INTEGRATION OF FUNCTIONS

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CONTENTS:

1.0 LEARNING OUTCOMES OF THE CHAPTER

After completion of the present chapter, you should be able to;

- Describe integration by using area under curve
- *Evaluate an indefinite integral using an anti-derivative*
- *Describe an indefinite integral and its application*
- Evaluate definite integrals and relationship between differentiation and integration
- Find the area between two curves by using definite integration.
- *Understanding economic application of integration*

1.1 AREA UNDER CURVES

- **Introduction**

There are two limiting processes of Calculus. First one is differentiation in which we study about the tangent to the curve or rate of change in one variable due to change in other variables. On the other hand, second one is integration, in which we study about the area under curve integration can be defined as:

"Integration is the process of finding the function from it's derivative and this function is called the integral of the function".

Basically, we use integration to find out area under a curve. We can also find the area under curve by geometrically. However, concept of integration and differentiation do not depend on geometry as analytically. A geometrical interpretation is used only to understand intuitively.

Let $y = f(x)$ be a continuous and positive function on the closed interval [a, b] in the figure (1). We have to find the area of given function on the closed interval [a, b]. Now the question is how do we compute area (A) under the given graph.

Further, suppose A(x) is the area that measures the area under curve $y = f(x)$ on the closed interval [a, x]

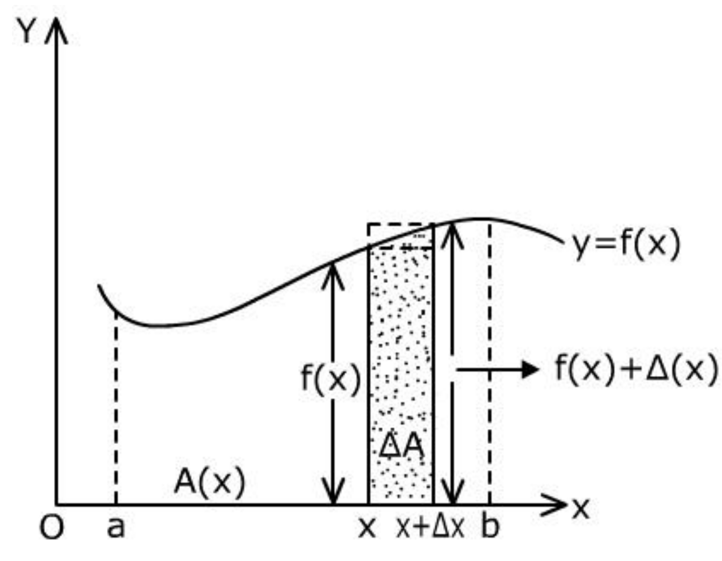


Figure 1

It is clear from the given

figure (1) that;

$$A(a) = 0$$

Because, there is no area from 'a' to 'a' and the total area can be defined as,

$$A = A(b)$$

Now, we suppose that 'x' increases by Δx amount. Then, $A(x+\Delta x)$ is the area under curve $y = f(x)$ over the closed interval $[a, x+\Delta x]$, Hence, the required area is given by;

$$A(x+\Delta x) - A(x)$$

It is the area {ΔA} under the curve $y = f(x)$ over the closed interval $[x, x+\Delta x]$. Let, Δ A be very small i.e. magnified and this area can not be exceed the area of rectangle with edges Δx and $f(x+\Delta x)$ and cannot be lesser than area of the rectangle with edges Δx and f(x). Hence, $\forall x > 0$, then;

$$f(x)\Delta x \leq A(x+\Delta x) - A(x) \leq f(x+\Delta x)\Delta x$$

$$OR,\ f(x) \leq \frac{A(x+\Delta x) - A(x)}{\Delta x} \leq f(x+\Delta x)$$

If we take $\Delta x \to 0$ in the above equation then the interval $[x, x+\Delta x]$ shrinks to the single point 'x' and the value $f(x+\Delta x)$ approaches $f(x)$. So, the function A (x) is differentiable and it measures the area under the curve $y = f(x)$ over the closed interval $[a, x]$. Then, the derivative of the function is given by;

$$A'(x) = f(x) \quad \{\forall x \in (a,b)\}$$

This proves that the derivatives of the area function A (x) is a curve height function {i.e. y = f (x)}

Now, suppose $F(x)$ is another continuous function with the function $y = f(x)$ as its derivative;

Then, $F'(x) = A'(x) = f(x) \quad \forall x \in (a,b)$

Because, $\frac{d}{dx}\left[A(x) - F(x)\right] = A'(x) - F'(x) = 0$

It must also be true that,

$A(x) = F(x) + C$ {C is some constant}

If A(a) = 0, then

A (a) = F(a) +C = 0

Or C = -F(a), put this value in above equation

$$\boxed{\begin{array}{l} A(x) = F(x) + C = F(x) - F(a)\{when, F'(x) = f(x)\} \\ At, x = b, then, A(x) = F(b) - F(a) \end{array}}$$

In short, the method for finding the area under the curve $y = f(x)$ and its domain (a,b) or above the x –axis from $x = a$ to $x = b$ has following steps;

- Find an arbitrary function F(x),that is continuous over the interval (a, b) such that

$$F'(x) = f(x) \qquad \forall x \in (a,b) \text{ ------------------- (i)}$$

- Then the required area of the function is given by

$$A(x) = F(b) - F(a) \qquad \text{-------------------------------(ii)}$$

What happens, if the function $y = f(x)$ has negative value in [a, b]. At this condition, the required area is $A(x) = -[f(b) - F(a)]$. Further, we know that, the area of a region is always positive. So $A(x)$ is also positive.

Example 1:

Find the area under the straight line $y = f(x) = x$ over the interval [0,1]

Solution:

We have to find the shaded area (A) in the given figure. According to above equation (i) and (ii) given above, we must find a function, that has x as its derivative.

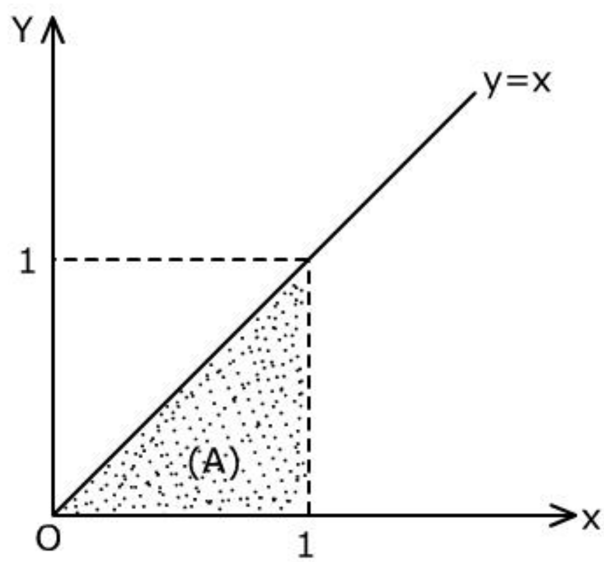


Figure 2

Then,

$$F(x) = \frac{x^2}{2}$$

$$\left\{ \because \frac{d}{dx}(ax^n) = anx^{n-1} = x, here, n = 2 \,\&\, a = \frac{1}{2} \right\}$$

$$F'(x) = 2\frac{x}{2} = x$$

Thus, the required area is given by;

$$A = F(1) - F(0)$$

$= \frac{1}{2} - 0 = \frac{1}{2}$, This answer is reasonable.

Example 2:

Compute the area under the parabola; $y = f(x) = x^2$ over the interval [a, b]

Solution:

We have estimated the shaded area A in the given figure (3). According to equation (i) and (ii) given above, we have to find a function, that has x as its derivative.

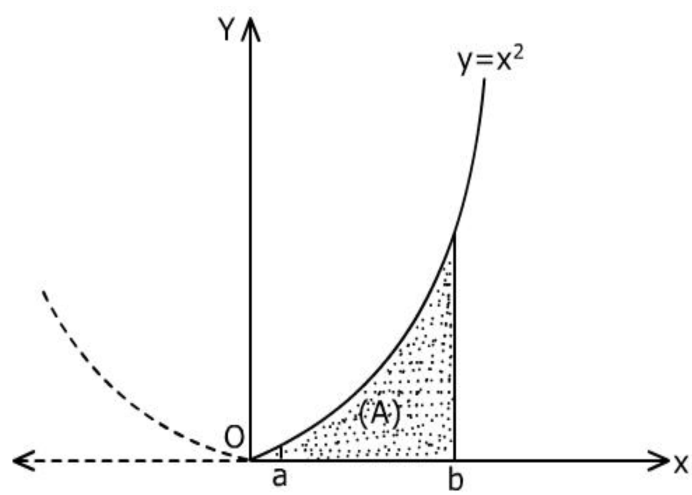


Figure 3

Let,

$$F(x) = \frac{1}{3}x^3$$

$$Then, F'(x) = f(x) = \frac{1}{3} \times 3x^2 = x^2$$

Thus, the required area is given by

$$A = F(b) - F(a) = \frac{1}{3}b^3 - \frac{1}{3}a^3$$

$$A = \frac{1}{3}\left[b^3 - a^3\right]$$

Example 3:

Compute the area 'A' under the straight line $y = f(x) = ax + b$ over the interval $[\alpha, \beta]$

Solution:

Let, the shaded area under the straight line be given by 'A', then from equations (i) and (ii) given above, we get;

$$F(x) = \frac{1}{2}ax^2 + bx\left\{\because \frac{d}{dx}(ax^n) = anx^{n-1} = ax + b\right\}$$

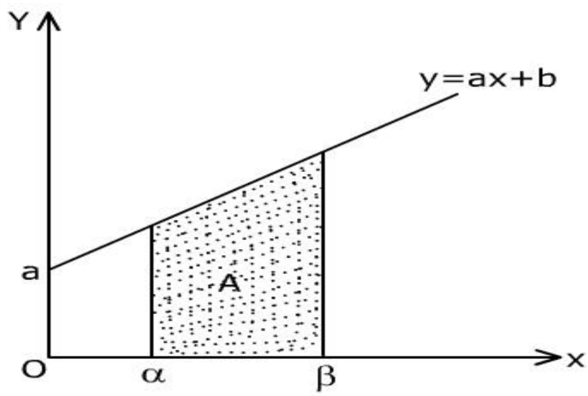


Figure 4

Then, $F'(x) = \frac{1}{2}.2ax + b.1$

$= ax + b$

So, the required area A is given by

$$A = F(b) - F(a)$$
$$A = F(\beta) - F(\alpha)$$

$$= \frac{1}{2}a\beta^2 + b\beta - \frac{1}{2}a\alpha^2 - b\alpha$$

$$= \frac{1}{2}a(\beta^2 - \alpha^2) + b(\beta - \alpha)$$

$$= (\beta - \alpha)\left\{\frac{a(\beta + \alpha) + 2b}{2}\right\}$$

Example 4: Find the shaded area 'A' of the function $y = f(x) = e^{x/3} - 3$ over the closed interval [0, 3 ln 3]

Solution: First we have to find the function $F(x)$, whose derivative is $e^{x/3} - 3$

By using the results of equation (i) and (ii) given above, we take the function,

$$F(x) = 3e^{x/3} - 3x \qquad \left\{ \because \frac{d}{dx}(e^x) = e^x \right\}$$

Then, $F'(x) = f(x) = e^{x/3} - 3$

So, the required area A is given by,

A = - [F(b) – F(a)]

= $-(3e^{\ln 3} - 3 \times 3 \ln 3 - 3e^{\circ}$

= - (9-9 ln 3-3) = 9 ln 3-6 (ignore –ve sign)

$\therefore$ A = 3.89 units

Problem Set

1. Find the area under the graph of polynomial $y = f(x) = x^3$ over the interval [0,1]

2. Find the bounded area of the graph of function $y = f(x) = \frac{1}{2}(e^x + e^{-x})$ over the interval (-1,1)

3. Find the area under straight line, $y = f(x) = cx + d$ over the interval [o,1]

4. Compute the area under the parabola $y = 4x^2$ over the interval [o,1]

Answer of the Problem Set

1. Area (A) = $\frac{1}{4}$

2. Area (A) = $\left(e - \frac{1}{e}\right)$

3. Area (A)= $\frac{1}{2}(a+b)$

4. Area (A)= $\frac{4}{3}$

1.2 INDEFINITE INTEGRALS

✓ **Introduction**

The previous section of the present chapter discusses the problem of finding an anti-derivative of the function $f(x)$ i.e. a function $F(x)$ whose derivative is $f(x)$.

$$F'(x) = f(x)$$

Anti-derivative is an appropriate name. Usually in practice, we call $F(x)$ an indefinite integral of $f(x)$. It is denoted by the symbol $\int$.

"If $f(x)$ is the differential coefficient of function $F(x)$,then $F(x)$ is the integral of $f(x)$"

By symbolically, if $$\frac{d}{dx}[F(x)] = f(x)$$

Then, $$\boxed{\int f(x)dx = F(x) + C}$$

Here 'C' is the constant term. We know that differentiation of constant term is zero. If integral constant 'C' can take any value then the integral is called indefinite integral.

✓ **Basic Rule of Integration**

Power Rule: It is defined as;

$$\boxed{\int x^n dx = \frac{1}{n+1} x^{n+1} + C} \qquad \{ n \neq -1 \}$$

Example: $$\int x\, dx = \frac{x^2}{2} + C$$

Exponential Rule: It is defined as;

$$\boxed{\int e^x dx = e^x + C}$$

And, $$\boxed{\int a^x dx = \frac{a^x}{\log_e a} + C} \qquad \{a > 0 \,\&\, a \neq 1\}$$

Examples:

$$\int e^{-x}dx = e^{-x} + C$$

$$\int e^{ax}dx = \frac{1}{a}e^{ax} + C \qquad \{a \neq 0\}$$

$$\int 2^{x}dx = \frac{2^{x}}{log_{e} 2} + C$$

Logarithmic Rule: It is defined as;

$$\boxed{\int \frac{1}{x} = \ln|x| + C}$$

Example:

$$\int \frac{1}{t}dt = \ln|t| + C$$

✓ **Some standard Results of Integration**

- **Constant multiple property**

$$\int af(x)dx = a\int f(x)dx \qquad \{ a \text{ is the real constant}\}$$

- **Integral of sum**

$$\int [f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx$$

In general,

$$\boxed{\int [a_1 f_1(x) + a_2 f_2(x) + ... + a_n fn(x)]dx = a_1\int f_1(x)dx + a_2\int f_2(x)dx + ... + a_n\int fn(x)dx}$$

- **Integral of Difference**

$$\int [F(x) - g(x)]dx = \int f(x)dx - \int g(x)dx$$

In general,

$$\boxed{\int [a_1 f(x) - a_2 f(x) - ...... - a_n fn(x)]dx = a_1\int f_1(x)dx - a_2\int f_2(x)dx - .... - a_n\int fn(x)dx}$$

- **Integral of Multiplication**

$$\boxed{\int f(x)g(x)dx = f(x)\int g(x)dx - \int [\frac{d}{dx} f(x) \int \mathrm{g(x)dx}]dx}$$

This property is also known as integration by part.

✓ **Some other results**

$$\boxed{\int \frac{1}{x^2+a^2}dx = log\left[x+\sqrt{x^2+a^2}\right]+C}$$

$$\boxed{\int \frac{1}{x^2-a^2}dx = log\left[x+\sqrt{x^2-a^2}\right]+C}$$

$$\boxed{\int \frac{1}{a^2-x^2}dx = \frac{1}{2a}\log\left[\frac{(a+x)}{(a-x)}\right]+C}$$

Example1:

Find the integral $\int (5x^4+3x^2+2x-1)dx$

Solution: $\int (5x^4+3x^2+2x-1)dx$

$= \int 5x^4dx + \int 3x^2dx + \int 2xdx - \int dx$

$= 5\int x^4dx + 3\int x^2dx + 2\int xdx - \int dx$

$= 5\frac{x^5}{5} + C_1 + \frac{3.x^3}{3} + C_2 + 2\frac{x^2}{2} + C_3 - x + C_4$

$= x^5 + x^3 + x^2 - x + C_1 + C_2 + C_3 + C_4$

$= x^5 + x^3 + x^2 - x + C$ $\qquad \{C = C_1 + C_2 + C_3 + C_4\}$

Example 2: Evaluate $\int (e^x + \frac{1}{x^3} + 1)dx$

Solution: $\int (e^x + \frac{1}{x^3} + 1)dx$

$$= \int e^x dx + \int x^{-3} dx + \int 1 dx$$

$$= e^x - \frac{1}{2}x^{-2} + x + c$$

$$= e^x - \frac{1}{2x^2} + x + c$$

Example 3: Find the integral $\int \frac{(x+1)^2 + 2x^{-1/2}}{\sqrt{x}} dx$

Solution: $\int \left[\frac{(x+1)^2 + 2x^{-1/2}}{\sqrt{x}} \right] dx$

$$= \int \left[\frac{x^2 + 2x + 1 + 2x^{-1/2}}{x^{1/2}} \right] dx$$

$$= \int (x^{3/2} + 2x^{1/2} + x^{-1/2} + 2\frac{1}{x}) dx$$

$$= \frac{2}{5}x^{5/2} + \frac{4}{3}x^{3/2} + 2x^{1/2} + 2\ln x + c$$

Example 4: Compute $\int \frac{x^2}{x+1} dx$

Solution: Let, $\int \frac{x^2}{x+1} dx$

$$= \int \left\{ \frac{x^2 - 1 + 1}{x+1} \right\} dx$$

$$= \int \frac{(x-1)(x+1)+1}{(x+1)} dx$$

$$= \int (x-1)dx + \int \frac{1}{(x+1)} dx$$

$$= \frac{x^2}{2} - x + \log|x+1| + c$$

Example5: Evaluate $\int \frac{dx}{\sqrt{x+c} - \sqrt{x-d}}$

Solution: Let, $\int \frac{dx}{\sqrt{x+c} - \sqrt{x-d}}$

$$= \frac{\sqrt{x+c} + \sqrt{x-d}\ )}{(\sqrt{x+c} - \sqrt{x-d}\ )(\sqrt{x+c+\sqrt{x-d}}} dx$$

$$= \int \frac{\sqrt{x+c} + \sqrt{x-d}\ )}{(x+c)-(x-d)} dx$$

$$= \frac{1}{(c+d)} \int (x+c)^{1/2} dx + \frac{1}{(c+d)} \int (x-d)^{1/2} dx$$

$$= \frac{1}{(c+d)} \quad \frac{2}{3}(x+c)^{3/2} + \frac{1}{c+d} \quad \frac{2}{3}(x-d)^{3/2} + c$$

$$= \frac{2}{3}\frac{1}{(c+d)} \quad \left[(x+c)^{3/2} + (x-d)^{3/2}\right] + c$$

Example 6: Find the integration $\int (6x+9)^8 dx$

Solution: By using substitution method,

Let $y = 6x+9$

Then, $dy = 6dx \quad or \quad dx = \frac{1}{6} dy$

So, we get, $\int (6x+9)^8 dx = \frac{1}{6}\int y^8 dy$

$$= \frac{1}{6} \quad \frac{y^9}{9} + c$$

Now putting the value; $y = 6x+9$, then

$$\int (6x+9)^8 dx = \frac{1}{54}(6x+9)^9 + C$$

Example 7: Evaluate $\int \frac{-x^2}{4-x^2} dx$

Solution: Let, $\int \frac{-x^2}{x^2-4} dx \quad = \quad \int \frac{-x^2+4-4}{4-x^2} dx$

$$= \int 1dx - 4\int \frac{1}{4-x^2} dx$$

$$= x - 4.\frac{1}{2\times 2}\log\left[\frac{2+x}{2-x}\right] + c \quad \text{(by the formulae)}$$

$$= x - log\left[\frac{2+x}{2-x}\right] + c$$

Example 8: Evaluate $\int x^2 e^{2x}$

Solution: Let, $I = \int x^2 e^{2x} dx$

By using the formulae for integration by part,

$$I = x^2 \int e^{2x} dx - \int \left\{ \frac{d}{dx} x^2 \int e^{2x} dx \right\} dx$$

$$= x^2 \frac{e^{2x}}{2} - \int \frac{2x.e^{2x}}{2} dx$$

$$= \frac{1}{2} x^2 e^{2x} - \int x e^{2x} dx$$

$$= \frac{1}{2} x^2 e^{2x} - \left[x \int e^{2x} dx - \int \left\{ \frac{d}{dx}.x \int e^{2x} dx \right\} \right] dx$$

$$= \frac{1}{2} x^2 e^{2x} - \left[\frac{xe^{2x}}{2} - \int 1.\frac{e^{2x}}{2} dx \right]$$

$$= \frac{1}{2}x^2e^{2x} - \left[\frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x}\right] + c$$

$$= \frac{1}{2}x^2e^{2x} - \frac{1}{2}xe^{2x} + \frac{1}{4}e^{2x} + c$$

$$= \frac{1}{2}e^{2x}\left[x^2 - x + \frac{1}{2}\right] + c$$

Example 9: Find $\int \frac{1}{4x^2-9}dx$

Solution: $\int \frac{1}{4x^2-9}dx = \int \frac{1}{(2x+3)(2x-3)}dx$

Now, let; $\frac{1}{(2x+3)(2x-3)} = \frac{A}{(2x-3)} + \frac{B}{(2x+3)}$...............$(i)$

$$= \frac{2Ax + 3A + 2Bx - 3B}{4x^2 - 9}$$

$$= \frac{2x(A+B) + 3(A-B)}{4x^2-9}$$

Now compare both sides of the equation; $2x(A+B)+3(a-B)=1$

Hence $2(A+B)=0$ or A=-B and 3(A-B)=1 or B=-1/6 and A=1/6, now by equation (i)

$$\therefore \int \frac{1}{4x^{2-9}}dx = \frac{1}{6}\int \frac{dx}{2x-3} - \frac{1}{6}\int \frac{dx}{2x+3}$$

$$= \frac{1}{12}\ln|2x-3| - \frac{1}{12}\ln|2x+3| + C$$

Example 10: Calculate Q(L), where Q'(L) = $6L^{1/3}$ and Q(0) = 0

Solution: Q(L) = $\int 6L^{1/3}dL = \frac{18}{4}L^{4/3} + c$

Given L=0, then Q(0)= 0+C or C=0

Then; Q(L) = 18/4L$^{4/3}$

Problem Set

1. Find the integrals of the following:

(i) $\int(4x^3 - 9x^2 + 2x + 2)dx$ (ii) $\int \frac{A}{r^{5/2}} dr$

(iii) $\int(3t^2 + 2t - e^t)dt$ (iv) $\int x\sqrt{x} \quad dx$

2. Prove that, $\int(ax+b)^\alpha dx = \frac{1}{a(\alpha+1)}(ax+b)^{\alpha+1} + c$

3. Find the integration (i) $\int \frac{1}{\sqrt{x}+2} dx$ (ii) $\int \frac{x}{2x^2+3} dx$

4. Calculate (i) $\int x\sqrt{x^2+1} \quad dx, x > 0$ (ii) $\int x\sqrt[3]{x-2} \quad dx$

5. If the marginal cost of producing x units for a manufacture product is MC=C′=2x+4 then find total cost function C(x). Given, fixed cost = 40

6. Evaluate $\int \frac{1}{2}(e^x + e^{-x})dx$

7. Given, $f''(t) = 1/t^2 + t^3 + 2 \;\; \forall t > 0 \text{ and } f(1) = 0, f'(1) = 1/4$ then find f(t).

8. Prove that,

$$\int t\sqrt{at+b}.dt = \frac{2}{15a^2}(3at-2b)(at+b)^{3/2} + c$$

9. Find the integration (i) $\int \log x \, dx$ (ii) $\int x^5 e^x dx$

10. Find the general form of the function f(x), whose third derivative is x and also given

$$f''(0) = f'(0) = f(0) = 0$$

11. Evaluate, (i) $\int \frac{1}{x^2 - a^2} dx$ (ii) $\int \frac{2x+1}{(x+1)(x-2)(x-3)}$

Answers of Problem Set

1. (i) $x^4 - 3x^3 + x^2 + 2x + c$

(ii) $-\dfrac{2A}{3r^{3/2}} + c$

(iii) $t^3 + t^2 - e^t + c$

(iv) $\dfrac{2}{5}x^{5/2} + c$

3. (i) $2[\sqrt{x} - 2\ln(\sqrt{x} + 2] + C$

(ii) $\dfrac{1}{4}\ln(2x^2 + 3] + C$

4. (i) $\dfrac{1}{3}\ln(x^2 + 1)^{3/2} + C$

(iv) $\dfrac{3}{14}(x-2)^{4/3}(2x+3) + c$

5. C$(x) = x^2 + 4x + 40$

6. $\dfrac{1}{2}(e^x - e^{-x})$

7. $t + \dfrac{1}{20}t^5 - \log|t|$

9. (i) $x\log x - x + c$

(ii) $x^5e^x - 5x^4 5e^x + 20x^3e^x - 60x^2e^x + 120xe^x + 120e^x + c$

10. $\dfrac{1}{24}x^4$

11. (i) $\dfrac{1}{2a}\log\left|\dfrac{x-a}{x+a}\right| + c$

(ii) $-\dfrac{1}{12(x+a)} - \dfrac{3}{5(x-2)} + \dfrac{7}{4(x-3)}$

1.3 THE DEFINITE INTEGRAL

✓ **Introduction**

Let F(x) be a continuous function over the interval [a, b] and it has a derivative $f(x)$ i.e. $F'(x) = f(x) \forall x \in (a,b)$. Then the difference, F(b)-F(a), is called the definite integral of function $f(x)$ over the interval [a, b]. In the first section of the present chapter, this difference, F(b)-F(a), does not depend on indefinite integrals. On the other hand, definite integral of $f(x)$ depends only on the function $f(x)$ and its interval [a, b]. Definite integral can be written as;

$$\boxed{\int_a^b f(x)dx = F(x)\Big|_a^b = \Big|F(x)\Big|_a^b = F(b) - F(a)}$$

where, $F'(x) = f(x) \forall x \in (a,b)$ and the number 'b' and 'a' are the upper and lower limits respectively.

✓ **Steps of Evaluating Definite Integral**

Let $I = \int_a^b f(x)dx$

- first, find the indefinite integral, $\int f(x)dx = F(x) + c$
- Substitute, x = b upper limit in this integral, i.e. F(b) +C
- Substitute, x = a lower limit in this integral i.e. F(a)+C
- Subtract, second {F(b)+c} from third {F(a)+C}

$$\boxed{\therefore \int_a^b f(x)dx = F(x)\Big|_a^b = \Big|F(x)\Big|_a^b = F(b) - F(a)}$$

Example 1: Find, $\int_a^b x \ dx$

Solution: Let $I = \int_a^b x \ dx$

$$I = \left|\frac{x^2}{2} + c\right|_a^b$$

$$= \left[\frac{b^2}{2} + c\right] - \left[\frac{a^2}{2} + c\right]$$

$$= \left[\frac{b^2}{2} - \frac{a^2}{2}\right] = \frac{1}{2}(b^2 - a^2)$$

- **Some Basic Properties of Definite Integral**

- $\int_a^b F(x)dx = -\int_b^a f(x)dx$
- $\int_a^b f(x)dx = \int_b^{c_1} f(x)dx + \int_{c_1}^{c_2} f(x)dx + \int_{c_2}^{b} f(x)dx \quad \{c_1, c_2 \in [a,b]\}$
- $\int_a^a f(x)dx = 0 \qquad \{F(a) - F(a) = 0\}$
- $\int_a^b f(x)dx = \int_a^b f(y)dy = \int_a^b f(z)dz$
- $\int_0^a f(x)dx = \int_0^a f(x-a)dx$
- $\int_{-a}^a f(x)dx = 2\int_0^a f(x)dx$
- $\frac{d}{dt}\int_{b(t)}^{a(t)} f(x)dx = f'(t) = f\{b(t)\}.b'(t) - f\{a(t)\}.a'(t)$
    - Every continuous function is integrable, if this function has an anti-derivative i.e.

$$\boxed{F'(x) = f(x),\ \forall x \in (a,b)}$$

Example 2: Find $\int_0^1 (2x + \frac{1}{x})dx$

Solution: Let $I = \int_1^2 (2x + \frac{1}{x})dx$

$$= \left| \frac{2x^2}{2} + \log x \right|_1^2$$

$$= \left| x^2 + \log x \right|_1^2$$

= [4 + log2]-[1+0]

= 3 + log 2

Example 3: Find the area of the parabola $x^2 = 4$ by between $x - axis$ and its ordinate at $x = 3$

Solution: The required area = $\int_0^3 ydx$

$$= \int_0^3 \frac{x^2}{4b}dx \qquad \left\{\because y = \frac{x^2}{4b}\right\}$$

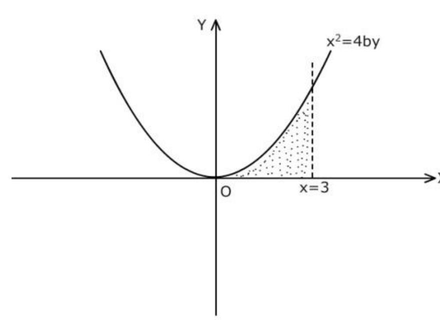


Figure 5

$$= \frac{1}{4b}\left[\frac{x^3}{3}\right]_0^3$$

$$= \frac{1}{4b}\left[\frac{27}{3} - 0\right] = \frac{9}{4b}$$

Example 4: find $\int_1^4 |x-2|dx$

Solution: Let

$$|x-2| = \begin{cases} x-2 & \text{If } x \geq 2 \\ -(x-2) & \text{If } x < 2 \end{cases}$$

Then $\int_1^4 |\text{x-2}|\, dx = \int_1^2 -(x-2)dx + \int_2^4 (x-2)dx$ {By property of Integration)

$$= \left[\frac{-x^2}{2} + 2x\right]_1^2 + \left[\frac{x^2}{2} - 2x\right]_2^4$$

$$= \left[\left(-\frac{4}{2} + 4\right) - \left(-\frac{1}{2} + 2\right)\right] + \left[\left(\frac{16}{2} - 8\right) - \left(\frac{4}{2} - 4\right)\right]$$

$$= 2 - \frac{3}{2} + 0 + 2. = \frac{5}{2}$$

Example 5: Find the area between the regions of parabola $y = x^2$ and straight line $y = |x|$ over the interval [-1,1]or $\{(x,y)x^2 \leq y \leq |x|\}$

Solution: Given $y = x^2$ and $y = |x|$ i.e. $y = x$ or $y = -x$

The required Area

= Area OAB + Area OCD

= 2 *Area OAB

(Because, curve is symmetrical about the y axis)

$$= 2\left[\int_0^1 x dx - \int_0^1 x^2 dx\right]$$

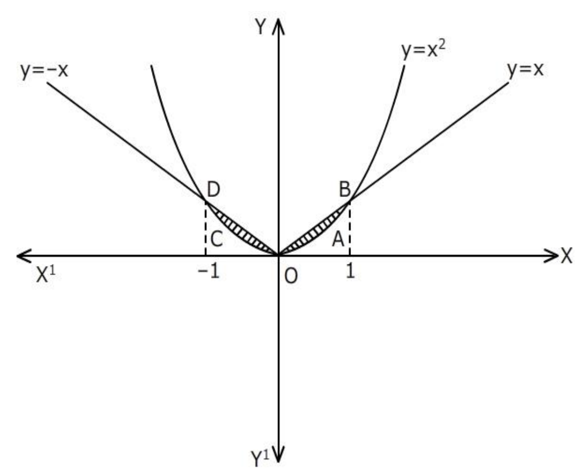


Figure 6

$$= 2\left[\left|\frac{x^2}{2}\right|_0^1 - \left|\frac{x^3}{3}\right|_0^1\right]$$

$$= 2\left[\left(\frac{1}{2}-0\right)-\left(\frac{1}{3}-0\right)\right]$$ = 2/3 square units

Example 6: Evaluate $\int_0^T \left(\frac{K}{T}\right) e^{-Qt} dt, where$ T> 0 and K and Q are positive constants.

Solution: Let W(T) = $\int_O^T \left(\frac{K}{T}\right) e^{-Qt} dt$

$$= \frac{K}{T}\int_O^T e^{-Qt} dt$$

$$= \frac{K}{T}\left[\frac{-e^{-Qt}}{Q}\right]_O^T$$

$$= \frac{K}{TQ}\left[(-e^{-QT})-(-e^{\circ})\right]$$

W(T) = $\frac{K}{TQ}\left[1-e^{-QT}\right]$

Example 7: Find the area included between the two parabola i.e. $y^2 = 4x$ and $x^2 = 4y$

Solution: Given, $y^2 = 4x \,\&\, x^2 = 4y$

Solving both, we get;

$$\left(\frac{x^2}{4}\right) = 4x$$

Or, $x(x^3 - 64) = 0$

So, $x = 0 \,\&\, 4$

The required area = Area OBCD

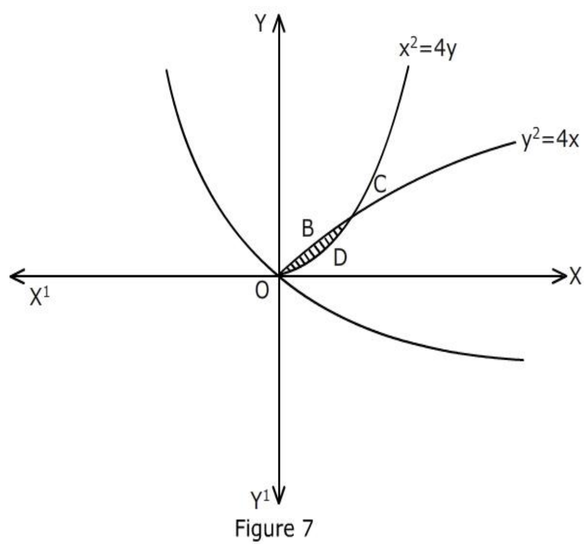


Figure 7

$$=\int_O^4\left(\sqrt{4x}-\frac{x^2}{4}\right)dx \qquad \left\{\because y^2=4x\ \&\ y=x^2/4\right\}$$

$$=\ 2\left[\frac{x^{3/2}}{3/2}-\frac{x^3}{12}\right]_0^4$$

= 5.3 square unit.

Example 8: Find $\dfrac{d}{dx}\displaystyle\int_x^{x^2} e^{-4^2}du$

Solution: By the direct property of integration, we get;

$$\frac{d}{dx}\int_{a(x)}^{b(x)} f(x)dx$$

$$=\ f\{(b(x)\}b'(x)-f\{a(x)\}a'(x)$$

Then, $$\frac{d}{dx}\int_x^{x^2} e^{4^2}du$$

$$=\ e^{-(x^2)^2}-2x-e^{-x^2}.1$$

$$=\ e^{-x^2}\left\{2xe^{-x^2}-1\right\}$$

Problem Set

1. Find the definite integral for the following:

 (i) $\int_{-1}^{1} e^x\ dx$ (ii) $\int_0^2\left(t^3-t^2\right)dt$ (iii) $\int_1^3\frac{3y}{10}dy$

2. Find, (i) $\dfrac{d}{dx}\displaystyle\int_0^x t^2dt$ (vi) $\dfrac{d}{du}\displaystyle\int_{-u}^{u} e^{-v^2}dv$ (iii) $\dfrac{d}{du}\displaystyle\int_{-u}^{u}\frac{1}{\sqrt{x^4+1}}dx$

3. Find the area of line $y=4x$ between $x-axis$ and the ordinate $x=4$

4. Find the area intercepted between the line $3x+2y=-12$ and the parable $y=\dfrac{3}{4}x^2$

5. Find the area between the parabolas; $y^2 = 4ax$ and $x^2 = 4ay, a > 0$

6. Prove that

$$\int_0^{2a} f(x)dx = 2\int_0^a f(x)dx, \text{ If } f(2a-x) = f(x)$$

$$= 0 \qquad \text{If } f(2a-x) = -f(x)$$

7. Evaluate

(i) $\int_0^1 (t + \sqrt{t} + \sqrt[4]{t})dt$ (ii) $I = \frac{1}{2000}\int_{1000}^{3000} f(t)dt$

Given $F(t) = 4000 - t - \frac{3000000}{t}$

8. Prove that $F(t^*) = \frac{1}{b-a}\int_a^b f(t)dt$

If $f(t)$ is continuous function over the interval [a,b] and $t^* \in (a,b)$
$\left\{Hint: Put\ F(t) = \int_a^t f(x)dx\right\}$

Answers of Problem Set

1. (i) $\frac{e^2-1}{e}$ (ii) $\frac{4}{3}$ (iii) $\frac{39}{10}$

2. (i) x^2 (ii) $2e^{-u^2}$ (iii) $\frac{1}{2\sqrt{u^4+1}}$

3. 32 sq. units

4. 27 sq. units

5. $\frac{16}{3}a^3$ sq. units

7. (i) $\frac{13}{12}$ (ii) $I \approx 352$

1.4 ECONOMIC APPLICATION OF INTEGRATION

✓ **Introduction**

Integration has an important role in economics. The present section shows the role of integration in economics by illustrating some important examples.

Important Results of Integration in Economics

- If $f(r)$ is the function of individuals income over the interval [a, b], then the no. of individuals with incomes in [a, b]

$$= n\int_a^b f(r)dr$$

- Total income of individuals = $n\int_a^b rf(r)dr$ $\{r \Rightarrow earning\}$

- The mean income of the individuals is given by

$$m = \frac{\int_a^b r\ f(r)dr}{\int_a^b f(r)dr}$$

Example 1: If the income distribution of population over interval [a, b] is given by, $f(r) = Ar^{-5/2}$ {A is a positive constant}, then determine mean income in the given group.

Solution: Let $\int_a^b f(r)dr = \int_a^b Ar^{-5/2}dr = A\left[-\frac{2}{3}r^{-3/2}\right]_a^b = \frac{2}{3}A\left(\left|a^{-3/2} - b^{-3/2}\right.\right)$

And $\int_a^b rf(r)dr = \int_a^b Ar.r^{-5/2}dr$

$= A \int_a^b r^{-3/2}dr = 2A\left[a^{-1/2} - b^{-1/2}\right]$

So, the mean income of the group is given by

$$m = \frac{2A(a^{-1/2} - b^{-1/2})}{2/3A(a^{-3/2} - b^{-3/2})} = 3\frac{(a^{-1/2} - b^{-1/2})}{(a^{-3/2} - b^{-3/2})}$$

Now, suppose b is very large then $b^{-1/2}$ and $b^{-3/2}$ close to zero, then m≈3a

Then, the mean income of the group is 3a.

✓ **Economic Application of Integration**

There are several other economic applications of integration. Some results are given below;

- **Total cost (TC)**

$$TC = \int MC(Q)dQ$$

Here; MC $\Rightarrow$ Marginal cost, Q $\Rightarrow$ output

- **Total Revenue (TR)**

TR= $\int MR(Q)dQ$, Here, MR $\Rightarrow$ Marginal Revenue

- **Consumer surplus (CS) and producer surplus (PS)**: These can be also calculated by using definite integral. Consumer surplus is given by;

$$CS = \int_o^x f(x)dx - p \times x$$

Here, $f(x) \Rightarrow$ demand of x commodity, P$\Rightarrow$ Price of x commodity

And, producer surplus is given by,

$$PS = x \times p - \int_o^x f(x)dx$$

- **The present discounted value is given by**;

PDV = $\int_o^T f(t)e^{-rt}dt$

- **The future discounted value is given by**;

FDV = $\int_o^T f(t)e^{r(T-t)}dt$

- **The discounted value at time is given by**;

DV = $\int_{t=S}^T f(t)e^{-r(t-s)}dt$

Example 2: Find total cost function from the given marginal cost function;

$$MC = f'(q) = 2 + 3q^{1/2} + 5/q^{-1/2}$$, Given; f(1) = 11

Solution: $\therefore F(q) = \int f'(q)dq = \int (2 + 3q^{1/2} + 5q^{-1/2})dq = 2q + 3.\frac{q^{3/2}}{3/2} + 5.\frac{q^{1/2}}{1/2} + c$

$$\therefore TC = 2q + 2q^{3/2} + c$$

$\therefore$ When q=1 then 11 = 2 + 2 + c, Then, c=7

$\therefore$ Total cost function F(q) = 22 + 3$q^{3/2}$ + 10 $q^{1/2}$+ 7

Example 3: If the marginal revenue function is given; $P_m = \left\{ \frac{\alpha\beta}{(x+\beta)^2} - \gamma \right\}$,

Then, show that $P = \left\{ \frac{\alpha\beta}{(x+\beta)} - \gamma \right\}$ is the demand law

Solution: $\because R = P.x$ and $MR = \frac{dR}{dx}$

$$\therefore R = \int MR.dx = \int \left(\frac{\alpha\beta}{(x+\beta)^2} - \gamma \right) dx$$

$$\therefore R = \alpha\beta \frac{(x+\beta)^{-1}}{-1} - \gamma x + A$$

$$\therefore R = P.x = -\frac{\alpha\beta}{(x+\beta)} - \gamma x + A,$$

We know that if output x=0 then revenue is also zero. Then A = α

$$\therefore R = P \times x = -\frac{\alpha\beta}{(x+\beta)} - \gamma x + \alpha = \frac{\alpha x}{(x+\beta)} - \gamma x$$

OR, $P = \frac{\alpha}{x+\beta} - \gamma$, Hence proved.

Example 4: If marginal revenue (MR) = $16 - q^2$, find the maximum total revenue, also find the total, average revenue demand.

Solution: When TR is maximum, then MR= 0

$$\therefore 16 - q^2 = 0 \Rightarrow q = \pm 4$$

$$\therefore TR = \int_0^4 MR \text{ dq} = \int_0^4 (16 - q^2) dq = \left[16q - \frac{q^3}{3} \right]_0^4 = \frac{128}{3}$$

Total Revenue (TR) = $\int (16 - x^2) dx = 16x - \frac{x^3}{3} + c$ when $x = 0$ then $c = 0$

Average Revenue (AR) = $\frac{TR}{q} = 16 - \frac{q^2}{3}$

Then, Demand (AR) =P = $16 - q^2/3$

Example 5: If marginal propensity to consume (MPC) function is given as follows; $\dfrac{dc}{dy} = 0.5 - 0.001y$, then find total consumption function. Given at income zero, c is 0.02.

Solution: $\therefore C = \int \dfrac{dc}{dy}.dy = \int (0.5 - 0.001y = 0.5y - \dfrac{0.001}{2}y^2 + A$

At $\therefore y = 0,$ then, C = 0.2, Hence, A = 0.2

$$\therefore C = 0.5y - 0.0005y^2 + 0.2$$

Example 6: The sales of a product is depicted by a function S(t) = $100e^{-0.5t}$, where t is number of years since the launching of the product, find

a) The total sales in the first three years
b) The sales in forth year &
c) The total sales in the future

Solution: a) $S(3) = \int_0^3 100e^{0.5t}dt = 155.40$

b) $S(4) - S(3),$

$S_4 = \int_3^4 \ 100e^{0.5t}dt = 17.6$

e) $S(\infty) = \int_0^\infty 100\ e^{0.5t}dt = 200$

Example 7: If the demand function is; $P = 30 - 2x - x^2$ and the demand is 3, what will be the consumer surplus (CS)?

Solution: Given, $P = 30 - 2x - x^2$

For $x = 3,$ then p = 20

$\therefore$ CS = $\int_0^3 (30 - 2x - x^2)dx - P \times x$

= $\left[30x - \dfrac{2x^2}{2} - \dfrac{x^3}{3}\right]_0^3 - 3 \times 20$

= 90-9-9-60=12 units

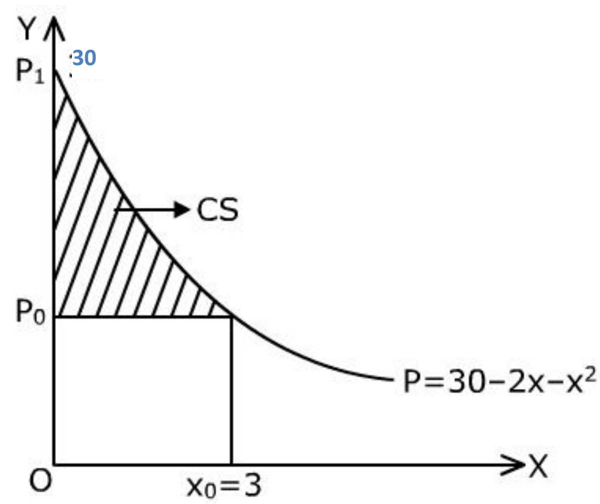


Figure 8

Example 8: The demand and supply laws are $P_d = (6-x)^2$ and $P_s=14+x$ respectively. Find the consumer surplus (CS), If;

(i) The demand and price are determined under perfect competition and;

(ii) The demand and price are determined under monopoly and the supply function is identified with marginal cost function.

Solution: (i) CS under perfect competition: at the equi8librium

$$(6-x)^2 = 14+x \Rightarrow x = 2$$

Then, P=14+x=16

$$\therefore CS = \int_0^2 (36-12x+x^2)dx - 16\times 2 = 56/3$$

(ii) CS under monopoly;

$$TR = P_d\, x = (36-12x+x^2)x = 36x-12x^2+x^3$$

$$\therefore MR = 36-24x+3x^2$$

And supply price: $P_s = 14+x$, supply function P_s=MC

To maximization of profit we know that,

MR=MC

$$36-24x+3x^2 = 14+x$$

i.e. $x = 1, or, 7.33$

At $x=1$, then, P_d=25

∵ Hence, $CS = \int_0^1 (36-12x+x^2)dx - 25x) = \frac{16}{3}$ unit

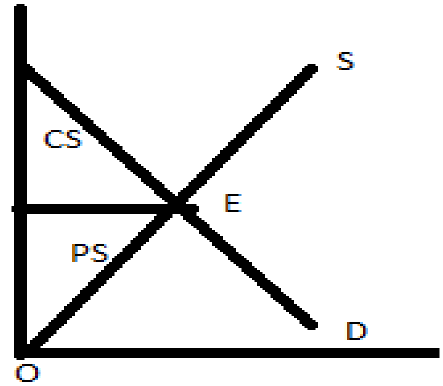
Similarly, we obtain CS at $x = 7.33$

Example 9: Obtain the producer surplus, when the demand and supply function is given;

$D = 20-4x$ and $S = 4+4x$

Solution: At equilibrium condition,

Demand(D) = Supply (S)



$$20-4x=4+4x$$

$$or, 8x = 16$$
$$then; x = 2$$
$$and, P = 4+8 = 12$$

And, P=4+8=12

Then, producer surplus (PS)

$$= P \times x - \int_0^2 (4+4x)dx = 24 - [4x+2x^2]_0^2$$
$$= 24 - 16 = 8 units$$

Problem Set

1. If the inverse demand function of commodity Q is given; $P = 3q^{-1/2}$ and presently 100 units are being sold, then find the consumer surplus.

Ans. 30

2. Let interest rate will vary and represent by r(t). What is the present value of a flow of income P(t) from t=a to t=b using this variable interest rate?

$$Ans. \int_a^b e^{-\int_a^t r(s)ds} P(t)dt$$

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