Dr.Deepak Gupta

BCA-II SEM

MATHEMATICS

UNIT-3

Partial Differentail Equations

Partial Differential equation.

Partial differential equation:

Let z be the dependent variable and x, y be two independent variables, then the differential equations involving the dependent variable and its Partial derivatives with respect to the independ variables, are called Partial differential equations.

$$P = \frac{\partial z}{\partial x}, \quad \frac{\partial z}{\partial y} = q, \quad \frac{\partial^2 z}{\partial x^2} = r, \quad \frac{\partial^2 z}{\partial y^2} = t$$

$$\frac{\partial^2 z}{\partial x \partial y} = s.$$

Formation of Partial differential equations by elimination of arbitrary Constants —:

Let $\quad f(x, y, z, a, b) = 0 \quad$ — (1)

where a and b are arbitrary constants

Differentiating (1) Partially with respect to 'x' and 'y', we

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z} \cdot P = 0 \quad — (2)$$

$$\frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} \cdot q = 0 \quad — (3)$$

Eliminating a and b from (1), (2) & (3), we get

$$\phi(x, y, z, P, q) = 0$$

<u>Ans</u>

Example 1. Eliminate the arbitrary Constants a and b from $z = ax + by$ and form a differential equation.

Sol. we have $$z = ax + by \quad \text{--- (i)}$$

Differentiating (i) Partially w.r.t. x and y respectively we get

$$\frac{\partial z}{\partial x} = a \quad \& \quad \frac{\partial z}{\partial y} = b$$

Putting the value of a & b in eq (i) we get

$$z = x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} \qquad \left[\begin{array}{l} p = \frac{\partial z}{\partial x} \\ q = \frac{\partial z}{\partial y} \end{array}\right.$$

$$\boxed{z = px + qy} \quad \text{Ans.}$$

Formation of Partial Differential equations by elimination of arbitrary functions —:

Example-2 Eliminate the arbitrary functions f and g from $z = f(x+iy) + g(x-iy)$ to obtain a PDE.

Sol. we have $$z = f(x+iy) + g(x-iy) \quad \text{--- (i)}$$

Differentiating (i) Partially w.r.t x and y respectively, we get

$$p = \frac{\partial z}{\partial x} = f'(x+iy) + g'(x-iy) \quad \text{--- (ii)}$$

and $$q = \frac{\partial z}{\partial y} = if'(x+iy) - ig'(x-iy) \quad \text{--- (iii)}$$

Differentiating again (ii) w.r.t. x and (iii) w.r.t y, we get

$$r = \frac{\partial^2 z}{\partial x^2} = f''(x+iy) + g''(x-iy) \quad \text{--- (iv)}$$

$$t = \frac{\partial^2 z}{\partial y^2} = i^2 f''(x+iy) - i^2 g''(x-iy) \quad \text{--- (v)} \qquad i^2 = -1$$

Adding (iv) & (v) we get $$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$$

Ans.

Partial differential equations of first order -:

A Partial differential equation which involve only the first order Partial derivatives p and q is called a Partial differential equation of first order.

$$f(x,y,z,p,q)=0$$

Lagrange's linear equation: A partial differential equation of the form $Pp+Qq=R$ where P, Q and R are function of x, y, z is called Lagrange's linear equation.

$$Pp+Qq=R$$

$$\boxed{\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}}$$

Example - Solve $2p+3q=1$

Sol. Comparing with $Pp+Qq=R$

$P=2 \quad Q=3 \quad R=1.$

Auxiliary equation is $\frac{dx}{2}=\frac{dy}{3}=\frac{dz}{1}$

Taking first two $\frac{dx}{2}=\frac{dy}{3}$, we get $3dx-2dy=0$

on integration

$3x-2y=c_1$

Simi taking last two

$\frac{dy}{3}=\frac{dz}{1} \Rightarrow dy=3dz$

$y-3z=c_2$

general sol $\phi(3x-2y,\ y-3z)=0$ Ans.

Example :- Find the general solution of

$$x(y^2-z^2)p + y(z^2-x^2)q = z(x^2-y^2)$$

Sol. Auxiliary equation is $\dfrac{dx}{x(y^2-z^2)} = \dfrac{dy}{y(z^2-x^2)} = \dfrac{dz}{z(x^2-y^2)}$

taking the Lagrange multipliers as x, y and z

$$\frac{x\,dx + y\,dy + z\,dz}{0} =$$

$$x\,dx + y\,dy + z\,dz = 0$$

On integration we get $x^2 + y^2 + z^2 = c_1$

Taking the Lagrange multipliers as $\frac{1}{x}, \frac{1}{y}$ and $\frac{1}{z}$ we get

$$\frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{0}$$

$$\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0$$

$$\log x + \log y + \log z = \log c_2$$

$$xyz = c_2$$

Hence $\phi(x^2+y^2+z^2,\ xyz) = 0$ ~~Ans~~ Ans.

Clairaut's form : $z = px + qy + f(p,q)$

If we eliminate the arbitrary constant a and b from the

$$\boxed{z = ax + by + f(a,b)}$$

$p \to a$

$q \to b$

Example :- write the Complete integral of $z = px + qy + pq$

Sol. we put $p = a$ & $q = b$.

$$\boxed{z = ax + by + ab}$$ Ans

type : $F(z, p, q) = 0$ —(1)

Let $z = f(x+ay)$ be a total sol. of (1)

Put $x + ay = u$ so that we have $z = f(u)$, $\frac{\partial u}{\partial x} = 1$

$$\frac{\partial u}{\partial y} = a$$

$$\therefore p = \frac{dz}{du} \cdot \frac{\partial u}{\partial x} = \frac{dz}{dx} \text{ \& } q = \frac{dz}{du}\frac{\partial u}{\partial y} = a\frac{dz}{du}$$

$$F\left(z, \frac{dz}{du}, a\frac{dz}{du}\right) = 0$$

Solving $\phi(z, a) = u + b = x + ay + b$ Ans

Example Find the Complete integral of $z = p^2 - q^2$

Sol. let $z = f(x+ay)$, $z = p^2 - q^2$ —(i)

$z = f(u)$ —(ii)

let $u = x + ay$

$$p = \frac{\partial z}{\partial x} = \frac{dz}{du}\frac{\partial u}{\partial x} = \frac{dz}{du}$$

$$q = \frac{\partial z}{\partial y} = \frac{dz}{du}\frac{\partial u}{\partial y} = a\frac{dz}{du}$$

Substituting the value of p and q in (i) we get

$$z = \left(\frac{dz}{du}\right)^2 - a^2\left(\frac{dz}{du}\right)^2 = \left(\frac{dz}{du}\right)^2(1-a^2) = z$$

$$\frac{dz}{du} = \sqrt{\frac{z}{1-a^2}}$$, separating the variables

$$\sqrt{1-a^2} \cdot \frac{dz}{z^{1/2}} = du$$

on integrating $2\sqrt{1-a^2} \cdot \sqrt{z} = u + b$

$$4(1-a^2)z = (x + ay + b)^2$$ Ans.

Charpit's method: Charpit's method is a general method for finding the complete solution of non-linear Partial differential equation of the first order of the form

$f(x, y, z, p, q) = 0$

$$\frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}} = \frac{dF}{0}$$

Example: Find a complete integral of the equation.

$$p^2x + q^2y = z \quad -(1)$$

Sol. The charpit's equation are

$$\frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}} = \frac{df}{0}$$

$$\frac{dx}{2px} = \frac{dy}{2qy} = \frac{dz}{2(p^2x+q^2y)} = \frac{dp}{p-p^2} = \frac{dq}{q-q^2}$$

$$\frac{p^2dx + 2px\,dp}{p^2x} = \frac{q^2dy + 2qy\,dq}{q^2y}$$

$$\Rightarrow p^2x = aq^2y \quad -(2)$$

Solving (1) & (2) we get $p = \left\{\frac{az}{(1+a)x}\right\}^{1/2}$, $q = \left\{\frac{z}{(1+a)y}\right\}^{1/2}$

$$dz = p\,dx + q\,dy$$

$$\left(\frac{1+a}{z}\right)^{1/2} dz = \left(\frac{a}{x}\right)^{1/2} dx + \left(\frac{1}{y}\right)^{1/2} dy$$

$$\{(1+a)z\}^{1/2} = (ax)^{1/2} + y^{1/2} + b \quad \text{Ans}$$

Linear Partial Differential equations with Constant Coefficients of Second order.

$$A_0 \frac{\partial^n z}{\partial x^n} + A_1 \frac{\partial^n z}{\partial x^{n-1}\partial y} + A_2 \frac{\partial^n z}{\partial x^{n-2}\partial^2 y} + \cdots + A_n \frac{\partial^n z}{\partial y^n} = F(x,y) \quad -(1)$$

$\frac{\partial z}{\partial x} = Dz$

$\frac{\partial z}{\partial y} = D'z$

Method of finding Complementary function-:

The equation obtain obtained by putting $D=m$ and $D'=1$ in $f(D,D')=0$ is called the auxiliary equation.

$$A_0 m^n + A_1 m^{n-1} + A_2 m^{n-2} + \cdots + A_n = 0 \quad -(2)$$

Case-1 The roots $m_1, m_2, \ldots m_n$ are distinct.

Then. CF $\phi_1(y+m_1x) + \phi_2(y+m_2x) + \cdots + \phi_n(y+m_nx)$

Case-II The roots of Auxiliary equation are repeated the

$m = m_1 = m_2 = m_3$

C.F $\phi_1(y+mx) + x\phi_2(y+mx) + x^2\phi_3(y+mx)$

Example-1 Solve $(D^2 - DD' - 2D'^2)Z = 0$

A.E $m^2 - m - 2 = 0$

$m = 1, -2$. [roots are distinct]

C.F $\phi_1(y-2x) + \phi_2(y+x)$

Since R.H.S is zero therefore $P.I = 0$

$Z = CF + PI$

$= \phi_1(y-2x) + \phi_2(y+x)$

Method of finding Particular integral (P.I)

Case .1. $\frac{1}{F(D,D')} e^{ax+by} = \frac{1}{f(a,b)} e^{ax+by}$, if $f(a,b) \neq 0$

Case -2 $\frac{1}{F(D,D')} x^r y^s = [F(D,D')]^{-1} x^r y^s$

Case-3. $\frac{1}{F(D^2, DD', D'^2)} \sin(ax+by) = \frac{1}{f(-a^2,-ab,-b^2)} \sin(ax+by)$ if $f(-a^2,-ab,-b^2) \neq 0$

Case-4. $\frac{1}{F(D^2, DD', D'^2)} \cos(ax+by) = \frac{1}{f(-a^2,-ab,-b^2)} \cos(ax+by)$ if $f(-a^2,-ab,-b^2) \neq 0$

Case-5. $\frac{1}{F(D,D')} e^{ax+by} \phi(x,y) = e^{ax+by} \frac{1}{f(D+a, D'+b)} \phi(x,y)$

Example : Solve $\frac{\partial^2 z}{\partial x^2} - 4\frac{\partial^2 z}{\partial x \partial y} + 4\frac{\partial^2 z}{\partial y^2} = e^{2x-y}$

Sol. $(D^2 - 4DD' + 4D'^2)z = e^{2x-y}$

AE $D \Rightarrow m$ & $D' \Rightarrow 1$ RHS $\Rightarrow 0$

$m^2 - 4m + 4 = 0$

$m = 2, 2$ (roots are repeated)

CF $\phi_1(y+2x) + x\phi_2(y+2x)$

P.I $\frac{1}{D^2 - 4DD' + 4D'^2} \cdot e^{2x-y}$

$= \frac{e^{2x-y}}{(2)^2 - 4(2)(-1) + 4(-1)^2} = \frac{e^{2x-y}}{16}$ $\begin{bmatrix} D \Rightarrow 2 \\ D' \Rightarrow -1 \end{bmatrix}$

$z = CF + P.I = \phi_1(y+2x) + x\phi_2(y+2x) + \frac{1}{16}e^{2x-y}$ Ans

Example : 3. $(D^2 - 2DD' + 2D'^2)z = \sin(x-y)$

Sol. AE $m^2 - 2m + 2 = 0$

$m = 1 \pm i$

CF $\phi_1\{y + (1+i)x\} + \phi_2\{y + (1-i)x\}$

P.I $\frac{1}{D^2 - 2DD' + 2D'^2} \sin(x-y)$

$= \frac{1}{-1 - 2 - 2} \sin(x-y) = -\frac{1}{5}\sin(x-y)$ $\begin{bmatrix} D^2 = -1 \\ DD' = -1 \times -1 \\ D'^2 = -b^2 = -1 \end{bmatrix}$

$z = CF + PI$

$= \phi_1\{y + (1+i)x\} + \phi_2\{y + (1-i)x\} - \frac{1}{5}\sin(x-y)$

Ans.

Example $\frac{\partial^2 z}{\partial x^2} + 3\frac{\partial^2 z}{\partial x \partial y} + 2\frac{\partial^2 z}{\partial y^2} = x+y$

$(D^2 + 3DD' + 2D'^2)z = x+y$

AE is $m^2 + 3m + 2 = 0$

$m = -2, -1$

CF $\phi_1(y-2x) + \phi_2(y-x)$

P.I $\frac{1}{D^2 + 3DD' + 2D'^2}(x+y)$

$= \frac{1}{D^2\left[1 + \left(\frac{3DD' + 2D'^2}{D^2}\right)\right]}(x+y)$

$= \frac{1}{D^2}\left[1 + \left(\frac{3DD' + 2D'^2}{D^2}\right)\right]^{-1}(x+y)$

$= \frac{1}{D^2}\left[1 - \left(3\frac{D'}{D} + 2\frac{D'^2}{D^2}\right) + \cdots\right](x+y)$

$= \frac{1}{D^2}\left[(x+y) - \frac{3}{D}(1)\right] = \frac{1}{D^2}(x+y-3x) = \frac{1}{D^2}(y-2x)$

$= \frac{yx^2}{2} - \frac{2x^3}{6}$

$z = CF + P.I = \phi_1(y-2x) + \phi_2(y-x) + \frac{x^2}{6}(3y-2x)$

Ans

ANSWERS

1. $\phi(3y - 4x, 3z - 2x) = 0$

2. $\phi\left(\dfrac{x}{y}, \dfrac{y}{z}\right) = 0$

3. $\phi(y - x, z + \cos x) = 0$

4. $\phi\left(\dfrac{\sin x}{\sin y}, \dfrac{\sin y}{\sin z}\right) = 0$

5. $\phi(xy, x - \log z) = 0$

6. $\phi\left(x + y, x - \dfrac{z}{\log (x + y)}\right) = 0$

7. $\phi\left(\dfrac{(x + y)^2}{y}, \dfrac{y}{z}\right) = 0$

8. $\phi\,(x - z, 2y - z^2) = 0$

9. $\phi\left(\dfrac{x}{y}, \dfrac{y}{\sqrt{z}}\right) = 0$

10. $\phi\,(x^2 - y^2, y^2 - z^2) = 0$

11. $\phi\left(\dfrac{a - x}{b - y}, \dfrac{b - y}{c - z}\right) = 0$

12. $\phi\,[2x + y, x^3 \sin (y - 2z) - z] = 0$

13. $\phi\left(x^2 - z^2, \dfrac{y}{x + y + z}\right) = 0$

14. $\phi\,(x^2 + y^2 + z^2, lx + my + nz) = 0$

15. $\phi\,(xyz, x + y + z) = 0$

16. $\phi\left(\dfrac{x}{y}, xy - z^2\right) = 0$

17. $\phi\left(\dfrac{1}{x} + \dfrac{1}{y}, \dfrac{z}{x + y}\right) = 0$

18. $\phi\,(x^2 + y^2 + z^2, x + 2y - z) = 0$

19. $\phi\,(x^2 + y^2 + z^2, 2x + 3y + 4z) = 0$

20. $\phi\,(xy, x^2 + y^2 + z^2) = 0.$

$\therefore$ The complete integral is given by $z = \text{C.F.} + \text{P.I.}_1 + \text{P.I.}_2 + \text{P.I.}_3$

$$z = \phi_1(y + x) + \phi_2(y + 2x) + \frac{e^{2x-y}}{12} - xe^{x+y} - \frac{1}{3}\cos(x + 2y)$$

Example 27. *Solve* $\dfrac{\partial^2 z}{\partial x^2} - 2\dfrac{\partial^2 z}{\partial x \partial y} + \dfrac{\partial^2 z}{\partial y^2} = \sin x$ **[UPTU 2004]**

Solution : The given equation can be written as

$$(D^2 - 2DD' + D'^2)\,z = \sin x$$

The auxiliary equation is $m^2 - 2m + 1 = 0$, *i.e.*, $(m-1)^2 = 0 \Rightarrow m = 1, 1$

$\therefore$
$$\text{C.F.} = \phi_1(y + x) + x\,\phi_2(y + x)$$

and
$$\text{P.I.} = \frac{1}{(D^2 - 2DD^2 + D'^2)}\sin x = -\sin x$$

Hence the solution is
$$z = \phi_1(y + x) + x\,\phi_2(y + x) - \sin x$$

PROBLEMS (A)

Solve the following partial differential equations :

1. $(D^2 - D'^2)\,z = 0$
2. $(4D^2 + 12DD' + 9D'^2)\,z = 0$
3. $r - a^2 t = 0$
4. $(D^3 - 6D^2D' + 11DD'^2 - 6D'^3)\,z = 0$
5. $(D^3 - 4D^2D' + 4DD'^2)\,z = 0.$
6. $25\dfrac{\partial^2 z}{\partial x^2} - 40\dfrac{\partial^2 z}{\partial x\,\partial y} + 16\dfrac{\partial^2 z}{\partial y^2} = 0$
7. $r + t + 2s = 0$
8. $(D^3 + DD'^2 - D^2D' - D'^3)\,z = 0$
9. $(D^2 - 2DD' + D'^2)\,z = e^{x+2y}$
10. $(D^2 - 2DD' + 2D'^2)\,z = 2e^{2x+4y}.$
11. $\dfrac{\partial^2 z}{\partial x^2} - 5\dfrac{\partial^2 z}{\partial x\,\partial y} + 6\dfrac{\partial^2 z}{\partial y^2} = e^{x+y}$
12. $(D^3 - 2D^2D')\,z = e^{x+2y}$
13. $(D^2 - 7DD'^2 - 6D'^3)\,z = e^{3x+y}$
14. $(2D^2 - 2DD' + D'^2)\,z = e^{x+y} + 2e^{3y}$
15. $(D^3 - 4D^2D' + 5DD'^2 - 2D'^3)\,z = e^{y+x} + e^{y-2x}$
16. $(D^2 + 3DD' - 4D'^2)\,z = \sin y$
17. $(D^2 + 3DD' + 2D'^2)\,z = \sin(x + 5y)$
18. $r + s - 6t = \cos(2x + y)$
19. $(D^2 - DD')\,z = \cos x \cos 2y$
20. $(D^2 - 4D'^2)\,z = \sin(2x + y)$
21. $(D^3 + D^2D' - DD'^2 - D'^3)\,z = 3\sin(x + y)$
22. $\dfrac{\partial^2 z}{\partial x^2} - a^2\dfrac{\partial^2 z}{\partial y^2} = E\sin cy$ (E, c constants)
23. $r + 2s + t = 2(y - x) + \sin(x - y)$
24. $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial x\,\partial y} - 6\dfrac{\partial^2 z}{\partial y^2} = \cos(2x + y)$
25. $(D^2 + DD' - 6D'^2)\,z = \cos(2x + 3y)$
26. $(D^2 - DD' - 6D'^2)\,z = xy$
27. $(D^2 - DD' - 2D'^2)\,z = 2x + 3y + e^{3x+4y}$
28. $(D^2 - 2DD' + D'^2)\,z = \cos(x - 3y) + e^{x-3y}$
29. $(D^2 - 4D'^2)\,z = (x + y) + e^{x+y} - 4\cos(x - y)$
30. $(D^3 + D^2D' - DD'^2 - D'^3)\,z = e^x\cos 2y$
31. $(D^2 + 2DD' + D'^2)\,z = x\sin y$
32. $(D^2 - 6DD' + 5D'^2)\,z = e^x\sinh y + xy$