Mittag-Leffler's Theorem

Let $\{z_j\}$ be a sequence of distinct complex numbers such that $\lim \frac{1}{|z_j|} = 0$ and for each j, let P_j be the rational function given by

$$P_j(z) = \frac{a^j_{-m_j}}{(z-z_j)^{m_j}} + \frac{a^j_{-m_j+1}}{(z-z_j)^{m_j+1}} + \cdots + \frac{a^j_{-1}}{(z-z_j)}$$

Then there exists polynomials h_j $(j=1,2,\ldots)$ such that

$$f(z) = \sum_{j=1}^{\infty} \{P_j(z) - h_j(z)\}$$

defines a meromorphic function with singularities precisely at $z_1, z_2, \ldots$ and principal parts $P_1, P_2, \ldots$ respectively.

<u>Proof</u>: Without loss of generality, we may assume that $z_j \neq 0$ for each j. The only pole of the rational function P_j is at $z_j \neq 0$. so it is holomorphic on $B(0;|z_j|)$. Let $g_j(z)$ be its Taylor's expansion on $B(0,|z_j|)$,

i.e. $g_j(z) = c_0^j + c_1^j z + c_2^j z^2 + \cdots\cdots$ —— (1)

Let n_j be such that

$$\left|\sum_{n=n_j+1}^{\infty} c_n^j z^n\right| < \frac{1}{2^j} \text{ for all } z \in cl\, B(0,|z_j|/2) \quad \text{—— (2)}$$

and let

$$h_j(z) = c_0^j + c_1^j z + c_2^j z^2 + \cdots + c_{n_j}^j z^{n_j}. \quad \text{—— (3)}$$

so from (1), (2) and (3), we have

$$|g_j(z) - h_j(z)| = \left|\sum_{n=n_j+1}^{\infty} c_n^j z^n\right| < \frac{1}{2^j}$$

Since $g_j(z)$ is the Taylor's expansion of $P_j(z)$, Therefore $|P_j(z) - h_j(z)| < \frac{1}{2^j}$ for all $j > m$

$\therefore \sum_{j=m+1}^{\infty} \{P_j(z) - h_j(z)\}$ converges uniformly on $B(0,R)$, hence it is holomorphic on $B(0,R)$.

But $\sum_{j=1}^{m} \{P_j(z) - h_j(z)\}$ is meromorphic on $B(0,R)$, being a finite sum and has poles at $z_1, z_2 \dots, z_m$ with Principal parts $P_1, P_2, \dots, P_m$.

Hence the infinite series $\sum_{j=1}^{\infty} \{P_j(z) - h_j(z)\}$ has only poles at $z_1, z_2 \dots, z_m$ in $B(0,R)$ with principal parts $P_1, P_2, \dots P_m$ respectively

$\therefore$ If $f_m = \sum_{j=1}^{m} \{P_j(z) - h_j(z)\}$,

$\Rightarrow$ sequence $\{f_m\}$ of meromorphic function converges uniformly on compact sets to f, where $f = \sum_{j=1}^{\infty} \{P_j(z) - h_j(z)\}$

Infinite Series Expression for cotangent function:

(An application of the Mittag-Leffler Theorem)

Consider a meromorphic function with simple poles at each integer j, with principal part

$$P_j(z) = \frac{1}{z-j}, \text{ at } j.$$

The series $\sum_{j=-\infty}^{\infty} P_j(z) = \sum_{j=-\infty}^{\infty} \frac{1}{z-j}$ does not converge uniformly for non-integer z with $|z| \le R$ $(R>0)$

$\therefore$ for any $j \neq 0$, $\left| \frac{1}{z-j} + \frac{1}{j} \right|$

$$= \left| \frac{z}{j(z-j)} \right|$$

$$= \frac{1}{j^2} \frac{|z|}{|(z/j)-1|} \le \frac{|z|}{j^2} \text{ for large } j$$

Hence for non-integer z with $|z| \le R$,

$\sum_{j \neq 0} \left\{ \frac{1}{z-j} + \frac{1}{j} \right\}$ converges uniformly for each $R>0$.

Hence, in the Mittag-Leffler's Theorem, let $h_j(z) = \frac{1}{j}$ for each non-zero integer j

$$\frac{1}{z} + \sum_{j \neq 0} \left\{ \frac{1}{z-j} + \frac{1}{j} \right\} = \frac{1}{z} + \sum_{j=1}^{\infty} \left\{ \frac{1}{z-j} + \frac{1}{j} + \frac{1}{z+j} - \frac{1}{j} \right\}$$

$$= \frac{1}{z} + \sum_{j=1}^{\infty} \frac{2z}{z^2 - j^2}. \quad \text{——— (1)}$$

which is meromorphic function with simple poles at integers and the principal part $\frac{1}{z-j}$ at each integer j.

~~Since~~ Now $\frac{d}{dz}\left(\frac{1}{z-j}+\frac{1}{j}\right) = -\frac{1}{(z-j)^2}$

Since term by term differentiation is valid.

$$\frac{d}{dz}\left\{\frac{1}{z}+\sum_{j\neq 0}\left(\frac{1}{z-j}+\frac{1}{j}\right)\right\}$$

$$= -\frac{1}{z^2}-\sum_{j\neq 0}\frac{1}{(z-j)^2} \qquad \text{——— (2)}$$

$$= -\sum_{j=-\infty}^{\infty}\frac{1}{(z-j)^2} = -\frac{\pi^2}{\sin^2 \pi z}$$ (Proved in factorization of $\sin \pi z$)

Since $\frac{d}{dz}(\pi \cot \pi z) = \frac{-\pi^2}{\sin^2 \pi z}$

$$\therefore \frac{d}{dz}(\pi \cot \pi z) = -\sum_{j=-\infty}^{\infty}\frac{1}{(z-j)^2}$$

or $\frac{d}{dz}(\pi \cot \pi z) = \frac{d}{dz}\left(\frac{1}{z}+\sum_{j\neq 0}\left(\frac{1}{z-j}+\frac{1}{j}\right)\right)$

(from above)

or $\frac{d}{dz}(\pi \cot \pi z) = \frac{d}{dz}\left(\frac{1}{z}+\sum_{j=1}^{\infty}\frac{2z}{z^2-j^2}\right)$, {from (1)}

Integrating w.r.t. 'z', we have

$$\pi \cot \pi z = c + \frac{1}{z} + \sum_{j=1}^{\infty}\frac{2z}{z^2-j^2} \qquad \text{——— (3)}$$

where c is constant,

changing z to $-z$, we ~~get~~ have

$$\pi \cot \pi(-z) = c + \frac{1}{-z} + \sum_{j=1}^{\infty}\frac{2(-z)}{z^2-j^2} \qquad \text{— (4)}$$

from (3) and (4), we get

$$c = 0$$

Hence $$\boxed{\pi \cot \pi z = \frac{1}{z} + \sum_{j=1}^{\infty} \frac{2z}{z^2 - j^2}}$$

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