LINEAR PROGRAMMING PROBLEM (LPP)

BY

Dr. SANJEEV KUMAR

Department of Economics, University of Lucknow, Lucknow

INTRODUCTION

- It is an optimization method applicable for the solution of optimization problem where objective function and the constraints are linear.
- It was first applied in 1930 by economist, mainly in solving resource allocation problem During World War II, the US Air force sought more effective procedure for allocation of resources George B. Dantzig, a member of the US Air Force formulate general linear problem for solving the resources allocation problem.

"A Linear Programming Problem is one that is concerned with finding the optimal value (maximum or minimum value) of a linear function (called objective function) of several variables (say x and y), subject to the conditions that the variables are non-negative and satisfy a set of linear inequalities (called linear constraints). The term linear implies that all the mathematical relations used in the problem are linear relations while the term programming refers to the method of determining a particular programme or plan of action."

ESSENTIALS OF LINEAR PROGRAMMING MODEL

For a given problem situation, there are certain essential conditions that need to be solved by using linear programming.

1. Limited resources :	limited number of labour, material equipment and finance
2. Objective :	refers to the aim to optimize (maximize the profits or minimize the costs).
3. Linearity :	increase in labour input will have a proportionate increase in output.
4. Homogeneity :	the products, workers' efficiency, and machines are assumed to be identical.
5. Divisibility :	it is assumed that resources and products can be divided into fractions. (in case the fractions are not possible, like production of one-third of a computer, a modification of linear programming called integer programming can be used).

FORMULATION OF LINEAR PROGRAMMING

Formulation of linear programming is the representation of problem situation in a mathematical form. It involves well defined decision variables, with an objective function and set of constraints.

Objective function:

The objective of the problem is identified and converted into a suitable objective function. The objective function represents the aim or goal of the system (i.e., decision variables) which has to be determined from the problem. Generally, the objective in most cases will be either to maximize resources or profits or, to minimize the cost or time.

Constraints:

When the availability of resources are in surplus, there will be no problem in making decisions. But in real life, organizations normally have scarce resources within which the job has to be performed in the most effective way. Therefore, problem situations are within confined limits in which the optimal solution to the problem must be found.

Non-negativity constraint

Negative values of physical quantities are impossible, like producing negative number of chairs, tables, etc., so it is necessary to include the element of non-negativity as a constraint

Solution

Decision variables completely describe the decisions to be made (in this case, by Manager). Manager must decide how many corrugated and ordinary cartons should be manufactured each week. With this in mind, he has to define:

xl be the number of corrugated boxes to be manufactured.
x2 be the number of carton boxes to be manufactured

Objective function is the function of the decision variables that the decision maker wants to maximize (revenue or profit) or minimize (costs). Manager can concentrate on maximizing the total weekly profit (z).

Here profit equals to **(weekly revenues) – (raw material purchase cost) – (other variable costs).**

Hence Manager's objective function is:

Maximize $z = 6X_2 + 4x_2$

Constraints show the restrictions on the values of the decision variables. Without constraints manager could make a large profit by choosing decision variables to be very large. Here there are three constraints:

Available machine-hours for each machine
Time consumed by each product

Sign restrictions are added if the decision variables can only assume nonnegative values (Manager can not use negative negative number machine and time never negative number)

All these characteristics explored above give the following **Linear Programming** (LP) problem

$$
\begin{aligned}
&\max z = 6x_1 + 4x_2 && \text{(The Objective function)}\\
&\text{s.t.}\quad 2x_1 + 3x_2 \le 120 && \text{(cutting timeconstraint)}\\
&\qquad 2x_1 + x_2 \le 60 && \text{(pinning constraint)}\\
&\qquad x_1, x_2 \ge 0 && \text{(Sign restrictions)}
\end{aligned}
$$

A value of (x_1,x_2) is in the **feasible region** if it satisfies all the constraints and sign restrictions.

This type of linear programming can be solve by two methods

1) Graphical method
2) Simplex algorithm method

Graphic Method

Step 1: *Convert the inequality constraint as equations and find co-ordinates of the line.*

Step 2: *Plot the lines on the graph.*

(***Note:*** If the constraint is ≥ type, then the solution zone lies away from the centre.
If the constraint is ≤ type, then solution zone is towards the centre.)

Step 3: *Obtain the feasible zone.*

Step 4: *Find the co-ordinates of the objectives function (profit line) and plot it on the* graph representing it with a dotted line.

Step 5: *Locate the solution point.*

(*Note: If the given problem is maximization,* **Z***max then locate the solution point at* the far most point of the feasible zone from the origin and if minimization, Zmin then locate the solution at the shortest point of the solution zone from the origin).

Step 6: Solution type

i. If the solution point is a single point on the line, take the corresponding values of x_1 and x_2.
ii. If the solution point lies at the intersection of two equations, then solve for x_1 and x_2 using the two equations.
iii. If the solution appears as a small line, then a multiple solution exists.
iv. If the solution has no confined boundary, the solution is said to be an unbound solution.

CONT…

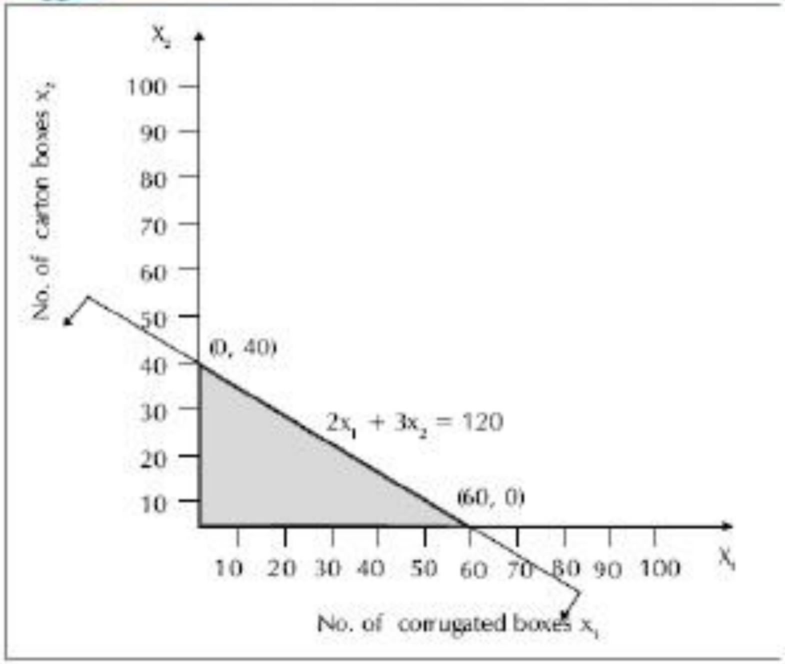

Figure 4.1: Graph Considering First Constraint

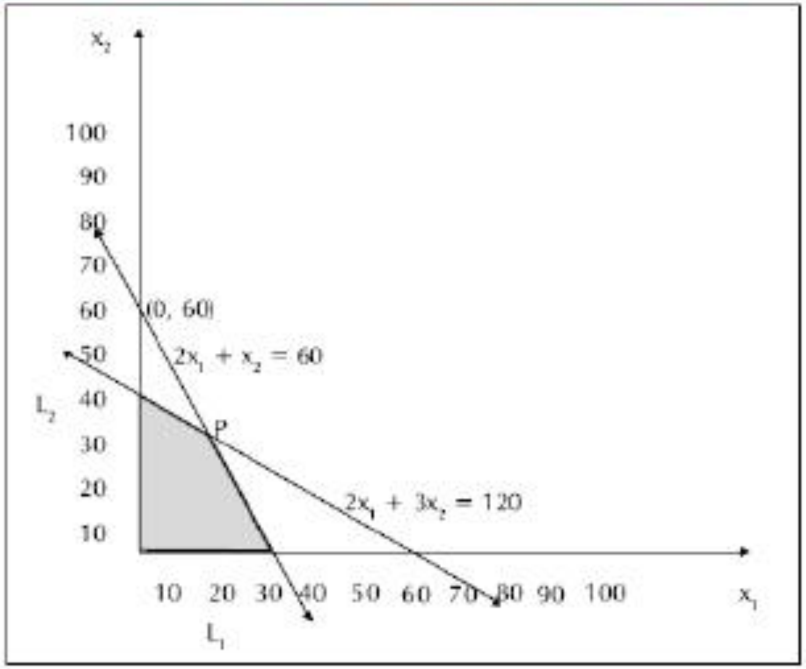

Figure 4.2: Graph Showing Feasible Area

The inequality constraint of the first line is (less than or equal to) ≤ type which means the feasible solution zone lies towards the origin.

(Note: If the constraint type is ≥ then the solution zone area lies away from the origin in the opposite direction). Now the second constraints line is drawn.

When the second constraint is drawn, you may notice that a portion of feasible area is cut. This indicates that while considering both the constraints, the feasible region gets reduced further. Now any point in the shaded portion will satisfy the constraint equations.

the objective is to maximize the profit. The point that lies at the furthermost point of the feasible area will give the maximum profit. To locate the point, we need to plot the objective function (profit) line.

Cont..

Objective function line (Profit Line)

Equate the objective function for any specific profit value Z,

Consider a Z-value of 60, i.e.,

$6x_1 + 4x_2 = 60$

Substituting $x_1 = 0$, we get $x_2 = 15$ and

if $x_2 = 0$, then $x_1 = 10$

Therefore, the co-ordinates for the objective function line are (0,15), (10,0) as indicated objective function line. The objective function line contains all possible combinations of values of xl and x2.

Therefore, we conclude that to maximize profit, 15 numbers of corrugated boxes and 30 numbers of carton boxes should be produced to get a maximum profit. Substituting

$x_1 = 15$ and $x_2 = 30$ in objective function, we get

$Z_{max} = 6x_1 + 4x_2$

$= 6(15) + 4(30)$

Maximum profit : Rs. 210.00

Example Problem

Problem: All products manufactured are shipped out of the storage area at the end of the day. Therefore, the two products must share the total raw material, storage space, and production time. **The company wants to determine how many units of each product to produce per day to maximize its total income.**

Solution

- The company has decided that it wants to maximize its sale income, which depends on the number of units of product I and II that it produces.
- Therefore, the decision variables, x1 and x2 can be the number of units of products I and II, respectively, produced per day.

- The object is to maximize the equation:

$$Z = 13x_1 + 11x_2$$

subject to the constraints on storage space, raw materials, and production time.

- Each unit of product I requires 4 ft^2 of storage space and each unit of product II requires 5 ft^2. Thus a total of $4x_1 + 5x_2$ ft^2 of storage space is needed each day. This space must be less than or equal to the available storage space, which is 1500 ft^2. Therefore,

$$4X_1 + 5X_2 \leq 1500$$

- Similarly, each unit of product I and II produced requires 5 and 3 1bs, respectively, of raw material. Hence a total of $5x_l + 3x_2$ Ib of raw material is used.

- This must be less than or equal to the total amount of raw material available, which is 1575 Ib. Therefore,

$$5x_1 + 3x_2 \leq 1575$$

- Prouct I can be produced at the rate of 60 units per hour. Therefore, it must take I minute or 1/60 of an hour to produce I unit. Similarly, it requires 1/30 of an hour to produce 1 unit of product II. Hence a total of $x_1/60 + x_2/30$ hours is required for the daily production. This quantity must be less than or equal to the total production time available each day. Therefore,

$$x_1 / 60 + x_2 / 30 \leq 7$$

or $$x_1 + 2x_2 \leq 420$$

- Finally, the company cannot produce a negative quantity of any product, therefore x_1 and x_2 must each be greater than or equal to zero.

- The linear programming model for this example can be summarized as:

Maximize

$$Z = 13x_1 + 11x_2$$

subject to:

$$\begin{aligned} 4x_1 + 5x_2 &\leq 1500 \\ 5x_1 + 3x_2 &\leq 1575 \\ x_1 + 2x_2 &\leq 420 \\ x_1 &\geq 0 \\ x_2 &\geq 0 \end{aligned}$$

…..Eq (4)

Graphical Solution to LP Problems

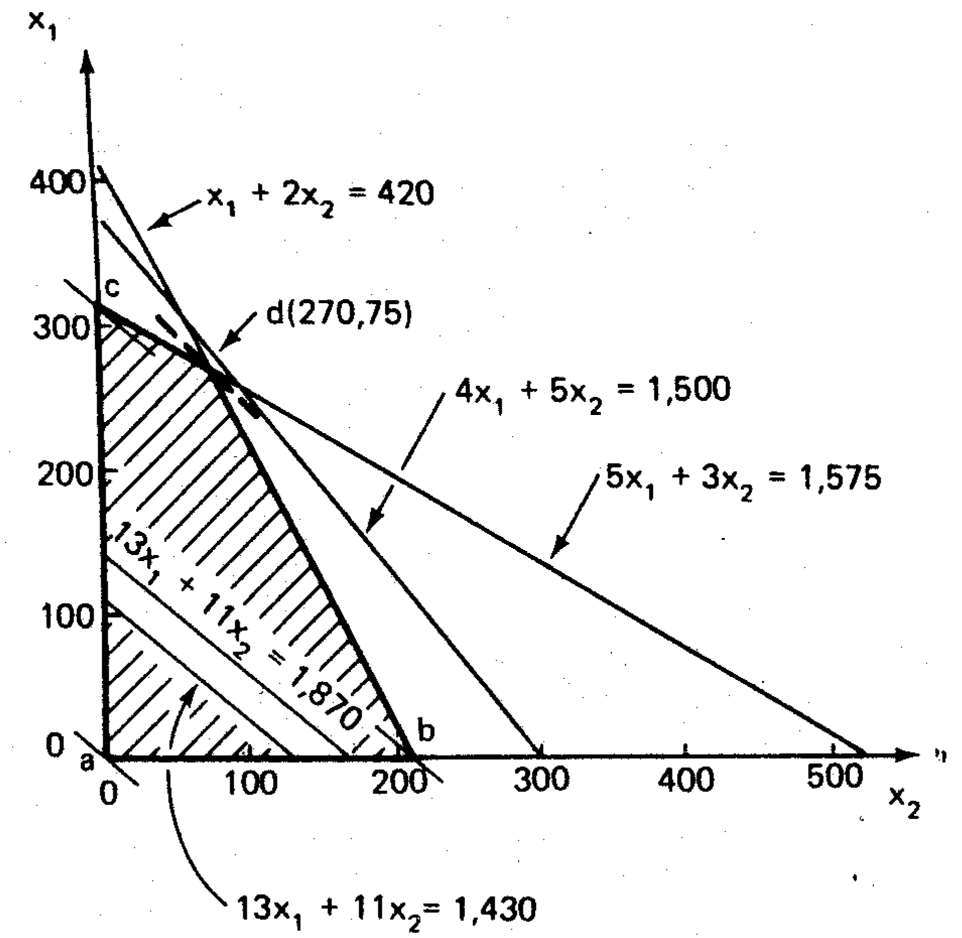

Figure Graphical solution for linear programming problem.

Simplex Method

In practice, most problems contain more than two variables and are consequently too large to be tackled by conventional means. Therefore, an algebraic technique is used to solve large problems using Simplex Method. This method is carried out through iterative process systematically step by step, and finally the maximum or minimum values of the objective function are attained.

The simplex method solves the linear programming problem in iterations to improve the value of the objective function. The simplex approach not only yields the optimal solution but also other valuable information to perform economic and 'what if' analysis.

ADDITIONAL VARIABLES USED IN SOLVING LPP

Three types of additional variables are used in simplex method such as,

(a) Slack variables (S_1, S_2, S_3.....S_n): Slack variables refer to the amount of unused resources like raw materials, labour and money.

(b) Surplus variables ($-S_1$, $-S_2$, $-S_3$.....$-S_n$): Surplus variable is the amount of resources by which the left hand side of the equation exceeds the minimum limit.

(c) Artificial Variables (a_1, a_2, a_3.. ...a_n): Artificial variables are temporary slack variables which are used for purposes of calculation, and are removed later.

The above variables are used to convert the inequalities into equality equations, as given in the Given Table below.

Cont…

Table 5.1: Types of Additional Variables

	Constraint Type		Variable added	Format
a)	Less than or equal to	$\leq$	Add Slack Variable	+S
b)	Greater than or equal to	$\geq$	Subtract surplus variable and add artificial variable	-S+a
c)	Equal to	=	Add artificial variable	+a

Procedure of simplex Method

Step 1: *Formulate the LP problem.*

Step 2: *Introduce slack /auxiliary variables.*
if constraint type is ≤ introduce + S
if constraint type is ≥introduce – S + a and
if constraint type is = introduce a

Step 3: *Find the initial basic solution.*

Step 4: *Establish a simplex table and enter all variable coefficients. If the objective function is maximization, enter the opposite sign co-efficient and if minimization, enter without changing the sign.*

Step 5: Take the most negative coefficient in the objective function, Zj to identify the key column (the corresponding variable is the entering variable of the next iteration table).

Step 6: Find the ratio between the solution value and the coefficient of the key column. Enter the values in the minimum ratio column.

Cont….

Step 7: *Take the minimum positive value available in the minimum ratio column to identify the key row. (The corresponding variable is the leaving variable of the table).*

Step 8: *The intersection element of the key column and key row is the pivotal element.*

Step 9: Construct the next iteration table by eliminating the leavin variable and introducing the entering variable.

Step 10: *Convert the pivotal element as 1 in the next iteration table and compute the other elements in that row accordingly. This is the pivotal equation row (not key row).*

Step 11: *Other elements in the key column must be made zero. For simplicity, form the equations as follows: Change the sign of the key column element, multiply with pivotal equation element and add the corresponding variable.*

Cont….

Step 12: *Check the values of objective function. If there are negative values, the solution is not an optimal one; go to step 5. Else, if all the values are positive, optimality is reached. Non-negativity for objective function value is not considered. Write down the values of $x_1, x_2, \ldots\ldots\ldots x_i$ and calculate the objective function for maximization or minimization.*

Note:

(i) If there are no x_1, x_2 variables in the final iteration table, the values of x_1 and x_2 are zero.

(ii) Neglect the sign for objective function value in the final iteration table.

Example

- Previous the packaging product mix problem is solved using simplex method.
- Maximize $Z = 6x_1 + 4x_2$
- Subject to constraints,
  - $2x_1+3x_2 \leq 120$ (Cutting machine) ....(i)
  - $2x_1+ x_2 \leq 60$ (Pinning machine) ....(ii)
  - where $x_1, x_2 \geq 0$
- Considering the constraint for cutting machine,
  - $2x_1+ 3x_2 \leq 120$
- To convert this inequality constraint into an equation, introduce a slack variable, S3 which represents the unused resources. Introducing the slack variable, we have the equation $2x_1+ 3x_2 + S_3 = 120$
- Similarly for pinning machine, the equation is

$$2x_1+ x_2 + S_4 = 60$$

Example cont….

If variables x1 and x2 are equated to zero,

i.e., x1 = 0 and x2 = 0, then

$S_3 = 120$

$S_4 = 60$

This is the basic solution of the system, and variables S_3 and S_4 are known as Basic Variables, S_B while x_1 and x_2 known as Non-Basic Variables. If all the variables are non negative, a basic feasible solution of a linear programming problem is called a Basic Feasible Solution.

Cont….

Rewriting the constraints with slack variables gives us,

$$Z_{max} = 6x_1 + 4x_2 + 0S_3 + 0S_4$$

Subject to constraints,

$2x_1 + 3x_2 + S_3 = 120$ ………………..(i)

$2x_1 + x_2 + S_4 = 60$ ………………..(ii)

where $x_1, x_2 \geq 0$

Which can shown in following simplex table form

Table 5.2: Co-efficients of Variables

Iteration Number	Basic Variables	Solution Value	X_1 K_C	X_2	S_3	S_4	Minimum Ratio	Equation
0	S_3	120	2	3	1	0	60	
	S_4	60	2	1	0	1	30	
	$-Z_j$	0	–6	–4	0	0		

Cont…

If the objective of the given problem is a maximization one, enter the co-efficient of the objective function Zj with opposite sign as shown in table. Take the most negative coefficient of the objective function and that is the key column Kc. In this case, it is -6.

Find the ratio between the solution value and the key column coefficient and enter it in the minimum ratio column.

The intersecting coefficients of the key column and key row are called the pivotal element i.e. 2.

The variable corresponding to the key column is the entering element of the next iteration table and the corresponding variable of the key row is the leaving element of the next iteration table (*In other words, x1 replaces S4 in the next iteration table. Given indicates the key column, key row and the pivotal element.*)

Cont..

Iteration Number	Basic Variables	Solution Value	X_1 K_C	X_2	S_3	S_4	Minimum Ratio	Equation
0	S_3	120	2	3	1	0	60	
K_r	S_4	60	2	1	0	1	30	
	$-Z_j$	0	-6	-4	0	0		

In the next iteration, enter the basic variables by eliminating the leaving variable (i.e., key row) and introducing the entering variable (i.e., key column).

Make the pivotal element as 1 and enter the values of other elements in that row accordingly.

In this case, convert the pivotal element value 2 as 1 in the next iteration table.

For this, divide the pivotal element by 2. Similarly divide the other elements in that row by 2. The equation is S4/2.

This row is called as Pivotal Equation Row Pe.

The other co-efficients of the key column in iteration Table 5.4 must be made as zero in the iteration Table 5.5.

For this, a solver, Q, is formed for easy calculation.

Cont...

Solver, $Q = S_B + (-K_c * P_e)$

The equations for the variables in the iteration number 1 of table 8 are,

For S_3 $Q = S_B + (-K_c * P_e)$

$= S_3 + (-2x P_e)$

$= S_3 - 2P_e$ ....(i)

For $-Z$, $Q = S_B + (-K_c * P_e)$

$= -Z + ((-6) * P_e)$

$= -Z + 6P_e$ ....(ii)

Using the equations (i) and (ii) the values of S3 and –Z for the values of Table 1 are found as shown in Table 5.4

Cont….

Iteration Number	Basic Variables	Solution Value	X_1 K_C	X_2 K_C	S_3	S_4	Minimum Ratio	Equation
0	S_3	120	2	3	1	0	60	
K_r	S_4	60	2	1	0	1	30	
	$-Z_j$	0	-6	-4	0	0		
I K_r	S_3	60	0	2	1	-1	30	$S_3 - 2P_e$
P_e	x_1	30	1	½	0	½	60	$S_{4/2}$
	$-Z_j$	100	0	-1	0	3		$-Z + 6P_e$

Using these equations, enter the values of basic variables S_B and objective function Z. If all the values in the objective function are non-negative, the solution is optimal.

Here, we have one negative value – 1. Repeat the steps to find the key row and pivotal equation values for the iteration 2 and check for optimality.

We get New Table as below:

Cont….

Table 5.5: Iteration Table

Iteration Number	Basic Variables	Solution Value	X_1	X_2	S_3	S_4	Minimum Ratio	Equation
0	S_3	120	2	3	1	0	60	
	S_4	60	2	1	0	1	30	
K_r	$-Z_j$	0	-6	-4	0	0		
1	S_3	60	0	2	1	-1	30	$S_3 - 2p_e$
K_r	x_1	30	1	½	0	½	60	$S_{4/2}$
P_e	$-Z_j$	100	0	-1	0	3		$-Z + 6P_e$
2 P_e	X_2	30	0	1	½	–		$S_{3/2}$
	x_1	15	1	0	–	1/2		$S_3 - P_{e/2}$
	$-Z_j$	210	0	0	1/4	¾		$-Z + P_e$
					½	5/2		

The solution is,

x_1 = 15 corrugated boxes are to be produced and

x_2 = 30 carton boxes are to be produced to yield a

Profit, Z_{max} = Rs. 210.00

Complications in Simplex Method

Example: Greater- Than-Or-Equal- To Constraints

Minimize

$$P = 1500y_1 + 1575y_2 + 420y_3$$

subject to:

$$4y_1 + 5y_2 + y_3 \geq 13$$

$$5y_1 + 3y_2 + 2y_3 \geq 11 \qquad \text{.......Eq. (1)}$$

$$\text{all } y_i \geq 0$$

- To start the solution, slack variables must first be assigned to convert all in-equalities to equalities. Let S_1 and S_2 be slack variables.
- Re- arrange the objective function so that all the variables are on the left-hand side of the equation.

Complications in Simplex Method

$$
\begin{aligned}
&P - 1500y_1 - 1575y_2 - 420y_3 = 0 \\
&4y_1 + 5y_2 + y_3 - S_1 = 13 \\
&5y_1 + 3y_2 + 2y_3 - S_2 = 11 \\
&\text{all } y_i \geq 0, \text{ all } S_i \geq 0
\end{aligned}
$$

…….Eq. (2)

- The negative signs for S_1 and S_2 make it no longer feasible to set all the decision variables (i.e., y_1, y_2, y_3) equal to zero as the initial solution.
- To assure a starting feasible solution, artificial variables can be added to the greater-than-or-equal-to constraints. Let W_1 and W_2 be two artificial variables. Hence the Eq. (2) becomes:

$$
\begin{aligned}
&4y_1 + 5y_2 + y_3 - S_1 + W_1 = 13 \\
&5y_1 + 3y_2 + 2y_3 - S_2 + W_2 = 11
\end{aligned}
$$

…….Eq. (3)

Complications in Simplex Method

- A starting feasible solution can be easily derived from Eq. (3) as follows:

$$y_1 = y_2 = y_3 = S_1 = S_2 = 0, \quad W_1 = 13, \quad \text{and} \quad W_2 = 11$$

- The objective function in Eq. (3) then becomes:

Minimize

$$P = 1500y_1 + 1575y_2 + 420y_3 + 5000W_1 + 5000W_2$$

…….Eq. (4)

- From Eq. (3):

$$W_1 = 13 - 4y_1 - 5y_2 - y_3 + S_1$$

$$W_2 = 11 - 5y_1 - 3y_2 - 2y_3 + S_2$$

- Substituting these expressions in Eq. (4) yields the following new expression for the objective function:

$$P = -43{,}500y_1 - 38{,}425y_2 - 14{,}580y_3 + 5000S_1 + 5000S_2 + 120{,}000$$

- The objective function may now be combined with Eq. (3) to express the problem model as follows:

$$\begin{aligned} P + 43{,}500y_1 + 38{,}425y_2 + 14{,}580y_3 - 5000S_1 - 5000S_2 &= 120{,}000 \\ 4y_1 + 5y_2 + y_3 - S_1 + W_1 &= 13 \\ 5y_1 + 3y_2 + 2y_3 - S_2 + W_2 &= 11 \end{aligned} \quad (9$$

- The coefficients and constants in Eq. (5) can now be arranged in the Tableau format as shown in next slide.

Complications in Simplex Method

Row Number	Basic Variable	P	y_1	y_2	y_3	S_1	S_2	W_1	W_2	Right-Hand Side	Upper Bound on Entering Variable
Initial tableau											
A1	P	1	43,500	38,425	14,580	−5,000	−5,000	0	0	120,000	
B1	y_6	0	4	5	1	−1	0	1	0	13	$\frac{13}{4}$
C1	y_7	0	5	3	2	0	−1	0	1	11	$\frac{11}{5}$
Second tableau											
A2	P	1	0	12,325	−2,820	−5,000	3,700	0	−8,700	24,300	
B2	y_6	0	0	$\frac{13}{5}$	$-\frac{3}{5}$	−1	$\frac{4}{5}$	1	$-\frac{4}{5}$	$\frac{21}{5}$	$\frac{21}{13}$
C2	y_1	0	1	$\frac{3}{5}$	$\frac{2}{5}$	0	$-\frac{1}{5}$	0	$\frac{1}{5}$	$\frac{11}{5}$	$\frac{11}{3}$
Third tableau											
A3	P	1	0	0	24.23	−259.62	−92.31	−4,740.38	−4,907.69	4,390.38	
B3	y_2	0	0	1	$-\frac{3}{13}$	$-\frac{5}{13}$	$\frac{4}{13}$	$\frac{5}{13}$	$-\frac{4}{13}$	$\frac{21}{13}$	
C3	y_1	0	1	0	$\frac{7}{13}$	$\frac{3}{13}$	$-\frac{5}{13}$	$-\frac{3}{13}$	$\frac{5}{13}$	$\frac{16}{13}$	$\frac{16}{7}$
Fourth tableau											
A4	P	1	−45	0	0	−270	−75	−4,730	−4,925	4,335	
B4	y_2	0	$\frac{3}{7}$	1	0	$-\frac{2}{7}$	$\frac{1}{7}$	$\frac{2}{7}$	$-\frac{1}{7}$	$\frac{15}{7}$	
C4	y_3	0	$\frac{13}{7}$	0	1	$\frac{3}{7}$	$-\frac{3}{7}$	$-\frac{3}{7}$	$\frac{5}{7}$	$\frac{16}{7}$	

Duality of LPP

- With every linear programming problem, there is associated another linear programming problem which is called the *dual* of the original (or the *primal*) problem.

Formulating the Dual problem

- Consider again the production mix problem of N. Dustrious Company.
- Suppose that the company is considering leasing out the entire production facility to another company, and it must decide on the minimum daily rental price that will be acceptable.
- This decision problem can also be formulated as a linear programming problem.

The primal Dual Relationship

Primal Problem

Maximize

$$Z = c_1 x_1 + c_2 x_2$$

subject to:

$$k_{11}x_1 + k_{12}x_2 \leq b_1$$

$$k_{21}x_1 + k_{22}x_2 \leq b_2$$

$$k_{31}x_1 + k_{32}x_2 \leq b_3$$

$$\text{all } x_i \geq 0$$

Dual Problem

Minimize

$$P = b_1 y_1 + b_2 y_2 + b_3 y_3$$

subject to:

$$k_{11}y_1 + k_{21}y_2 + k_{31}y_3 \geq c_1$$

$$k_{12}y_2 + k_{22}y_2 + k_{32}y_3 \geq c_2$$

$$\text{all } y_i \geq 0$$

Example on Dual LPP

- The primal problem can now take the following standard form:

Maximize

$$Z = 12x_1 + 4x_2$$

subject to:

$$\begin{aligned} 4x_1 + 7x_2 &\leq 56 \\ -2x_1 - 5x_2 &\leq -20 \\ 5x_1 + 4x_2 &\leq 40 \\ -5x_1 - 4x_2 &\leq -40 \\ x_1 &\geq 0 \\ x_2 &\geq 0 \end{aligned}$$

- The dual of this problem can now be obtained as follows:

Minimize

$$P = 56y_1 - 20y_2 + 40y_3 - 40y_4$$

subject to:

$$\begin{aligned} 4y_1 - 2y_2 + 5y_3 - 5y_4 &\geq 12 \\ 7y_1 - 5y_2 + 4y_3 - 4y_4 &\geq 4 \\ \text{all } y_i &\geq 0 \end{aligned}$$

Thank you