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BCA-IV SEM

Discrete Mathematics-BCA-401

Propositions

- Proposition A statement that is either true or false but not both is called a proposition.

• Examples of propositions –

"1 + 1 = 2" · · · True

"1 + 1 > 3" · · · False

"Singapore is in Europe." · · · False

• Examples (which are not propositions)

"1 + 1 > x" · · · ×

"What a great book!" . . . ×

"Is Singapore in Asia?" . . .×

EXAMPLE: We know what sentences are (I hope):

1. John is going to the store.
2. That guy is going to the store.
3. John, go to the store.
4. Did John go to the store?

- *Declarative* sentences are <u>propositions</u>.

1. i.e. Sentences that assert a fact that could either be true or false.
2. i.e. Something you could make into a question.
3. Of the above, only (1) is a proposition as it is: we need all the details.
4. (2) would be a proposition if we knew who “that guy” is.

- Above, (3) is a command: it's not true or false.
- (4) is a question, so definitely not a statement.

EXAMPLE: There are statements in math like “10−4=6” and “1+1=3”.

One of those is true and one is false, but they are both propositions.

Propositional logic

- **Proposition :** A proposition is classified as a declarative sentence which is either true or false.

  eg: *1) It rained yesterday.*

- **Propositional symbols/variables**: P, Q, S, ... (**atomic sentences**)
- Sentences are combined by **Connectives**:

$\wedge$ ...and	[conjunction]
$\vee$ ...or	[disjunction]
$\Rightarrow$ ...implies	[implication / conditional]
$\Leftrightarrow$ ..is equivalent	[biconditional]
$\neg$ ...not	[negation]

- **Literal**: atomic sentence or negated atomic sentence

Propositional logic (PL)

Sentence or well formed formula

- A sentence (well formed formula) is defined as follows:
  - A symbol is a sentence
  - If S is a sentence, then $\neg$S is a sentence
  - If S is a sentence, then (S) is a sentence
  - If S and T are sentences, then $(S \vee T)$, $(S \wedge T)$, $(S \rightarrow T)$, and $(S \leftrightarrow T)$ are sentences
  - A sentence results from a finite number of applications of the above rules

Logic Basics

- A proposition can be negated.
- That is, if p is true, its negation is false; if p is false, its negation is true.

- Some examples with natural language statements:
- e.g. in English: the negation of “John is going to the store.” is “John is not going to the store.”
- Some are less easy to negate: “I will not go to the store any day this week.” is negated to “I will go to the store some day this week.”
- We'll write “¬p” for the negation of p.
- So we could say things like: “if p is the proposition ‘2+2=4’, then its negation is ¬p, ‘2+2≠4’.”
- “¬p” is itself a proposition: the “¬” takes a proposition and makes a new one.
- We will use a truth table to give all of the true/false values for predicates.

- Here is a truth table for negation

p	$\neg p$
T	F
F	T

Conjunction

- When writing a truth table, we have to list all of the possible combinations of values for the propositions (p, q, …) in it.
- Since “¬p” contains only “p”, there are only two rows.
- Negation is the first way we have to manipulate propositions.
- We will need others.

- When propositions are manipulated to make another proposition, we call the result a compound proposition.
- Conjunction is a way to combine two propositions.
- The conjunction of p and q is written $p \wedge q$.
- $p \wedge q$ is true if both p and q are true.
- In other words, $\wedge$ means “and”.
- In a truth table:

p	*q*	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Disjunction

- Notice that we got every possible combination of values for “p” and “q” in the truth table.
- Disjunction is another way to combine two propositions.
- The disjunction of p and q is written $p \vee q$.
- $p \vee q$ is true if p is true, or q is true, or both are true.
- In other words, $\vee$ means “or”.
- In a truth table:

p	*q*	$p \vee q$
T	T	T

T	F	T
F	T	T
F	F	F

Conditionals

- The other sentence we couldn't easily translate before: "If the store is open today, then John will go."
- That's a conditional statement or an implication.
  i.e. it expresses "If (something is true), then (something else)."
  We will write p→q for the conditional "If p then q."

P	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

- Examples of p→q: "if p then q"; "q whenever p"; "p implies q"; "q follows from p"; "q only if p".
- We could also have written p→q using only ∨, ∧, and ¬.
- For a conditional proposition p→q…
- q→p is its converse.
- ¬p→¬q is its inverse.
- ¬q→¬p is its contrapositive.
- Draw a truth table for ¬p∨q if you don't believe me.

identical last columns:

p	q	$p \to q$
T	T	**T**
T	F	**F**
F	T	**T**
F	F	**T**

p	q	$\neg q$	$\neg p$	$\neg q \to \neg p$
T	T	F	F	**T**
T	F	T	F	**F**
F	T	F	T	**T**
F	F	T	T	**T**

“exclusive or”

- We actually could have expressed “exclusive or” with the operators we had:
- The last two columns are the same. So, we can say that the two are equivalent, and write $p \oplus q \equiv (p \lor q) \land \neg(p \land q)$.

P	q	$p \lor q$	$\neg(p \land q)$	$(p \lor q) \land \neg(p \land q)$	$p \oplus q$
T	T	T	F	F	F
T	F	T	T	T	T
F	T	T	T	T	T
F	F	F	T	F	F

Laws of Algebra of Propositions

- **Idempotent:**

  $p \vee p \equiv p$ $\quad\quad$ $p \wedge p \equiv p$

- **Commutative:**

  $p \vee q \equiv q \vee p$ $\quad\quad$ $p \wedge q \equiv q \wedge p$

- **Complement:**

  $p \vee \sim p \equiv T$ $\quad\quad$ $p \wedge \sim p \equiv F$

- **Double Negation:**

  $\sim(\sim p) \equiv p$

- **Associative:**

  $p \vee (q \vee r) \equiv (p \vee q) \vee r$

  $p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$

- **Distributive:**

  $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

  $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

- **Absorbtion:**

  $p \vee (p \wedge q) \equiv p$

  $p \wedge (p \vee q) \equiv p$

- **Identity:**

  $p \vee T \equiv T$ $\quad\quad$ $p \wedge T \equiv p$

  $p \vee F \equiv p$ $\quad\quad$ $p \wedge F \equiv F$

- **De Morgan's**

  $\sim(p \vee q) \equiv \sim p \wedge \sim q$

  $\sim(p \wedge q) \equiv \sim p \vee \sim q$

- **Equivalence of Contrapositive:**

  $p \rightarrow q \equiv \sim q \rightarrow \sim p$

- **Others:**

  $p \rightarrow q \equiv \sim p \vee q$

  $p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$

NOTE: USING TRUTH TABLE VERIFY ALL LAW OF ALGEBRA OF PROPOSITION

Logical Equivalence of Conditional and Contrapositive

The easiest way to check for logical equivalence is to see if the truth tables of both variants have *identical last columns*:

p	q	$p \rightarrow q$
T	T	**T**
T	F	**F**
F	T	**T**
F	F	**T**

p	q	$\neg q$	$\neg p$	$\neg q \rightarrow \neg p$
T	T	F	F	**T**
T	F	T	F	**F**
F	T	F	T	**T**
F	F	T	T	**T**

Tautologies and contradictions

- A **tautology** is a sentence that is True under all interpretations.
- An **contradiction** is a sentence that is False under all interpretations.

p	$\neg p$	$p \vee \neg p$
F	T	T
T	F	T

p	$\neg p$	$p \wedge \neg p$
F	T	F
T	F	F

Tautology by truth table

p	q	$\neg p$	$p \vee q$	$\neg p \wedge (p \vee q)$	$[\neg p \wedge (p \vee q)] \rightarrow q$
T	T	F	T	F	T
T	F	F	T	F	T
F	T	T	T	T	T
F	F	T	F	F	T

EXAMPLE 2.5.1. *Use the logical equivalences above to show that* $\neg(p \vee \neg(p \wedge q))$ *is a contradiction.*

Solution.

$\neg(p \vee \neg(p \wedge q))$

$\Leftrightarrow \neg p \wedge \neg(\neg(p \wedge q))$ *De Morgan's Law*

$\Leftrightarrow \neg p \wedge (p \wedge q)$ *Double Negation Law*

$\Leftrightarrow (\neg p \wedge p) \wedge q$ *Associative Law*

$\Leftrightarrow F \wedge q$ *Contradiction*

$\Leftrightarrow F$ *Domination Law and Commutative Law*

NOTE: A **contingency** is a proposition that is neither a tautology nor a contradiction.

Propositional Logic - one last proof

- Show that $[p \wedge (p \rightarrow q)] \rightarrow q$ is a tautology.
- We use $\equiv$ to show that $[p \wedge (p \rightarrow q)] \rightarrow q \equiv \mathrm{T}$.

$[p \wedge (p \rightarrow q)] \rightarrow q$

$\equiv [p \wedge (\neg p \vee q)] \rightarrow q$	substitution for $\rightarrow$
$\equiv [(p \wedge \neg p) \vee (p \wedge q)] \rightarrow q$	distributive
$\equiv [\mathrm{F} \vee (p \wedge q)] \rightarrow q$	complement
$\equiv (p \wedge q) \rightarrow q$	identity
$\equiv \neg(p \wedge q) \vee q$	substitution for $\rightarrow$
$\equiv (\neg p \vee \neg q) \vee q$	DeMorgan's
$\equiv \neg p \vee (\neg q \vee q)$	associative
$\equiv \neg p \vee \mathrm{T}$	complement
$\equiv \mathrm{T}$	identity

NOTE: YOU CAN PROVE THE ABOVE PROBLEM BY USING TRUTH TABLE ALSO.

First-order logic

- First-order logic (FOL) models the world in terms of
  - **Objects,** which are things with individual identities
  - **Properties** of objects that distinguish them from other objects
  - **Relations** that hold among sets of objects
  - **Functions,** which are a subset of relations where there is only one "value" for any given "input"
- Examples:
  - Objects: Students, lectures, companies, cars ...
  - Relations: Brother-of, bigger-than, outside, part-of, has-color, occurs-after, owns, visits, precedes, ...
  - Properties: blue, oval, even, large, ...
  - Functions: father-of, best-friend, second-half, one-more-than ...

FOL Provides

- **Variable symbols**
  - E.g., x, y, foo
- **Connectives**
  - Same as in PL: not ($\neg$), and ($\wedge$), or ($\vee$), implies ($\rightarrow$), if and only if (biconditional $\leftrightarrow$)
- **Quantifiers**
  - Universal $\forall$**x** or **(Ax)**
  - Existential $\exists$**x** or **(Ex)**

Quantifiers

- **Universal quantification**
  - $(\forall x)P(x)$ means that P holds for **all** values of x in the domain associated with that variable
  - E.g., $(\forall x)\ \text{dolphin}(x) \rightarrow \text{mammal}(x)$
- **Existential quantification**
  - $(\exists\ x)P(x)$ means that P holds for **some** value of x in the domain associated with that variable
  - E.g., $(\exists\ x)\ \text{mammal}(x) \wedge \text{lays-eggs}(x)$
  - Permits one to make a statement about some object without naming it

Quantifier Scope

- Switching the order of universal quantifiers *does not* change the meaning:
  - $(\forall x)(\forall y)P(x,y) \leftrightarrow (\forall y)(\forall x)\ P(x,y)$
- Similarly, you can switch the order of existential quantifiers:
  - $(\exists x)(\exists y)P(x,y) \leftrightarrow (\exists y)(\exists x)\ P(x,y)$
- Switching the order of universals and existentials *does* change meaning:
  - Everyone likes someone: $(\forall x)(\exists y)\ \text{likes}(x,y)$
  - Someone is liked by everyone: $(\exists y)(\forall x)\ \text{likes}(x,y)$

Connections between All and Exists

We can relate sentences involving $\forall$ and $\exists$ using De Morgan's laws:

$$(\forall x)\ \neg P(x) \leftrightarrow \neg(\exists x)\ P(x)$$

$$\neg(\forall x)\ P \leftrightarrow (\exists x)\ \neg P(x)$$

$$(\forall x)\ P(x) \leftrightarrow \neg\ (\exists x)\ \neg P(x)$$

$$(\exists x)\ P(x) \leftrightarrow \neg(\forall x)\ \neg P(x)$$

Quantified inference rules

- Universal instantiation
  $\forall x\ P(x)\ \therefore\ P(A)$
- Universal generalization
  - $P(A) \wedge P(B) \ldots\ \therefore\ \forall x\ P(x)$
- Existential instantiation
  $\exists x\ P(x)\ \therefore P(F)$
- Existential generalization
  - $P(A)\ \therefore\ \exists x\ P(x)$

EXERCISE-1

1. If $p \equiv$ Sam is a teacher, $q \equiv$ John is an honest boy, then translate the following into logical sentences :
   (*a*) $\sim (p \wedge q)$, (*b*) $p \vee \sim q$, (*c*) $\sim p \Leftrightarrow q$, (*d*) $p \Rightarrow \sim q$.
2. Change the following sentence into symbols :
   (*a*) 'If I do not have car or I do not wear good dress then I am not a millionaire'.
   (*b*) Everyone who is healthy can do all kinds of work. (*Anna, 2004S*)
3. Prepare truth tables for the following statements (*a*) $(p \Rightarrow q) \wedge \sim q$, (*b*) $(p \Leftrightarrow q) \wedge (r \vee q)$.
4. Write down the truth table of
   (*a*) $p \vee q$ (*Madras, 1997*) (*b*) $p \wedge (p \wedge q)$ (*Madras, 2005S*)
5. Verify that the following are tautologies :
   (*a*) $p \to [q \to (p \wedge q)]$ (*Anna, 2005*)
   (*b*) $(p \wedge q \Rightarrow r) \Leftrightarrow (p \Rightarrow r) \vee (q \Rightarrow r)$ (*c*) $(p \Rightarrow q \wedge r) \Rightarrow (\sim r \Rightarrow \sim p)$.
6. Show that $Q \vee (P \wedge \sim Q) \vee (\sim P \wedge \sim Q)$ is a tautology. (*Anna, 2004S ; Madras, 2003S*)
7. Over the universe of positive integers
   $p(n)$: n is prime and $n < 32$.
   $q(n)$: n is a power of 3.
   $r(n)$: n is a divisor of 27.
   (*i*) What are the truth sets of these propositions ?
   (*ii*) Which of the three propositions implies one of the others ?
8. Given the propositions over the natural numbers
   $p : n < 4$, $q : 2n > 17$ and $r : n$ is a divisor of 18, what are the truth sets of
   (*i*) q, (*ii*) $p \wedge q$,
   (*iii*) r, (*iv*) $q \to r$.
9. Show that (*a*) $\sim Q, P \to Q \Rightarrow \sim P$. (*Madras, 1999*)
   (*b*) $(P \to R) \wedge (Q \to R) \Leftrightarrow (P \vee Q) \to R$. (*Madras, 2003*)
10. Construct the truth table for (*i*) $(\sim p \to q) \wedge (q \rightleftarrows p)$. (*Madras, 2001*)
    (*ii*) $\rceil[P \vee (Q \wedge R)] \rightleftarrows (P \vee Q) \wedge (P \vee R)$. (*Bharthian, Msc. 2001*)
    (*Andhra, 2004*)
11. Prove that the following statement is a contradiction :
    $S = [(p \vee q) \wedge (p \vee \sim q) \wedge (\sim p \vee q) \wedge (\sim p \vee \sim q)]$.
12. If p, q, r are three statements then prove that
    (*a*) $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ (*b*) $(p \Rightarrow q) \vee r \equiv (p \vee r) \Rightarrow (q \vee r)$
    (*c*) $\sim (p \vee q) \equiv \sim p \wedge \sim q$.
13. Define conjunction, conditional, biconditional and negation, with examples.

EXERCISE-2

1. If $A = \{1, 2, 3, 4, 5\}$ be the universal set, determine the truth values of each of the following statements :
   (*a*) $(\forall\, x \in A)\ (x + 2 < 10)$ (*b*) $\exists\, x \in A\ (x + 2 = 10)$
2. Negate each of the following statements :
   (*a*) $\forall\, x,\ x^3 = x$; (*b*) $\forall\, x,\ x + 5 > x$
   (*c*) Some students are 26 or older. (*d*) All students live in the hostels.
3. What is the truth value of $\forall\, x\, P(x)$ where $P(x)$ is a statement '$x^2 < 10$' and the universe of discourse consists of positive integers not exceeding 4.
4. Use universal quantifier to state 'the sum of any two rational numbers is rational'.
5. Over the universe of real numbers, use quantifier to say that the equation $a + x = b$ has a solution for all values of a and b. (*Madras, 1999*)
6. Translate the following statements involving quantifiers, into formulae :
   (*a*) All rationals are reals. (*b*) No rationals are reals.
   (*c*) Some rationals are reals. (*d*) Some rationals are not reals.
7. Show that $Q \vee (P \wedge {\sim} Q) \vee ({\sim} P \wedge {\sim} Q)$ is a tautology. (*Madras, 1998*)

NOTE: Follow the link : Lecture 1-5

https://www.youtube.com/watch?v=xlUFkMKSB3Y