- *Declarative* sentences are <u>propositions</u>.

1. i.e. Sentences that assert a fact that could either be true or false.
2. i.e. Something you could make into a question.
3. Of the above, only (1) is a proposition as it is: we need all the details.
4. (2) would be a proposition if we knew who “that guy” is.

- Above, (3) is a command: it's not true or false.
- (4) is a question, so definitely not a statement.

EXAMPLE: There are statements in math like “10−4=6” and “1+1=3”.

One of those is true and one is false, but they are both propositions.

Propositional logic

- **Proposition :** A proposition is classified as a declarative sentence which is either true or false.

  eg: *1) It rained yesterday.*
- **Propositional symbols/variables**: P, Q, S, ... (**atomic sentences**)
- Sentences are combined by **Connectives**:

$\wedge$ ...and	[conjunction]
$\vee$ ...or	[disjunction]
$\Rightarrow$...implies	[implication / conditional]
$\Leftrightarrow$..is equivalent	[biconditional]
$\neg$ ...not	[negation]

- **Literal**: atomic sentence or negated atomic sentence

Propositional logic (PL)

Sentence or well formed formula

- A sentence (well formed formula) is defined as follows:
  - A symbol is a sentence
  - If S is a sentence, then $\neg$S is a sentence
  - If S is a sentence, then (S) is a sentence
  - If S and T are sentences, then (S $\vee$ T), (S $\wedge$ T), (S $\rightarrow$ T), and (S $\leftrightarrow$ T) are sentences
  - A sentence results from a finite number of applications of the above rules

Logic Basics

- A proposition can be negated.
- That is, if p is true, its negation is false; if p is false, its negation is true.

- Some examples with natural language statements:
- e.g. in English: the negation of "John is going to the store." is "John is not going to the store."
- Some are less easy to negate: "I will not go to the store any day this week." is negated to "I will go to the store some day this week."
- We'll write "¬p" for the negation of p.
- So we could say things like: "if p is the proposition '2+2=4', then its negation is ¬p, '2+2≠4'."
- "¬p" is itself a proposition: the "¬" takes a proposition and makes a new one.
- We will use a truth table to give all of the true/false values for predicates.

- Here is a truth table for negation

p	$\neg p$
T	F
F	T

Conjunction

- When writing a truth table, we have to list all of the possible combinations of values for the propositions (p, q, …) in it.
- Since "¬p" contains only "p", there are only two rows.
- Negation is the first way we have to manipulate propositions.
- We will need others.

- When propositions are manipulated to make another proposition, we call the result a compound proposition.
- Conjunction is a way to combine two propositions.
- The conjunction of p and q is written p∧q.
- p∧q is true if both p and q are true.
- In other words, ∧ means “and”.
- In a truth table:

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Disjunction

- Notice that we got every possible combination of values for “p” and “q” in the truth table.
- Disjunction is another way to combine two propositions.
- The disjunction of p and q is written p∨q.
- p∨q is true if p is true, or q is true, or both are true.
- In other words, ∨ means “or”.
- In a truth table:

p	q	$p \vee q$
T	T	T

T	F	T
F	T	T
F	F	F

Conditionals

- The other sentence we couldn't easily translate before: “If the store is open today, then John will go.”
- That's a conditional statement or an implication.
  i.e. it expresses “If (something is true), then (something else).”
  We will write $p \to q$ for the conditional “If p then q.”

P	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

- Examples of $p \to q$: “if p then q”; “q whenever p”; “p implies q”; “q follows from p”; “q only if p”.
- We could also have written $p \to q$ using only $\lor$, $\land$, and $\neg$.
- For a conditional proposition $p \to q$…
- $q \to p$ is its converse.
- $\neg p \to \neg q$ is its inverse.
- $\neg q \to \neg p$ is its contrapositive.
- Draw a truth table for $\neg p \lor q$ if you don't believe me.

identical last columns:

p	q	$p \to q$
T	T	**T**
T	F	**F**
F	T	**T**
F	F	**T**

p	q	$\neg q$	$\neg p$	$\neg q \to \neg p$
T	T	F	F	**T**
T	F	T	F	**F**
F	T	F	T	**T**
F	F	T	T	**T**

“exclusive or”

- We actually could have expressed “exclusive or” with the operators we had:
- The last two columns are the same. So, we can say that the two are equivalent, and write $p \oplus q \equiv (p \lor q) \land \neg (p \land q)$.

P	q	$p \lor q$	$\neg(p \land q)$	$(p \lor q) \land \neg(p \land q)$	$p \oplus q$
T	T	T	F	F	F
T	F	T	T	T	T
F	T	T	T	T	T
F	F	F	T	F	F

Laws of Algebra of Propositions

- **Idempotent:**

  $p \vee p \equiv p$ $\qquad$ $p \wedge p \equiv p$

- **Commutative:**

  $p \vee q \equiv q \vee p$ $\qquad$ $p \wedge q \equiv q \wedge p$

- **Complement:**

  $p \vee \sim p \equiv T$ $\qquad$ $p \wedge \sim p \equiv F$

- **Double Negation:**

  $\sim(\sim p) \equiv p$

- **Associative:**

  $p \vee (q \vee r) \equiv (p \vee q) \vee r$

  $p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$

- **Distributive:**

  $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

  $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

- **Absorbtion:**

  $p \vee (p \wedge q) \equiv p$

  $p \wedge (p \vee q) \equiv p$

- **Identity:**

  $p \vee T \equiv T$ $\qquad$ $p \wedge T \equiv p$

  $p \vee F \equiv p$ $\qquad$ $p \wedge F \equiv F$

- **<u>De Morgan's</u>**

  $\sim(p \vee q) \equiv \sim p \wedge \sim q$

  $\sim(p \wedge q) \equiv \sim p \vee \sim q$

- **<u>Equivalence of Contrapositive:</u>**

  $p \rightarrow q \equiv \sim q \rightarrow \sim p$

- **<u>Others:</u>**

  $p \rightarrow q \equiv \sim p \vee q$

  $p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$

NOTE: USING TRUTH TABLE VERIFY ALL LAW OF ALGEBRA OF PROPOSITION

Logical Equivalence of Conditional and Contrapositive

The easiest way to check for logical equivalence is to see if the truth tables of both variants have *identical last columns*:

p	q	$p \to q$
T	T	**T**
T	F	**F**
F	T	**T**
F	F	**T**

p	q	$\neg q$	$\neg p$	$\neg q \to \neg p$
T	T	F	F	**T**
T	F	T	F	**F**
F	T	F	T	**T**
F	F	T	T	**T**

Tautologies and contradictions

- A **tautology** is a sentence that is True under all interpretations.
- An **contradiction** is a sentence that is False under all interpretations.

p	$\neg p$	$p \vee \neg p$
F	T	T
T	F	T

p	$\neg p$	$p \wedge \neg p$
F	T	F
T	F	F

Tautology by truth table

p	q	$\neg p$	$p \vee q$	$\neg p \wedge (p \vee q)$	$[\neg p \wedge (p \vee q)] \to q$
T	T	F	T	F	T
T	F	F	T	F	T
F	T	T	T	T	T
F	F	T	F	F	T

EXAMPLE 2.5.1. *Use the logical equivalences above to show that* $\neg(p \vee \neg(p \wedge q))$ *is a contradiction.*

Solution.

$$\begin{array}{ll}
\neg(p \vee \neg(p \wedge q)) & \\
\Leftrightarrow \neg p \wedge \neg(\neg(p \wedge q)) & \textit{De Morgan's Law} \\
\Leftrightarrow \neg p \wedge (p \wedge q) & \textit{Double Negation Law} \\
\Leftrightarrow (\neg p \wedge p) \wedge q & \textit{Associative Law} \\
\Leftrightarrow F \wedge q & \textit{Contradiction} \\
\Leftrightarrow F & \textit{Domination Law and Commutative Law}
\end{array}$$

NOTE: A **contingency** is a proposition that is neither a tautology nor a contradiction.

Propositional Logic - one last proof

- Show that $[p \wedge (p \rightarrow q)] \rightarrow q$ is a tautology.
- We use $\equiv$ to show that $[p \wedge (p \rightarrow q)] \rightarrow q \equiv \mathrm{T}$.

$[p \wedge (p \rightarrow q)] \rightarrow q$

$\equiv [p \wedge (\neg p \vee q)] \rightarrow q$	substitution for $\rightarrow$
$\equiv [(p \wedge \neg p) \vee (p \wedge q)] \rightarrow q$	distributive
$\equiv [\mathrm{F} \vee (p \wedge q)] \rightarrow q$	complement
$\equiv (p \wedge q) \rightarrow q$	identity
$\equiv \neg(p \wedge q) \vee q$	substitution for $\rightarrow$
$\equiv (\neg p \vee \neg q) \vee q$	DeMorgan's
$\equiv \neg p \vee (\neg q \vee q)$	associative
$\equiv \neg p \vee \mathrm{T}$	complement
$\equiv \mathrm{T}$	identity

NOTE: YOU CAN PROVE THE ABOVE PROBLEM BY USING TRUTH TABLE ALSO.

First-order logic

- First-order logic (FOL) models the world in terms of
  - **Objects,** which are things with individual identities
  - **Properties** of objects that distinguish them from other objects
  - **Relations** that hold among sets of objects
  - **Functions,** which are a subset of relations where there is only one "value" for any given "input"
- Examples:
  - Objects: Students, lectures, companies, cars ...
  - Relations: Brother-of, bigger-than, outside, part-of, has-color, occurs-after, owns, visits, precedes, ...
  - Properties: blue, oval, even, large, ...
  - Functions: father-of, best-friend, second-half, one-more-than ...

FOL Provides

- **Variable symbols**
  - E.g., x, y, foo
- **Connectives**
  - Same as in PL: not ($\neg$), and ($\wedge$), or ($\vee$), implies ($\rightarrow$), if and only if (biconditional $\leftrightarrow$)
- **Quantifiers**
  - Universal $\forall$**x** or **(Ax)**
  - Existential $\exists$**x** or **(Ex)**

Quantifiers

- **Universal quantification**
  - $(\forall x)P(x)$ means that P holds for **all** values of x in the domain associated with that variable
  - E.g., $(\forall x)\ \text{dolphin}(x) \rightarrow \text{mammal}(x)$
- **Existential quantification**
  - $(\exists\ x)P(x)$ means that P holds for **some** value of x in the domain associated with that variable
  - E.g., $(\exists\ x)\ \text{mammal}(x) \wedge \text{lays-eggs}(x)$
  - Permits one to make a statement about some object without naming it

Quantifier Scope

- Switching the order of universal quantifiers *does not* change the meaning:
  - $(\forall x)(\forall y)P(x,y) \leftrightarrow (\forall y)(\forall x)\ P(x,y)$
- Similarly, you can switch the order of existential quantifiers:
  - $(\exists x)(\exists y)P(x,y) \leftrightarrow (\exists y)(\exists x)\ P(x,y)$
- Switching the order of universals and existentials *does* change meaning:
  - Everyone likes someone: $(\forall x)(\exists y)\ \text{likes}(x,y)$
  - Someone is liked by everyone: $(\exists y)(\forall x)\ \text{likes}(x,y)$

Connections between All and Exists

We can relate sentences involving $\forall$ and $\exists$ using De Morgan's laws:

$$(\forall x)\ \neg P(x) \leftrightarrow \neg(\exists x)\ P(x)$$

$$\neg(\forall x)\ P \leftrightarrow (\exists x)\ \neg P(x)$$

$$(\forall x)\ P(x) \leftrightarrow \neg\ (\exists x)\ \neg P(x)$$

$$(\exists x)\ P(x) \leftrightarrow \neg(\forall x)\ \neg P(x)$$

Quantified inference rules

- Universal instantiation

  $\forall x\ P(x) \therefore P(A)$
- Universal generalization
  - $P(A) \wedge P(B) \ldots \therefore \forall x\ P(x)$
- Existential instantiation

  $\exists x\ P(x) \therefore P(F)$
- Existential generalization
  - $P(A) \therefore \exists x\ P(x)$

EXERCISE-1

1. If $p \equiv$ Sam is a teacher, $q \equiv$ John is an honest boy, then translate the following into logical sentences :
   (a) $\sim (p \wedge q)$, (b) $p \vee \sim q$, (c) $\sim p \Leftrightarrow q$, (d) $p \Rightarrow \sim q$.
2. Change the following sentence into symbols :
   (a) 'If I do not have car or I do not wear good dress then I am not a millionaire'.
   (b) Everyone who is healthy can do all kinds of work. (Anna, 2004S)
3. Prepare truth tables for the following statements (a) $(p \Rightarrow q) \wedge \sim q$, (b) $(p \Leftrightarrow q) \wedge (r \vee q)$.
4. Write down the truth table of
   (a) $p \vee q$ (Madras, 1997) (b) $p \wedge (p \wedge q)$ (Madras, 2005S)
5. Verify that the following are tautologies :
   (a) $p \rightarrow [q \rightarrow (p \wedge q)]$ (Anna, 2005)
   (b) $(p \wedge q \Rightarrow r) \Leftrightarrow (p \Rightarrow r) \vee (q \Rightarrow r)$ (c) $(p \Rightarrow q \wedge r) \Rightarrow (\sim r \Rightarrow \sim p)$.
6. Show that $Q \vee (P \wedge \sim Q) \vee (\sim P \wedge \sim Q)$ is a tautology. (Anna, 2004S ; Madras, 2003S)
7. Over the universe of positive integers
   $p(n)$: n is prime and $n < 32$.
   $q(n)$: n is a power of 3.
   $r(n)$: n is a divisor of 27.
   (i) What are the truth sets of these propositions ?
   (ii) Which of the three propositions implies one of the others ?
8. Given the propositions over the natural numbers
   $p : n < 4$, $q : 2n > 17$ and $r : n$ is a divisor of 18, what are the truth sets of
   (i) q, (ii) $p \wedge q$,
   (iii) r, (iv) $q \rightarrow r$.
9. Show that (a) $\sim Q, P \rightarrow Q \Rightarrow \sim P$. (Madras, 1999)
   (b) $(P \rightarrow R) \wedge (Q \rightarrow R) \Leftrightarrow (P \vee Q) \rightarrow R$. (Madras, 2003)
   (Madras, 2001)
10. Construct the truth table for (i) $(\sim p \rightarrow q) \wedge (q \rightleftarrows p)$. (Bharthian, Msc. 2001)
    (ii) $\rceil [P \vee (Q \wedge R)] \rightleftarrows (P \vee Q) \wedge (P \vee R)$. (Andhra, 2004)
11. Prove that the following statement is a contradiction :
    $S = [(p \vee q) \wedge (p \vee \sim q) \wedge (\sim p \vee q) \wedge (\sim p \vee \sim q)]$.
12. If p, q, r are three statements then prove that
    (a) $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ (b) $(p \Rightarrow q) \vee r \equiv (p \vee r) \Rightarrow (q \vee r)$
    (c) $\sim (p \vee q) \equiv \sim p \wedge \sim q$.
13. Define conjunction, conditional, biconditional and negation, with examples.

EXERCISE-2

1. If $A = \{1, 2, 3, 4, 5\}$ be the universal set, determine the truth values of each of the following statements :

   (*a*) $(\forall\, x \in A)\ (x + 2 < 10)$
   (*b*) $\exists\, x \in A\ (x + 2 = 10)$

2. Negate each of the following statements :

   (*a*) $\forall\, x,\ x^3 = x$;
   (*b*) $\forall\, x,\ x + 5 > x$
   (*c*) Some students are 26 or older.
   (*d*) All students live in the hostels.

3. What is the truth value of $\forall\, x\, P(x)$ where $P(x)$ is a statement '$x^2 < 10$' and the universe of discourse consists of positive integers not exceeding 4.

4. Use universal quantifier to state 'the sum of any two rational numbers is rational'.

5. Over the universe of real numbers, use quantifier to say that the equation $a + x = b$ has a solution for all values of a and b. (*Madras, 1999*)

6. Translate the following statements involving quantifiers, into formulae :

   (*a*) All rationals are reals.
   (*b*) No rationals are reals.
   (*c*) Some rationals are reals.
   (*d*) Some rationals are not reals.

7. Show that $Q \vee (P \wedge \sim Q) \vee (\sim P \wedge \sim Q)$ is a tautology. (*Madras, 1998*)

2.7. Normal or Canonical Forms.

Definition 2.7.1. *Every compound proposition in the propositional variables p, q, r, ..., is uniquely equivalent to a proposition that is formed by taking the disjunction of conjunctions of some combination of the variables $p, q, r, \ldots$ or their negations. This is called the* **disjunctive normal form** *of a proposition.*

Discussion

The *disjunctive normal form* of a compound proposition is a natural and useful choice for representing the proposition from among all equivalent forms, although it may not be the simplest representative. We will find this concept useful when we arrive at the module on Boolean algebra.

2.8. Examples.

Example 2.8.1. *Construct a proposition in disjunctive normal form that is true precisely when*

(1) *p is true and q is false*
Solution. $p \wedge \neg q$
(2) *p is true and q is false or when p is true and q is true.*

Solution. $(p \wedge \neg q) \vee (p \wedge q)$
(3) *either p is true or q is true, and r is false*
Solution. $(p \vee q) \wedge \neg r \Leftrightarrow (p \wedge \neg r) \vee (q \wedge \neg r)$ *(Distributive Law)*

(Notice that the second example could be simplified to just p.)

Discussion

The methods by which we arrived at the disjunctive normal form in these examples may seem a little *ad hoc*. We now demonstrate, through further examples, a sure-fire method for its construction.

2.9. Constructing Disjunctive Normal Forms.

EXAMPLE 2.9.1. *Find the disjunctive normal form for the proposition* $p \to q$*.*

Solution. *Construct a truth table for* $p \to q$*:*

p	q	$p \to q$	
T	T	T	$\leftarrow$
T	F	F	
F	T	T	$\leftarrow$
F	F	T	$\leftarrow$

$p \to q$ *is true when either*
p *is true and* q *is true, or*
p *is false and* q *is true, or*
p *is false and* q *is false.*
The disjunctive normal form is then

$$(p \wedge q) \vee (\neg p \wedge q) \vee (\neg p \wedge \neg q)$$

Discussion

This example shows how a truth table can be used in a systematic way to construct the disjunctive normal forms. Here is another example.

EXAMPLE 2.9.2. *Construct the disjunctive normal form of the proposition*

$$(p \to q) \wedge \neg r$$

Solution. *Write out the truth table for* $(p \to q) \wedge \neg r$*:*

p	q	r	$p \to q$	$\neg r$	$(p \to q) \wedge \neg r$
T	T	T	T	F	F
T	T	F	T	T	T
T	F	T	F	F	F
F	T	T	T	F	F
T	F	F	F	T	F
F	T	F	T	T	T
F	F	T	T	F	F
F	F	F	T	T	T

The disjunctive normal form will be a disjunction of three conjunctions, one for each row in the truth table that gives the truth value T for $(p \to q) \wedge \neg r$. These rows have been boxed. In each conjunction we will use p if the truth value of p in that row is T and $\neg p$ if the truth value of p is F, q if the truth value of q in that row is T and $\neg q$ if the truth value of q is F, etc. The disjunctive normal form for $(p \to q) \wedge \neg r$ is then

$$(p \wedge q \wedge \neg r) \vee (\neg p \wedge q \wedge \neg r) \vee (\neg p \wedge \neg q \wedge \neg r),$$

because each of these conjunctions is true only for the combination of truth values of p, q, and r found in the corresponding row. That is, $(p \wedge q \wedge \neg r)$ has truth value T only for the combination of truth values in row 2, $(\neg p \wedge q \wedge \neg r)$ has truth value T only for the combination of truth values in row 6, etc. Their disjunction will be true for precisely the three combinations of truth values of p, q, and r for which $(p \to q) \wedge \neg r$ is also true.

Terminology. The individual conjunctions that make up the disjunctive normal form are called **minterms**. In the previous example, the disjunctive normal form for the proposition $(p \to q) \wedge \neg r$ has three minterms, $(p \wedge q \wedge \neg r)$, $(\neg p \wedge q \wedge \neg r)$, and $(\neg p \wedge \neg q \wedge \neg r)$.

2.10. Conjunctive Normal Form. The **conjunctive normal form** of a proposition is another "canonical form" that may occasionally be useful, but not to the same degree as the disjunctive normal form. As the name should suggests after our discussion above, the conjunctive normal form of a proposition is the equivalent form that consists of a "conjunction of disjunctions." It is easily constructed indirectly using disjunctive normal forms by observing that if you negate a disjunctive normal form you get a conjunctive normal form. For example, three applications of De Morgan's Laws gives

$$\neg[(p \wedge \neg q) \vee (\neg p \wedge \neg q)] \Leftrightarrow (\neg p \vee q) \wedge (p \vee q).$$

Thus, if you want to get the conjunctive normal form of a proposition, construct the disjunctive normal form of its *negation* and then negate again and apply De Morgan's Laws.

EXAMPLE 2.10.1. *Find the conjunctive normal form of the proposition* $(p \wedge \neg q) \vee r$.

Solution.

(1) Negate: $\neg[(p \wedge \neg q) \vee r] \Leftrightarrow (\neg p \vee q) \wedge \neg r$.

(2) Find the disjunctive normal form of $(\neg p \vee q) \wedge \neg r$:

p	q	r	$\neg p$	$\neg r$	$\neg p \vee q$	$(\neg p \vee q) \wedge \neg r$
T	T	T	F	F	T	F
T	T	F	F	T	T	T
T	F	T	F	F	F	F
F	T	T	T	F	T	F
T	F	F	F	T	F	F
F	T	F	T	T	T	T
F	F	T	T	F	T	F
F	F	F	T	T	T	T

The disjunctive normal form for $(\neg p \vee q) \wedge \neg r$ is

$$(p \wedge q \wedge \neg r) \vee (\neg p \wedge q \wedge \neg r) \vee (\neg p \wedge \neg q \wedge \neg r).$$

(3) The conjunctive normal form for $(p \wedge \neg q) \vee r$ is then the negation of this last expression, which, by De Morgan's Laws, is

$$(\neg p \vee \neg q \vee r) \wedge (p \vee \neg q \vee r) \wedge (p \vee q \vee r).$$

NOTE: Follow the link from Lecture 1-11

https://www.youtube.com/watch?v=xlUFkMKSB3Y