E-Content for Lie Algebras

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Basic Concepts

Go through the following definitions:

Definition

A subset B of an associative algebra A over a field F is said to be *weakly closed* if for all ordered pairs (a, b), $a, b \in B$, there exists $\gamma(a, b) \in F$ such that $a \times b = ab + \gamma(a, b)ba \in B$.

Definition

A subset S of a weakly closed set B is said to be a *subsystem* if $c \times d \in S$, for all $c, d \in S$.

Definition

A subsystem S of a weakly closed set B is said to be a *left ideal* (respectively, *ideal*) if $a \times c \in S$, (respectively, $a \times c$, $c \times a \in S$) for all $a \in B$, $c \in S$.

Now work out the following examples:

Example

Let B be a subalgebra of A_L. Then $ab - ba \in B$ for all $a, b \in B$. Take $\gamma(a, b) = -1$ for all $a, b \in B$ so that B is weakly closed set.

Example

If B is a subalgebra of A_L with basis $\{e, f, h\}$ such that $[e, f] = h$, $[e, h] = 2e$, $[f, h] = -2f$, then $S = Fe \cup Ff \cup Fh$ is a subsystem of B.

Example

Let $S_n(F)$ denote the set of all symmetric $n \times n$ matrices over F and let $a, b \in S_n(F)$. Then $(ab + ba)^t = ba + ab$ implies that $ab + ba \in S_n(F)$. Hence, with $\gamma(a, b) = 1$ for all $a, b \in S_n(F)$, we have that $S_n(F)$ is a weakly closed set in $M_n(F)$, the set of all $n \times n$ matrices over F.

Example

Let $B = SS_n(F) \cup S_n(F)$, where $SS_n(F)$ denotes the set of all $n \times n$ skew-symmetric matrices over F. Then we have the following:
for $a, b \in S_n(F)$, $(ab + ba)^t = ba + ab$ implies $ab + ba \in S_n(F)$,
for $a \in SS_n(F)$, $b \in S_n(F)$, $(ab - ba)^t = -ba + ab$ implies $ab - ba \in S_n(F)$,
for $a \in S_n(F)$, $b \in SS_n(F)$, $(ab - ba)^t = -ba + ab$ implies $ab - ba \in S_n(F)$,
for $a, b \in SS_n(F)$, $(ab - ba)^t = ba - ab$ implies $ab - ba \in SS_n(F)$.

Choose

$$\gamma(a, b) = \begin{cases} 1 & \text{if } a, b \in S_n(F), \\ -1 & \text{otherwise.} \end{cases}$$

Clearly, B is a weakly closed set in $M_n(F)$ and $S_n(F)$ is an ideal of B.

Example

If B is a weakly closed set in A such that $\gamma(a, b) = 0$ for all $a, b \in B$, then B is a multiplicative semigroup in A.

Notations: If A is an algebra over F, then we mean that A is an associative algebra with 1, otherwise we say that A is associative algebra. Let A be an algebra (associative algebra) over F and $C \subseteq A$ then the subalgebra $C^{\dagger}$ (respectively, C^{*}) of A containing 1 generated by C (respectively, subalgebra of A generated by C) is called the *enveloping algebra* (respectively, the *enveloping associative algebra*) of C in A.

Lecture 1

Definition

Let A be an associative algebra over a field F and let B_1 and B_2 be subspaces of A. Then
$B_1 B_2 =$ subspace spanned by $\{b_1 b_2 \mid b_1 \in B_1, b_2 \in B_2\}$. Thus $A^1 = A$ and $A^k = A^{k-1}A$ for all $k \geq 2$. We say that A is nilpotent if there exists a positive integer n such that $A^n = 0$, that is, every product of k elements in A is zero. Also A is said to be nil if every element of A is nilpotent, that is, for every $a \in A$, there exists $n_a \in \mathbb{N}$ such that $a^{n_a} = 0$. Clearly, every nilpotent algebra is nil but not conversely.

Definition

Let A be a finite dimensional algebra over a field F. A nilpotent ideal R of A of maximal dimension is called the radical of A. Further, A is called semisimple if $R = 0$. Note that $\frac{A}{R}$ is always semisimple. If R is radical of A and I is a nilpotent ideal of A then $R + I$ is nilpotent, and so $R + I = R$. Hence, $I \subset R$. Therefore R contains every nilpotent ideal of A.

Lemma

Let B be a weakly closed set in an associative algebra A. If $w \in B$ then $w_B = \{w\}^* \cap B$ is a subsystem of B such that $w_B^* = \{w\}^*$.

Proof.

Clearly, $\{w\}^* = \{a_1 w + a_2 w^2 + \cdots + a_n w^n \mid n \in N, a_i \in F\}$. Then for $f_1, f_2 \in \{w\}^* \cap B$, $f_1 \times f_2 \in \{w\}^* \cap B$. Therefore w_B is a subsystem of B. Also, $\{w\}^* \subseteq w_B^*$, as $w \in w_B$. Further, $w_B \subseteq \{w\}^*$ implies $w_B^* \subseteq \{w\}^*$. Hence, $w_B^* = \{w\}^*$. $\square$

Lemma

If S is a subsystem of B and $w \in B$ be such that $s \times w \in S^*$ for all $s \in S$, then $S^* w \subseteq wS^* + S^*$.

Proof.

Let $\alpha \in S^*$. Then α is a linear combination of monomials $s_1 s_2 \cdots s_r$, $s_i \in S$. We prove, by induction on r, that $\alpha w \in wS^* + S^*$.
If $s \in S$, then $sw = -\gamma(s, w)ws + s \times w \in wS^* + S^*$. Therefore, the result is true for $r = 1$. Let the result be true for $r - 1$. Now

$$\begin{aligned} s_1 s_2 \cdots s_r w &= s_1 s_2 \cdots s_{r-1}(-\gamma(s_r, w)ws_r + s_r \times w) \\ &= -\gamma(s_r, w)(s_1 s_2 \cdots s_{r-1} w)s_r + s_1 s_2 \cdots s_{r-1}(s_r \times w) \\ &\in (wS^* + S^*)s_r + S^* \subseteq wS^* + S^*. \end{aligned}$$

$\square$

Lecture 2

Lemma

If S is a subsystem of B such that S^* is nilpotent and $S^* \neq B^*$. Then there exists $w \in B$ such that $w \notin S^*$ but $s \times w \in S^*$ for all $s \in S$.

Proof.

If $B \subseteq S^*$, then $B^* \subseteq S^* \subseteq B^*$ or $S^* = B^*$. So $B \nsubseteq S^*$ and there exists $w_1 \in B$ such that $w_1 \notin S^*$.
If $s \times w_1 \in S^*$ for all $s \in S$, then take $w = w_1$. Otherwise, there exists $s_1 \in S$ such that $w_2 = s_1 \times w_1 \notin S^*$. So, $w_2 \in B \setminus S^*$.
If $s \times w_2 \in S^*$ for all $s \in S$, take $w = w_2$, otherwise there exists $s_2 \in S$ such that $w_3 = s_2 \times w_2 \in B \setminus S^*$.
Continuing like this, either we get the required element w in a finite number of steps; or we get an infinite sequence $w_{i+1} = s_i \times w_i \in B \setminus S^*$, $s_i \in S, w_i \in B$. □

Proof Continued

In the second case $w_k = s_{k-1} \times (s_{k-2} \times (s_{k-3} \times (\cdots (s_1 \times w_1))) \cdots)$. Since S^* is nilpotent, so there exist $n \in \mathbb{N}$ such that any product of n elements of S^* is 0. Now

$$
\begin{aligned}
w_2 &= s_1 \times w_1 = s_1 w_1 + \gamma(s_1, w_1) w_1 s_1 \\
w_3 &= s_2 \times w_2 = s_2 w_2 + \gamma(s_2, w_2) w_2 s_2 \\
&= s_2 s_1 w_1 + \gamma(s_1, w_1) s_2 w_1 s_1 + \gamma(s_2, w_2) s_1 w_1 s_2 + \gamma(s_2, w_2)\gamma(s_1, w_1) w_1 s_1 s_2 \\
w_4 &= s_3 \times w_3 = s_3 w_3 + \gamma(s_3, w_3) w_3 s_3 \\
&= s_3 s_2 s_1 w_1 + \gamma(s_1, w_1) s_3 s_2 w_1 s_1 + \gamma(s_2, w_2) s_3 s_1 w_1 s_2 \\
&\qquad + \gamma(s_2, w_2)\gamma(s_1, w_1) s_3 w_1 s_1 s_2 + \gamma(s_3, w_3)\{s_2 s_1 w_1 s_3 + \gamma(s_1, w_1) s_2 w_1 s_1 \\
&\qquad\qquad + \gamma(s_2, w_2) s_1 w_1 s_2 s_3 + \gamma(s_2, w_2)\gamma(s_1, w_1) w_1 s_1 s_2 s_3\}.
\end{aligned}
$$

In general, w_{2n} is a linear combination of terms $c_1 c_2 \cdots c_j w_1 d_1 d_2 \cdots d_k$ where $c_i, d_i \in S$, $j + k = 2n - 1$. So, either $j \geq n$ or $k \geq n$. Hence, either $c_1 c_2 \cdots c_j = 0$ or $d_1 d_2 \cdots d_k = 0$. So, $w_{2n} = 0 \in S^*$, a contradiction.

Theorem

Let V be a vector space over a field F, $\dim_F(V) < \infty$. Let B be a weakly closed set in $L(V)$ such that every element of B is nilpotent, that is, for every $W \in B$ there exists $n \in N$ such that $W^n = 0$. Then B^* is nilpotent.

Proof.

The proof is by induction on $\dim_F(V)$. If $\dim_F(V) = 0$ or $B = \{0\}$, then the theorem is obvious. Therefore let $\dim_F(V) > 0$ and let $B \neq \{0\}$.
Let $\Omega = \{S \mid S \text{ is a subsystem of } B \text{ and } S^* \text{ is nilpotent}\}$. Let $\tilde{S} \in \Omega$ be such that $\dim_F(\tilde{S}^*)$ is maximal.
If $0 \neq W \in B$, then by Lemma 11, $W_B = \{W\}^* \cap B$ is a subsystem and $W_B^* = \{W\}^* = \{a_1 w + a_2 w^2 + \cdots + a_n w^n \mid n \in \mathbb{N}, a_i \in F\}$. This implies $\{W\}^*$ is nilpotent, and so $W_B^* \in \Omega$. As $W \neq 0$, we have $W_B^* = \{W\}^* \neq \{0\}$. Therefore $\tilde{S}^* \neq \{0\}$. □

Proof Continued

Let $\tilde{S}^*(V)$ denote the space spanned by $\{T(x)|T \in \tilde{S}^*, x \in V\}$. If $\tilde{S}^*(V) = V$, then for any $x \in V$, $x = \sum_i T_i(x_i)$, $x_i \in V$, $T_i \in \tilde{S}^*$. Also $x_i = \sum_j U_j(y_j)$, for some $y_j \in V$, $U_j \in \tilde{S}^*$ implies $x = \sum_i \sum_j T_i(U_j(y_j))$. On repeating this, we get $x = \sum T_{i_1} T_{i_2} T_{i_3} \dots T_{i_r}(z_i)$, $z_i \in V$, $T_{i_j} \in \tilde{S}^*$. Since $\tilde{S}^*$ is nilpotent, we have $x = 0$ which gives $V = \{0\}$, a contradiction. Therefore $\{0\} \subsetneq \tilde{S}^*(V) \subsetneq V$, which implies $0 < \dim_F(\tilde{S}^*(V)) < \dim_F(V)$. Let $\bar{S} = \{T \in B | T(\tilde{S}^*(V)) \subseteq \tilde{S}^*(V)\}$. If $T, U \in \bar{S}$, then $T \times U = TU + \gamma(T, U)UT$ and $(T \times U)(x) = U(T(x)) + \gamma(T, U)T(U(x)) \in \tilde{S}^*(V)$ for all $x \in \tilde{S}^*(V)$. This gives $T \times U \in \bar{S}$. Thus $\bar{S}$ is subsystem of B and $\tilde{S} \subseteq \bar{S}$.

Proof Continued

If $T \in \bar{S}$, then consider $T^* = T|_{\tilde{S}^*(V)} \in L(\tilde{S}^*(V))$ and $\bar{T} \in L(\frac{V}{\tilde{S}^*(V)})$ defined by $\bar{T}(v + \tilde{S}^*(V)) = T(v) + \tilde{S}^*(V)$ for all $v \in V$. Then $B_1 = \{T^* | T^* = T|_{\tilde{S}^*(V)}$ for some $T \in \bar{S}\}$, $B_2 = \{\bar{T}|$ for some $T \in \bar{S}\}$ are weakly closed systems of nilpotent linear transformations on $\tilde{S}^*(V)$ and $\frac{V}{\tilde{S}^*(V)}$. Since $\dim_F(\tilde{S}^*(V))$, $\dim_F\left(\frac{V}{\tilde{S}^*(V)}\right) < \dim_F(V)$, so by induction B_1^*, B_2^* are nilpotent. Therefore there exists $p, q \in \mathbb{N}$ such that $\bar{T}_1 \bar{T}_2 \cdots \bar{T}_p = 0$ on $\frac{V}{\tilde{S}^*(V)}$ for $T_i \in \bar{S}$; and $U_1^* U_2^* \cdots U_q^* = 0$ on $\tilde{S}^*(V)$ for $U_i \in \bar{S}$. This gives $T_1 T_2 \ldots T_p(v) \in \tilde{S}^*(V)$ and $U_1 U_2 \cdots U_q T_1 T_2 \cdots T_p(v) = 0$. Therefore $\bar{S}$ is nilpotent. Hence $\bar{S} \in \Omega$. By maximality of $\dim_F(\tilde{S})$, we have $\tilde{S}^* = \bar{S}^*$.

Proof Continued

If $\tilde{S}^* \neq B^*$, then by Lemma 13, there exists $W \in B$ such that $W \notin \tilde{S}^*$ but $T \times W \in \tilde{S}^*$ for all $T \in \tilde{S}$. By Lemma 12, $\tilde{S}^* W \subseteq W\tilde{S}^* + \tilde{S}^*$ implies that for all $v \in V$ and $T \in \tilde{S}^*$, we have $TW(v) = (WT_1 + T_2)(v)$, $T_i \in \tilde{S}^*$. This gives $W(T(v)) = T_1(W(v)) + T_2(v)$ and therefore $W(\tilde{S}^*(V)) \subseteq \tilde{S}^*(V)$.
So, $W \in \bar{S}$. Now since $W \notin \tilde{S}^*$, we get $\dim_F(\bar{S}^*) > \dim_F(\tilde{S}^*)$, a contradiction. Hence, $\tilde{S}^* = B^*$ and B^* is nilpotent.

This completes the proof.

Example

Let $T_n(F)$ be the set of all $n \times n$ upper triangular matrices over a field F. Verify that $T_n(F)$ is a subalgebra of $M_n(F)$ and $\dim(T_n(F)) = \frac{1}{2}n(n+1)$.
A basis of $T_n(F)$ is $\mathcal{B} = \{E_{ij} | i \leq j\}$, where $E_{ij} = (e_{ij})_{n\times n}$ such that
$e_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$
Clearly, $T_n(F)_L$ is Lie subalgebra of $M_n(F)_L$. Let $A, B \in T_n(F)$. Then we can write $A = \sum_{i=1}^n \sum_{j=i}^n a_{ij}E_{ij}$ and $B = \sum_{i=1}^n \sum_{j=i}^n b_{ij}E_{ij}$. Now

$$\begin{aligned} [A, B] &= AB - BA \\ &= \sum_{i=1}^n \sum_{j=i}^n a_{ij}E_{ij} \sum_{r=1}^n \sum_{s=r}^n b_{rs}E_{rs} - \sum_{i=1}^n \sum_{j=i}^n b_{ij}E_{ij} \sum_{r=1}^n \sum_{s=r}^n a_{rs}E_{rs}. \end{aligned}$$

Example . . .

As $E_{ij}E_{rs} = \begin{cases} E_{is} & \text{if } j = r, \\ 0 & \text{if } j \neq r, \end{cases}$ we have, therefore

$$[A, B] = \sum_{i=1}^{n} \sum_{s=i}^{n} \left(\sum_{r=1}^{n} a_{ir} b_{rs} E_{is} \right) - \sum_{i=1}^{n} \sum_{s=i}^{n} \left(\sum_{r=1}^{n} b_{ir} a_{rs} E_{is} \right).$$

Now if $i = s$, then

$$\sum_{r=1}^{n} (a_{ir} b_{rs} - b_{ir} a_{rs}) = \sum_{r=1}^{n} (a_{sr} b_{rs} - b_{sr} a_{rs}).$$

For $r > s$, $b_{rs} = 0 = a_{rs}$, for $r < s$, $a_{sr} = 0 = b_{sr}$, and for $r = s$, we have $a_{ss} b_{ss} - b_{ss} a_{ss} = 0$.
Therefore, $\sum_{r=1}^{n} (a_{sr} b_{rs} - b_{sr} a_{rs}) = 0$, and hence $[A, B] = C = (c_{ij})$, where $c_{ij} = 0$ for $i \geq j$, that is, $C \in N_n(F)$, the set of all $n \times n$ strictly upper-triangular matrices over F.

Example . . .

Now, if $A \in N_n(F)$ then $A = (a_{ij})$, $a_{ij} = 0$ for $i \geq j$, $j = 1, 2, \ldots, n$. This gives

$$\begin{aligned} A^2 &= (b_{ij}) \text{ such that } b_{ij} = 0 \text{ for } i \geq j-1, \quad j = 2, \ldots, n. \\ A^3 &= (c_{ij}) \text{ such that } c_{ij} = 0 \text{ for } i \geq j-2, \quad j = 3, \ldots, n, \end{aligned}$$

and continuing we get $A^n = 0$ as $A^n = (d_{ij})$ such that $d_{ij} = 0$ for $i \geq j - (n-1) = j - n + 1$, $j = n$. That is, $d_{1n} = 0$. So $N_n(F)$ consists of nil-triangular matrices and $T_n(F)^{(1)} \subseteq N_n(F)$.

Hence, $T_n(F)_L$ is a solvable Lie algebra but it is not a nilpotent Lie algebra. By Theorem 14, $N_n(F)^*$ is nilpotent associative algebra. (For, if $\dim_F(V) = n$ then $L(V) \simeq M_n(F)$ and $N_n(F)$ is a weakly closed set in $M_n(F)$ such that every element of $N_n(F)$ is nilpotent. Now use the previous theorem.)

Lecture 5

Theorem

Let V and B be as in Theorem 14. Then there exists a basis $\mathcal{B}$ for V such that for all $T \in B$, $[T]_{\mathcal{B}} \in N_n(F)$.

Proof.

Assume $V \neq \{0\}$. Let $B^*(V)$ be the space spanned by $\{T(v) | v \in V, T \in B^*\}$. Then $B^*(V) \neq V$ as in the proof of Theorem 14. $(\tilde{S}(V) \neq V)$.
Also, $B^{*2}(V) = B^*(B^*(V))$ and as above, if $B^*(V) \neq 0$, then $B^{*2}(V) \subsetneq B^*(V)$.
Thus we get a chain

$$V \supsetneq B^*(V) \supsetneq B^{*2}(V) \supsetneq \cdots \supsetneq B^{*(N-1)}(V) \supsetneq B^{*N}(V) = \{0\},$$

where $N \in \mathbb{N}$ such that $B^{*N}(V) = \{0\}$ but $B^{*(N-1)}(V) \neq \{0\}$. □

Proof . . .

Let $\mathcal{B}_1 = \{e_1, e_2, \ldots, e_{n_1}\}$ be a basis for $B^{*(N-1)}(V)$. Extend this to obtain a basis $\mathcal{B}_2 = \{e_1, e_2, \ldots, e_{n_1+n_2}\}$ for $B^{*(N-2)}(V)$. Continuing like this we obtain a basis $\mathcal{B} = \mathcal{B}_N = \{e_1, e_2, \ldots, e_{n_1+n_2+\cdots+n_N}\}$ for V,
$n_1 + n_2 + \cdots n_N = m = \dim_F(V)$.
Since $B(B^{*(N-k)}(V)) \subseteq B^{*(N-k+1)}(V)$, so for any $T \in B$:
$T(e_1) = T(e_2) = \cdots = T(e_{n_1}) = 0$;
$T(e_{n_1+1}), T(e_{n_1+2}), \ldots, T(e_{n_1+n_2}) =$ linear combination of elements of $\mathcal{B}_1$;
$T(e_{n_1+n_{2+1}}), T(e_{n_1+n_2+2}), \ldots, T(e_{n_1+n_2+n_3}) =$ linear combination of elements of $\mathcal{B}_2$;
and so on
$T(e_{n_1+n_2+\cdots+n_{N-1}+1}), \ldots, T(e_{n_1+n_2+\cdots+n_N}) =$ linear combination of elements of $\mathcal{B}_{N-1}$.

Proof . . .

This gives

$$[T]_{\mathcal{B}} = \begin{pmatrix} 0_{n_1} & * & . & . & . & . & * \\ & 0_{n_2} & . & . & . & . & * \\ & & . & & & & . \\ & & & . & & & . \\ & & & & . & & * \\ & & & & & 0_{n_{N-1}} & * \\ & & & & & & 0_{n_N} \end{pmatrix} \in N_n(F),$$

where $*$ denote entries which may be nonzero.

This completes the proof.

Lemma

Let B be a weakly closed subset of an associative algebra A over a field F and let I be an ideal of B. Then

1. $I^{*k}B^* \subseteq B^*I^{*k} + I^{*k}$;
2. $B^*I^{*k} \subseteq I^{*k}B^* + I^{*k}$;
3. $(B^*I^*)^k \subseteq B^*I^{*k}$;
4. $(I^*B^*)^k \subseteq I^{*k}B^*$.

Proof.

1. We prove by induction. By Lemma 2, $I^*w \subseteq wI^* + I^*$ for all $w \in B$, as I is an ideal. Therefore, if $w_1, w_2, \ldots, w_r \in B$, then
$I^*w_1w_2\cdots w_r \subseteq B^*I^* + I^*$. This implies $I^*B^* \subseteq B^*I^* + I^*$.
Assume that $I^{*k-1}B^* \subseteq B^*I^{*k-1} + I^{*k-1}$.

□

Proof . . .

Now, for $a_1, a_2, \ldots, a_k \in I^*$, $w \in B^*$, we have

$$\begin{aligned} a_1 a_2 \cdots a_k w &= a_1 a_2 \cdots a_{k-1}(a_k w) \subseteq a_1 a_2 \cdots a_{k-1}(B^* I^* + I^*) \\ &\subseteq (B^* I^{*k-1} + I^{*k-1}) I^* + I^{*k-1}) I^* \\ &\subseteq B^* I^{*k} + I^{*k}. \end{aligned}$$

2. Let $w \in B$, and let $a \in I$. Then $wa + \gamma(w, a)aw = w \times a \in I$. This gives $wI^* \subseteq I^* w + I^*$, and so $B^* I^* \subseteq I^* B^* + I^*$. Thus the result is true for $k = 1$. Rest follows by applying induction on k.
3. It is clearly true for $k = 1$. Assume that $(B^* I^*)^k \subseteq B^* I^{*k}$. Then

$$\begin{aligned} (B^* I^*)^{k+1} &= (B^* I^*)^k (B^* I^*) \subseteq B^* I^{*k} B^* I^* \\ &\subseteq B^*(B^* I^{*k} + I^{*k}) I^* \subseteq B^* I^{*k+1} \end{aligned}$$

4. Similar to (3) above.

Theorem

Let B and V be as in Theorem 14. Let I be an ideal of B such that every element of I is nilpotent. Then $I^* \subseteq R$, the radical of B^*. Hence, $I \subseteq R$.

Proof.

Clearly, $B^*I^* + I^*$ is a two sided ideal of associative algebra B^* as
$B^*(B^*I^* + I^*) \subseteq B^*I^*$ and
$(B^*I^* + I^*)B^* \subseteq B^*(B^*I^* + I^*) + B^*I^* + I^* \subseteq B^*I^* + I^*$.
Also $(B^*I^* + I^*)^k \subseteq B^*I^* + I^{*k}$.
By Theorem 14, I^* is nilpotent. Let $n \in \mathbb{N}$, such that $I^{*n} = \{0\}$. Then $(B^*I^* + I^*)^n \subseteq B^*I^*$ and $(B^*I^*)^n \subseteq B^*I^{*n} = \{0\}$. This implies $(B^*I^* + I^*)^{n^2} = \{0\}$. Therefore, $B^*I^* + I^*$ is a nilpotent ideal of B^*.
So, $I \subseteq I^* \subseteq B^*I^* + I^* \subseteq R$, as every nilpotent ideal of an associative algebra is contained in its radical. $\square$

Lecture 7

Theorem

(Engel's Theorem on abstract Lie algebras) *If L is a finite dimensional Lie algebra, then L is nilpotent if and only if ad_a is nilpotent for all $a \in L$.*

Proof.

As L is nilpotent, there exists $n \in \mathbb{N}$ such that $L^n = \{0\}$. Therefore $[\cdots[[a_1, a_2], a_3], \ldots, a_n] = 0$ for all $a_i \in L$. In particular, for all $x, a \in L$, we have

$$[\cdots[[x, \underbrace{a], a], \cdots a}_{(n-1)\text{-times}}] = 0.$$

Hence, $ad_a^{n-1} = 0$ for all $a \in L$, and so ad_a is nilpotent.
Conversely, let L be finite dimensional and let ad_a be nilpotent for all $a \in L$. Now $Inn(L) = \{ad_a | a \in L\}$.This set is a Lie algebra of nilpotent linear operators on L. (as $[ad_a, ad_b] = ad_{[a,b]} \in Inn(L)$). □

Proof . . .

Now $Inn(L)$ is a weakly closed set in $L(L)$ and $\dim_F(L) < \infty$ such that every element of $Inn(L)$ is nilpotent. Therefore by Theorem 14, $Inn(L)^*$ is a nilpotent associative algebra. So, there exists $m \in \mathbb{N}$ such that $ad_{a_1} ad_{a_2} \cdots ad_{a_m} = 0$ for all $a_i \in L$. This gives $[\cdots[[a, a_1], a_2], \ldots, a_m] = 0$ for all $a, a_1, \ldots, a_m \in L$. Hence, $L^{m+1} = \{0\}$ and L is nilpotent. This completes the proof.

Theorem

(Engel's Theorem on Lie algebra of linear transformations) *If L is a Lie algebra of linear transformations on a finite dimensional vector space V, (that is, L is a Lie subalgebra of $L(V)_L$) and every $T \in L$ is nilpotent then L^* is nilpotent.*

Proof.

As L is a Lie subalgebra of $L(V)_L$, so L is a weakly closed set in $L(V)$. Now apply Theorem of Lecture 3. □

Lemma

(Fitting) *Let V be a vector space over a field F, $\dim_F(V) < \infty$. Let $T \in L(V)$. Then $V = V_{0T} \oplus V_{1T}$ where each V_{iT} is invariant under T such that if $T_i = T|_{V_{iT}}$, then T_0 is nilpotent and T_1 is an automorphism of V_{1T}.*

Proof.

Clearly, $V \supseteq T(V) \supseteq T^2(V) \supseteq \cdots$ is a sequence of subspaces of V. As $\dim_F(V) < \infty$, there exists $r \in \mathbb{N}$ such that
$V_{1T} = T^r(V) = T^{(r+1)}(V) = \cdots$.
Let $W_i = \{z \in V | T^i(z) = 0\}$. Then $W_1 \subseteq W_2 \subseteq \cdots$ is a sequence of subspaces of V. As $\dim_F(V) < \infty$, there exists $s \in \mathbb{N}$ such that
$V_{0T} = W_s = W_{s+1} = \cdots$.
Let $t = \max(r, s)$. Then $V_{0T} = W_t$ and $V_{1T} = T^t(V)$. $\square$

Proof . . .

Let $x \in V$, then $T^t(V) = T^{2t}(V)$ implies $T^t(x) = T^{2t}(y)$ for some $y \in V$. This gives

$$x = (x - T^t(y)) + T^t(y) \in V_{0T} + V_{1T}.$$

Hence, $V = V_{0T} + V_{1T}$.
Let $z \in V_{0T} \cap V_{1T}$. Then $z = T^t(v)$ for some $v \in V$ and $T^t(z) = 0$, that is, $T^{2t}(v) = 0$. Therefore $v \in W_{2t} = W_t = V_{0T}$, and so $z = T^t(v) = 0$.
Therefore $V = V_{0T} \oplus V_{1T}$.
Since $V_{0T} = W_t$, we have $T^t = 0$ on V_{0T}, that is, $T_0 = T|_{V_{0T}}$ is nilpotent. Also $V_{1T} = T^r(V) = T^{r+1}(V)$ implies that for all $v \in V$ there exists $w \in V$ such that $T^r(v) = T^{r+1}(w) = T(T^r(w))$. Hence, $T_1 = T|_{V_{1T}}$ is surjective; and as $\dim_F(V_{1T}) < \infty$, T_1 is also injective. Therefore, T_1 is an automorphism of V_{1T}.

This completes the proof.

Note that the subspace V_{0T} is called Fitting null component of V relative to T and the subspace V_{1T} is called Fitting one component of V relative to T.

Theorem

(Primary Decomposition Theorem) *Let V be a vector space over a field F, $\dim_F(V) < \infty$ and let $T \in L(V)$. If $m_T(x) = p_1^{r_1}(x)p_2^{r_2}(x)\cdots p_k^{r_k}(x)$ is minimal polynomial of T where $p_i(x)$'s are its monic irreducible factors and r_i's are positive integers, then $V = V_{p_1T} \oplus V_{p_2T} \oplus \cdots \oplus V_{p_kT}$, where $V_{p_iT} = \{v \in V | p_i^{r_i}(T)(v) = 0\}$ are $T-$invariant subspaces of V. Also if for each i, $T_i = T|_{V_{p_iT}}$, then minimal polynomial of T_i is $p_i^{r_i}(x)$.*

The T-invariant subspaces V_{p_iT}, described in the above theorem, are called primary components of V corresponding to T. Note that if $p_i(x) = x$, then $T_i^{r_i} = 0$ or $T^{r_i} = 0$ on V_{p_iT}. This implies $V_{p_iT} \subseteq V_{0T}$.

Further, if $p_j(x) \neq x$, then $x \nmid p_j(x)$, and so T_j is an isomorphism. Therefore $V_{p_j T} = T(V_{pjT}) = \cdots$, and so $V_{p_j T} \subseteq V_{1T}$.

Hence, $\sum_{p_j(x) \neq x} V_{p_j T} \subseteq V_{1T}$. Therefore $V = V_{0T} \oplus V_{1T} = V_{xT} \oplus \sum_{p_j(x) \neq x} V_{p_j T}$. That is, Fitting null component $V_{0T} = V_{xT} =$ characterstic space of characterstic root 0 of T; and Fitting one component $V_{1T} = \sum_{p_j(x) \neq x} V_{p_j T}$.

Next we shall study nilpotent Lie algebras of linear transformations on a finite dimensional vector space V over a field F.

Let A be an associative algebra over a field F. For $a \in A$ inner derivation $ad_a : A \to A$ is given by $ad_a(x) = [x, a]$ for all $x \in A$.
Define inductively

$$x^{(0)} = x,\ x^{(1)} = [x^{(0)}, a] = ad_a(x),\ x^{(k)} = ad_a^{(k-1)}(x).$$

Then

$$\begin{aligned} xa &= ax + ad_a(x) = ax^{(0)} + x^{(1)}, \\ xa^2 &= (xa)a = (ax^{(0)} + x^{(1)})a = a(ax^{(0)} + x^{(1)}) + ax^{(1)} + x^{(2)} \\ &= a^2x + \binom{2}{1} ax^{(1)} + x^{(2)}. \end{aligned}$$

Let $xa^{k-1} = a^{k-1}x + \binom{k-1}{1} a^{k-2}x^{(1)} + \cdots + \binom{k-1}{k-2} ax^{(k-2)} + x^{(k-1)}$, then

$$
\begin{aligned}
xa^k &= (xa^{k-1})a \\
&= a^{k-1}(ax^{(0)} + x^{(1)}) + \binom{k-1}{1} a^{k-2}(ax^{(1)} + x^{(2)}) + \cdots \\
&\quad + \binom{k-1}{k-2} a(ax^{(k-2)} + x^{(k-1)}) + ax^{(k-1)} + x^{(k)} \\
&= a^k x + \binom{k}{1} a^{k-1} x^{(1)} + \binom{k}{2} a^{k-2} x^{(2)} + \cdots + \binom{k}{k-1} ax^{(k-1)} + x^{(k)}.
\end{aligned}
$$

Similarly,

$$
a^k x = xa^k - \binom{k}{1} x^{(1)} a^{k-1} + \binom{k}{2} x^{(2)} a^{k-2} + \cdots \pm x^{(k)}.
$$

Lemma

Let V be a vector space over F, $\dim_F(V) < \infty$ and let $T, U \in L(V)$ such that there exists $N \in \mathbb{N}$ satisfying $[\cdots[[U, \underbrace{T], T], \cdots, T}_{N\text{-times}}] = 0$. Then V_{0T}, V_{1T} are invariant under U.

Lecture 10

Proof.

Let $v \in V_{0T}$. Then $T^m(v) = 0$ for some m. Therefore for $k = N + m - 1$,

$$
\begin{aligned}
T^k(U(v)) &= UT^k(v) \\
&= (T^k U + \binom{k}{1} T^{k-1} U^{(1)} + \binom{k}{2} T^{k-2} U^{(2)} \\
&\qquad + \cdots + \binom{k}{N-1} T^{k-N+1} U^{(N-1)} + \ldots + \binom{k}{k-1} TU^{(k-1)} + \\
&= U(T^k(v)) + \binom{k}{1} U^{(1)}(T^{k-1}(v)) + \cdots + \binom{k}{N-1} U^{(N-1)}(T^m(\\
&\qquad + \binom{k}{N} U^{(N)}(T^{m-1}(v)) + \cdots + U^{(k)}(v).
\end{aligned}
$$

Here $U^{(0)} = U$, $U^{(1)} = ad_T(U^{(0)}) = [U, T]$, $U^{(r)} = ad_T^{(r-1)}(U)$, $r \geq 1$. So $U^{(N)} = 0$. $\square$

Proof . . .

Therefore for $m = k - N + 1 \le j \le k$, $T^j(v) = 0$ and for $j > N - 1$, $U^{(j)} = 0$. Hence, $T^k(U(v)) = 0$, and so $U(v) \in V_{0T}$.
Now let $v \in V_{1T} = T^t(V) = T^{t+1}(V) = \cdots = T^{t+N-1}(V)$. Then there exists $w \in V$ such that $v = T^{t+N-1}(w)$. Now

$$
\begin{aligned}
U(v) &= U(T^{t+N-1}(w)) = T^{t+N-1}U(w) \\
&= (UT^{t+N-1} - \binom{t+N-1}{1} U^{(1)} T^{t+N-2} + \binom{t+N-1}{2} U^{(2)} T^{t+N-3} \\
&\quad + \cdots + (-1)^{N-1}\binom{t+N-1}{N-1} U^{(N-1)} T^t + (-1)^N \binom{t+N-1}{N} U^{(N} \\
&\quad + \cdots \pm U^{(t+N-1)})(w)
\end{aligned}
$$

Proof . . .

$$U(v) = T^{t+N-1}(U(w)) - \binom{t+N-1}{1} T^{t+N-2}(U^{(1)}(w))$$
$$+ \binom{t+N-1}{2} T^{t+N-3}(U^{(2)}(w)) + \cdots + (-1)^{N-1}\binom{t+N-1}{N-1} T$$
$$+ (-1)^{N}\binom{t+N-1}{N} T^{t-1}(U^{(N)}(w)) + \cdots \pm U^{(t+N-1)}(w).$$

So, for $j \geq N$, $U^{(j)} = 0$ and for $j < N$, $T^{t+j}(U^{(N-j-1)}(w)) \in T^{t+j}(V) = V_{1T}$. Hence, $U(v) \in V_{1T}$. This completes the proof.

Dear Students, The e-content on Unit 4 will be uploaded next week.

<u>STAY SAFE</u>