P-1

## Unit-III

### Lie Group:

Let $G \to$ grp, which is also a diff. Manifold.
Let $\phi_1, \phi_2$ be maps given by
$\phi_1 : G \times G \to G$ & $\phi_2 : G \to G$
defined by
$\phi_1(g_1, g_2) = g_1 g_2$ & $\phi_2(g) = g^{-1}$
$\forall g, g_1, g_2 \in G$.

If $\phi_1, \phi_2$ are both are differentiable, Then G is ~~ya~~ a Lie Group.

### Examples

**Question:**

(1) ~~Show that~~ The set $R^+ = \{x \in R \mid x > 0\}$ is a Lie grp with respect to mult^n.

Pf. $G \equiv R^+$
Let $x, y \in R^+$
Let $\phi_1 : R^+ \times R^+ \to R^+$ defined by $\phi_1(x, y) = xy$ $\forall x, y \in R^+$

& $\phi_2 : R^+ \to R^+$, defined by $\phi_2(x) = \frac{1}{x}$, $\forall x \in R^+$

Obviously $\phi_1, \phi_2$ are diff'ble, $\Rightarrow R^+$ is Lie grp.

(2) R is a Lie grp wrt addition.

Let $G \equiv R$
Let $x, y \in R$
Define $\phi_1 : R \times R \to R$
s.t. $\phi_1(x, y) = x + y$
& $\phi_2 : R \to R$,
s.t. $\phi_2(x) = -x$
$\forall x, y \in R$

Obviously $\phi_1, \phi_2$ are diff $\Rightarrow$ R is Lie grp.

P-2.

$GL(n,\mathbb{R}) \equiv \mathbb{R}^{n^2}$ is also a lie grp wrt matx multi.
|||
Set of all non Singular Matx.

$\rightarrow \phi_1 : GL(n,\mathbb{R}) \times GL(n,\mathbb{R}) \longrightarrow GL(n,\mathbb{R})$

$\left.\begin{array}{l} \phi_1(A,B) = A \cdot B \\ \& \ \phi_2 : GL(n,\mathbb{R}) \rightarrow GL(n,\mathbb{R}) \\ \phi_2(A) = A^{-1} \end{array}\right\}$ $\forall A, B, \in GL(n,\mathbb{R})$ where $A = [a_{ij}]$, $B = [b_{ij}]$

As $\phi_1, \phi_2$ are diff ble $\Rightarrow GL(n,\mathbb{R})$ is lie grp.

## Left Translation & Right translation:

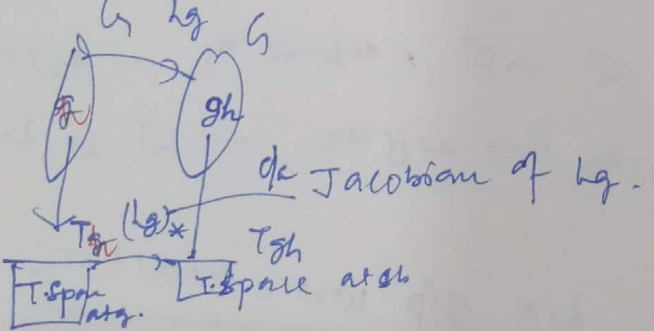
Let $G \rightarrow$ lie grp. & $g \in G$.

Then The mapping $L_g : G \rightarrow G$ defined as $L_g(h) = gh$, $\forall h \in G$, is ca

Left translation.

Similarly, The right translation $R_g : G \rightarrow G$ is defined $R_g(h) = hg$, $\forall h \in G$.

Note: ①




~~$(L_g)_* : T_g \rightarrow T_{gh}$ is define by~~

If $L_g \circ R_{g^{-1}} = I_g$, find $I_g(h) = ?$

$\because I_g(h) = (L_g \circ R_{g^{-1}})(h)$
$= L_g(R_{g^{-1}}(h))$
$= L_g(hg^{-1}) = ghg^{-1}$

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# Lie Algebra

Lie Algebra over $\mathbb{R}$ is a Real vector Sp $V$, equipped with bilinear map $[\cdot,\cdot] : V \times V \to V$ satisfying —

1. $[x,x] = 0$
2. $[x,y] = -[y,x]$
3. $[[x,y],z] + [[y,z],x] + [[z,x],y] = 0$

Thus, The Space of vector fields on $M$, equipped with ~~Lie bracket~~ $[\cdot,\cdot]$ as a bin op$^n$, is called a Lie Algebra..

i.e $(T_m M, [\cdot,\cdot]) \to$ Lie Algebra.

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**Homomorphism & Isomorphism:**

Let $G_1, G_2 \to$ Lie Grp.

If diff map $\phi : G_1 \to G_2$ satisfying $\phi(gh) = \phi(g)\cdot\phi(h)$, $\forall g,h \in G_1$

Then $\phi$ is called Homomorphism.

Further If $\phi$ is a bijective Homo, Then $\phi$ is called Isomorphism from $G_1$ into $G_2$.

Note $\phi[x,y] = [\phi x, \phi y]$, (if $\phi$ is Homo)

**Def$^n$:** Left Invariant vector field: A vector field $X$, on a Lie grp $G$, is called Left Invariant field if

$$(L_g)_* X = X,$$

07.2010
**Quest:** The Set of all Left Invariant vector field over a Lie grp form a Lie algebra over $G$.

**Pf:** Let $V \equiv$ Set of all vector fields over Lie grp $G$.
$\Rightarrow V(\mathbb{R}) \to$ vector sp over field of real Nos.

Let $\mathcal{G} \equiv$ Set of all left Inv. vector fields over $G$.

Then $\mathcal{G} \subset V$

Let $X, Y \in \mathcal{G}$, $a, b \in R$

$\Rightarrow X, Y$ are ~~left~~ Inv. field over $G$.

$\Rightarrow (L_g)_* X = X$ & $(L_g)_* Y = Y$

Consider $(L_g)_*(\alpha X+\beta Y) = (L_{g*})(\alpha X) + (L_g)_*(\beta Y)$

$= \alpha (L_{g*}) X + \beta (L_{g*}) Y$

$= \alpha X + \beta Y$, $\forall \alpha, \beta \in R$

$\Rightarrow \alpha X+\beta Y$ is ~~left~~ Inv

$\Rightarrow \alpha X+\beta Y \in \mathcal{G}$.

$\Rightarrow \mathcal{G}$ is Sub sp of $V$.

$= \mathcal{G}$ is vector sp. over $R$.

TPT: $\mathcal{G}$ is Lie algebra over $G$.

For,

Let $X, Y \in \mathcal{G}$ –

$\Rightarrow (L_g)_* X = X$, $(L_g)_* Y = Y$

Now $(L_{g*})[X, Y] = [(L_g)_* X, (L_g)_* Y]$

$= [X, Y]$

$\Rightarrow [X, Y] \in \mathcal{G}$.

We know Lie bracket satisfy.

Properties ① $[X, X] = 0$

② $[X, Y] = -[Y, X]$

& ③ $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$

Therefore $\mathcal{G}$ is Lie Algebra.

## Basis of Lie Algebra

Let $G \to$ Lie grp
& $\mathfrak{G} \to$ Lie algebra of left Inv vector fields over $G$.

Let $B = \{x_1, x_2, \ldots x_n\}$ be a basis of Lie algebra $\mathfrak{G}$.

Let $x_i, x_j \in \mathfrak{G} \Rightarrow [x_i, x_j] \in \mathfrak{G}$.

Thus $[x_i, x_j] = c_{ij}^1 x_1 + c_{ij}^2 x_2 + c_{ij}^3 x_3 + \cdots c_{ij}^n x_n$
$$= \sum c_{ij}^r x_r$$
$$= c_{ij}^r x_r.$$

① & $[x_i, x_i] = 0. \Rightarrow c_{ii}^r x_r = 0$
$\Rightarrow \boxed{c_{ii}^r = 0}$ $\because x_r \neq 0$
$\forall i$

② $[x_i, x_j] = -[x_j, x_i]$
$\Rightarrow c_{ij}^r x_r = -c_{ji}^r x_r \Rightarrow c_{ij}^r = -c_{ji}^r$, $\because x_r \neq 0$

③ For basis vector $x_i, x_j, x_k$, we have

$$[[x_i, x_j], x_k] + [[x_j, x_k], x_i] + [[x_k, x_i], x_j] = 0$$
$$\Rightarrow [c_{ij}^r x_r, x_k] + [c_{jk}^r x_r, x_i] + [c_{ki}^r x_r, x_j] = 0$$
$$\Rightarrow c_{ij}^r [x_r, x_k] + c_{jk}^r [x_r, x_i] + c_{ki}^r [x_r, x_j] = 0$$
$$\Rightarrow c_{ij}^r c_{rk}^s x_s + c_{jk}^r c_{ri}^s x_s + c_{ki}^r c_{rj}^s x_s = 0$$
$$\Rightarrow \boxed{c_{ij}^r c_{rk}^s + c_{jk}^r c_{ri}^s + c_{ki}^r c_{rj}^s = 0},$$ $\because x_s \neq 0$ & $\{x_1, x_2, \ldots x_n\}$ is L.I.

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**Th:** If the Lie grp $G$ is of the dimension $n$, Then the Lie algebra over $G$, $\mathcal{G}$ is also of dim $n$.

**Proof:** Let $G \rightarrow$ Lie grp & $\dim(G) = n$. (Given)

& $\mathcal{G} \rightarrow$ Lie algebra over $G$.

Let $e \in G \Rightarrow T_e \equiv$ Tang. Sp. at pt $e$

i.e $\dim(T_e) = n$.

**TPT:** $\dim(\mathcal{G}) = n$ ?

For,

Define a linear map.

$$i_L : \mathcal{G} \rightarrow T_e \text{ s.t}$$

$$\boxed{i_L(x) = x_e}, \ \forall x \in \mathcal{G}$$

**Claim:** $i_L$ is Bijective Homomorph i.e $\mathcal{G} \cong T_e$.

For,

① **$i_L$ is 1-1:**

If $i_L(x) = i_L(y)$

$= x_e = y_e$


$\Rightarrow (L_g)_* x_e = (L_g)_* y_e$    Where $g \in G$, $L_g \to$ left translation, $(L_g)_* \to$ Jacobian

$\Rightarrow x_{ge} = y_{ge}$

$\Rightarrow x_g = y_g$    [e & g → be pts]

$= \boxed{x = y}$    $\forall g \in G$.

$\Rightarrow i_L$ is 1-1

② **$i_L$ is Onto:**

Let $x_e \in T_e$ Then $\exists\, x \in \mathcal{G}$ s.t $i_L(x) = x_e$.

Hence $i_L$ is Onto.

③ **$i_L$ is linear trasfn / Homo**

Let $\alpha, \beta \in \mathbb{R}$, $x, y \in \mathcal{G}$

$i_L(\alpha x + \beta y) = (\alpha x + \beta y)_e = \alpha x_e + \beta y_e = \alpha\, i_L(x) + \beta\, i_L(y)$

$\Rightarrow i_L$ is lin. trasf.

$\boxed{\dim G = \dim \mathcal{G}}$

Theorem

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## Lie transformation grp:

Let $G \to$ Lie grp.

$M \to C^\infty$ n-Manifold.

Then $G$ is a Lie transformation grp of $M$ if there is a diff$^{ble}$ Map

$$\psi : M \times G \to M \text{ defined as}$$

$$\psi(x,g) = xg \quad \forall x \in M, g \in G.$$

$$\text{s.t. } (xg)h = x(gh), \; h \in G.$$

$$\& \quad xe = x.$$

In this Case, we say that $G$ acts on $M$ to the right or $G$ acts differentiably to $M$ from Right.

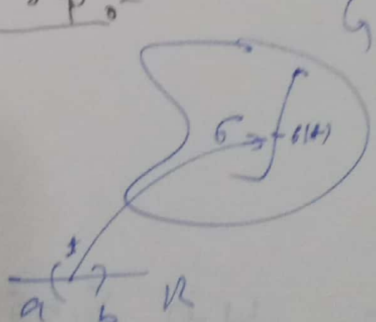
Similarily action of $G$ to $M$ from left can be defined by $\phi : G \times M \to M$. given by

$$\phi(g,x) = gx$$

$$\text{s.t. } h(gx) = (hg)x$$

$$\& \quad ex = x.$$

## One parameter Subgrp of Lie grp:-




Let $G \to$ Lie grp.

Let $\sigma$ be a $C^\infty$ curve in $G$.

i.e $\sigma : (a,b) \to G$, is a diff$^{ble}$ Mapping where $(a,b) \subset \mathbb{R}$

Consider $H = \{\sigma(t) \mid t \in \mathbb{R}\} \Rightarrow$ Set of pts on the curve $\sigma$ in $G$.

obviously $H \subset G$.

claim: $H$ is Subgrp of $G$.

For, Define '$\circ$' in $H$ as

$$\sigma(s) \circ \sigma(t) = \sigma(s+t) \quad \text{as } s,t \in \mathbb{R}$$

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So H is closed under opr 'o'.

Existence of Identity:

since $0 \in R$

$\sigma(0) \in H$

let $\sigma(t) \in H$

$\sigma(0) \circ \sigma(t) = \sigma(0+t) = \sigma(t)$

& $\sigma(t) \circ \sigma(0) = \sigma(t+0) = \sigma(t)$

$\Rightarrow$ $\sigma(0)$ is identity elt of H.

Existence of Inverse:

let $t \in R$

$\Rightarrow -t \in R$

i.e $\sigma(t) \in H$ & $\sigma(-t) \in H$

& $\sigma(t) \circ \sigma(-t) = \sigma(t-t) = \sigma(0)$

& $\sigma(-t) \circ \sigma(t) = \sigma(-t+t) = \sigma(0)$

Thus $\sigma(-t)$ is Inv elt of $\sigma(t)$ in H.

Associativity:

$\{\sigma(s) \circ \sigma(t)\} \circ \sigma(r) = \{\sigma(s+t)\} \circ \sigma(r)$

$= \sigma\{(s+t)+r\}$ $\because r, s, t \in R$ Assc

$= \sigma\{s+(t+r)\}$

$= \sigma(s) \circ \sigma(t+r)$

$= \sigma(s) \circ \{\sigma(t) \circ \sigma(r)\}$

Therefore H is Subgrp of G. wrt 'o'.

H is c/a one parameter subgrp of the grp G.

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Some problems:

Ex ① In a die grp $G$, $\forall g \in G$, P.T. the map $I_g : G \to G$ defined as

$$I_g(h) = ghg^{-1}$$ is an automorphism of $G$.

Pf: Claim: $I_g : G \to G$ is an Isomorphism

For:

① $I_g$ is 1-1:

let $h_1, h_2 \in G$.

If ~~let~~ $I_g(h_1) = I_g(h_2)$

$$\Rightarrow g h_1 g^{-1} = g h_2 g^{-1}$$

$$\Rightarrow g h_1 = g h_2$$

$$\Rightarrow \boxed{h_1 = h_2}$$

② $I_g$ is Onto:

let $h \in G$ (CoDomain)

let $I_g(x) = h$, where $x \in G$ Domain

$$g x g^{-1} = h$$

$$\boxed{x = g^{-1} h g.}$$

i.e $I_g(g^{-1} h g) = h$.

So $I_g$ is onto.

③ $I_g$ Preserves the composition:

let $h_1, h_2 \in G$.

$$I_g(h_1 h_2) = g(h_1 h_2) g^{-1}$$

$$= g h_1 (g^{-1} g) h_2 g^{-1}$$

$$= (g h_1 g^{-1})(g h_2 g^{-1})$$

$$= I_g(h_1) \cdot I_g(h_2)$$

Proved:

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Ex ② P.T.

① $L_g \circ L_h = L_{gh}$ ② $L_g \cdot R_h = R_h L_g$ ③ $R_g R_h = R_{hg}$

Pf: ① Let $t \in G$

$$(L_g \circ L_h)(t) = L_g(L_h(t))$$
$$= L_g(ht)$$
$$= ght$$
$$= L_{gh}(t) \quad , \forall t$$

$$\Rightarrow \boxed{L_g \circ L_h = L_{gh}}$$

② $L_g R_h(t) = L_g(th) = gth$
$$= R_h(gt)$$
$$= R_h \cdot L_g(t) \quad \forall t$$

$$\Rightarrow L_g R_h = R_h L_g$$

③ $R_g R_h(t) = R_g(th) = (th)g = t(hg)$ (By Ass)
$$= R_{hg}(t) \quad , \forall t.$$

$$\Rightarrow R_g R_h = R_{hg}.$$

Ex ③: PT: ① $C_x \circ C_y = -C_y \circ C_x$.

② If $\alpha \in F^p(A)$, $\beta \in F^q(A)$, Then $(\alpha \wedge \beta) = (-1)^{pq}(\beta \wedge \alpha)$

Pf ① LHS: let $x_1, x_2, \ldots x_{p-1} \in T^{1,0}(A)$, $\omega \in F^p(A)$

$$\{(C_x \circ C_y)\omega\}(x_1, x_2, \ldots x_{p-1}) = C_x(C_y\omega)(x_1, x_2, \ldots x_{p-1})$$
$$= C_x\{\omega(y, x_1, x_2, \ldots x_{p-1})\}$$
$$= \omega(x, y, x_1, x_2, \ldots x_{p-1})$$
$$= -\omega(y, x, x_1, x_2, \ldots x_{p-1})$$
$$= -(C_y \circ C_x\omega)(x_1, x_2, \ldots x_{p-1})$$

or $\boxed{C_x \circ C_y = -C_y \circ C_x}$

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# Unit - IV

## Principal Fibre Bundle:

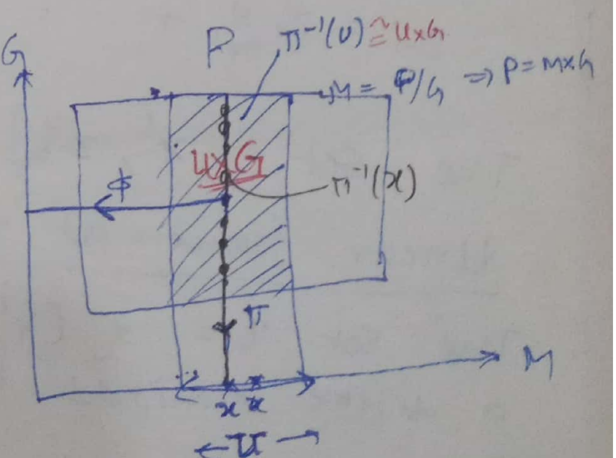
A set $\{P, M, G, \pi\}$ is called principal fibre bundle if P & M are diff'ble Manifold, G is a Lie ~~grp~~ transformation on M from the right. $\pi$ is a Projection map, satisfying following conditions: —

① G acts on P differentiably to the right, i.e $\exists$ a diff'ble map from $P \times G \to P$ s.t
$(u,g) = ug$ , $\forall u \in P$, $g \in G$. $ug \in P$
and $(ug)h = u(gh)$ , $\forall h \in G$.

② M is a Quotient Manifold, i.e $M \cong P/G$ and $\pi$ is a Projection Map $\pi : P \to P/G$ $(\pi : P \to M)$ & is diff'ble.

③ For each $x \in M$, & for every nbd U of x such that $\pi^{-1}(U)$ is isomorphic to $U \times G$ by the diffeomorphism
$\psi : \pi^{-1}(U) \to U \times G$
s.t $\psi(u) = (\pi(u), \phi(u))$ , where $\phi : \pi^{-1}(U) \to G$ satisfying $\phi(ug) = \phi(u)g$ $\forall u \in \pi^{-1}(U)$ $\forall g \in G$.

We call $P \to$ Total Space or Bundle Space
$M \to$ Base Space
$G \to$ Structure ~~grp~~.
$\pi \to$ Projection Map.




$\forall x \in M$, $\pi^{-1}(x)$ is a closed submanifold of P, called fibre over x.

or

If $u \in \pi^{-1}(x)$; Then $\pi^{-1}(x)$ is the set of point $(u,g)$, $g \in G$ $= ug$ and is called fibre through u.

## Linear frame:

Let $M \to$ Diff$^{ble}$ Manifold.

& Let $x \in M$,

$U \to$ nbd of $x$.

& $(x^1, x^2, \ldots x^n) \to$ LCS in $U$

The Canonical basis vectors in $U$ are $\left\{\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \ldots \frac{\partial}{\partial x^n}\right\}$

Let $\{Z_1, Z_2, Z_3, \ldots Z_n\}$ be another basis in $U$. Then

we can write –

$$Z_1 = Z_1^1 \frac{\partial}{\partial x^1} + Z_1^2 \frac{\partial}{\partial x^2} + Z_1^3 \frac{\partial}{\partial x^3} + \cdots + Z_1^n \frac{\partial}{\partial x^n}$$

$$Z_2 = Z_2^1 \frac{\partial}{\partial x^1} + Z_2^2 \frac{\partial}{\partial x^2} + Z_2^3 \frac{\partial}{\partial x^3} + \cdots + Z_2^n \frac{\partial}{\partial x^n}$$

$$\vdots$$

$$Z_n = Z_n^1 \frac{\partial}{\partial x^1} + \cdots \cdots + Z_n^n \frac{\partial}{\partial x^n}$$

The set $(x^i, Z_b^a)$, $1 \leq i \leq n$, $1 \leq a, b \leq n$ is called

Linear frame at $x$.

The set $L = \{(x^i, Z_b^a)\}$ of all linear frames is a diff$^{ble}$ manifold of dim. $n+n^2$

Doubt Claim: $\{L, M, GL(n,\mathbb{R}), \pi\} \to$ Principal fibre bundle

f.g.a

linear frame bundle.

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(i) First of all, we'll show that $Gl(n,R)$ acts diff'ly on $L$ to the right.

~~Let $u \in L$, $g \in Gl(n,R)$ given by~~ $u \equiv (x^i, z^a_b)$
$g \equiv g^a_b$

Pn, Define a map; from $L \times Gl(n,R) \to$ ~~$Gl(n,R)$~~ $L$

~~set~~ $(u,g) \to ug$.

s.t $$\left((x^i, z^a_b), g^b_c\right) = \left(x^i, z^a_b g^b_c\right) \in L$$

where $(x^i, z^a_b) \in L$
$\&$ $g^b_c \in Gl(n,R)$

Observe: $(ug)h = u(gh)$

i.e $$\left((x^i, z^a_b) \cdot g^b_c\right) \cdot h^c_d = \left(x^i, z^a_b g^b_c\right) h^c_d$$
$$= \left(x^i, z^a_b \left(g^b_c h^c_d\right)\right)$$
$$= (x^i, z^a_b)\left(g^b_c h^c_d\right)$$
$$= u(gh)$$

So $Gl(n,R)$ acts differentiably on $L$ to the right.

(ii) We can take $M = L/Gl(n,R)$, $\&$ $\pi$ as projection Map

$\pi : L \to M$ s.t

$u = (x^i, z^a_b) \in L$, $\pi(x^i, z^a_b) = x^i$, which is diff'le map.

(iii) Let $x \in M$, $U \to$ nbd of $x$ in $M$.

Then $\pi^{-1}(U) \subset L \cong M \times Gl(n,R)$

Let $\pi^{-1}(U) = \{(x^i, z^i_b) \mid x^i \in M, z^i_b \in Gl(n,R)\}$

Consider a map from;

$(x^i, z^i_a) \xrightarrow{Id} (x^i, z^i_a)$ which is a identity Map

from $\pi^{-1}(U) \xrightarrow{Id} U \times Gl(n,R)$. obviously Id. map is isomap

So $\pi^{-1}(U) \cong U \times Gl(n,R)$. So the set $\{ - \}$ is principle fibre bundle

## Tangent Bundle

Let $M \to n$ dim diff'ble manifold at each pt $p \in M$,
Then, $\exists$ n dimensional tangent Sp $T_pM$, then

$TM = \bigcup_{p \in M} T_pM$, Coll^n of all Tangent Spaces of $M$ is called Tangent bundle of $M$

$TM \to$ Diff'ble Manifold of dim $\underline{\underline{2n}}$.

The manifold $M$, over which $TM$ is defined is called base space

**Quest:** Prove that, Tangent Bundle $TM$ is always orientable.

**Pf:** Let $M$ be a diff'ble Manifold. and $TM \to$ Tangent Bundle
Let $\pi$ is the Projection Map $\pi : TM \to M$.
Let $x \in M$. & $U$ be nbd of pt $x$ in $M$.
Let $\{x^1, x^2, \dots x^n\} \to$ local coord. Syst in $U$
we can also write this coordinate —
$(x^h)$. , $h = 1, 2, \dots n$.

Then $\pi^{-1}(U)$ is the ~~local~~ coord. ~~Syst~~ nbd in $TM$, with Coord Syst $\{x^1, x^2, \dots x^n, y^1, y^2, \dots y^n\}$ or $(x^h, y^h)$, $h = 1, 2, \dots n$.

Let $U' \to$ another ~~loc~~ nbd of $x$ in $M$ with LCS $(x^{1'}, x^{2'}, x^{3'} \dots x^{n'})$ or $x^{h'}$, $h' = 1', 2', \dots n'$

Then $\pi^{-1}(U')$ is ~~co~~ Coord nbd in $TM$, with local Coord Syst
$\{x^{1'}, x^{2'}, x^{3'}, \dots x^{n'}, y^{1'}, y^{2'}, \dots y^{n'}\}$, or $(x^{h'}, y^{h'})$
$h' = 1', 2', \dots n'$.

Since $U \cap U' \neq \phi$

$\Rightarrow \pi^{-1}(U) \cap \pi^{-1}(U') \neq \phi$

Now, for the points, which ~~lying~~ in the region — $\pi^{-1}(U) \cap \pi^{-1}(U')$
we have

$$\left.\begin{aligned} x^{h'} &= x^{h'}(x^h) \\ \& \; y^{h'} &= \frac{\partial x^{h'}}{\partial x^h} y^h \end{aligned}\right\} \quad \text{— (A)}$$

P-5 Putting $y^i = x^{n+i}$

or $y^1 = x^{n+1}$

$y^2 = x^{n+2}, \cdots y^n = x^{2n}$

So Local coord. Syst in $\pi^{-1}(U)$ is $(x^p)$, $p = 1, 2, \cdots n, n+1, \cdots 2n$

& local coord syst in $\pi^{-1}(U')$ is $(x^{p'})$, $p' = 1', 2', \cdots n', n+1', \cdots 2n'$

Eqn (A), may also be expressed as —

$$\left.\begin{aligned} x^{h'} &= x^{h'}(x^h) \\ \& \ y^{h'} &= \frac{\partial x^{h'}}{\partial x^i} y^i \end{aligned}\right\} \text{ — (B)}$$

Now, The Jacobian Matx of transformation, for the pts lie in the region $\pi^{-1}(U) \cap \pi^{-1}(U')$ is given by.

$$\left(\frac{\partial x^{p'}}{\partial x^p}\right) = \frac{\partial (x^{h'}, y^{h'})}{\partial (x^h, y^h)} = \begin{bmatrix} \frac{\partial x^{h'}}{\partial x^h} & \frac{\partial x^{h'}}{\partial y^h} \\ \frac{\partial y^{h'}}{\partial x^h} & \frac{\partial y^{h'}}{\partial y^h} \end{bmatrix}.$$

Since $\frac{\partial x^{h'}}{\partial y^h} = \frac{\partial}{\partial y^h}\left(x^{h'}(x^h)\right)$

$= 0$

$$\frac{\partial y^{h'}}{\partial x^h} = \frac{\partial}{\partial x^h}\left(\frac{\partial x^{h'}}{\partial x^i} y^i\right) = \frac{\partial^2 x^{h'}}{\partial x^h \partial x^i} y^i$$

$$\frac{\partial y^{h'}}{\partial y^h} = \frac{\partial}{\partial y^h}\left(\frac{\partial x^{h'}}{\partial x^h} y^h\right) = \frac{\partial x^{h'}}{\partial x^h} + 0$$

$$= \frac{\partial x^{h'}}{\partial x^h}$$

Thus $$\frac{\partial (x^{h'}, y^{h'})}{\partial (x^h, y^h)} = \begin{bmatrix} \frac{\partial x^{h'}}{\partial x^h} & 0 \\ \frac{\partial^2 x^{h'}}{\partial x^h \partial x^i} y^i, & \frac{\partial x^{h'}}{\partial x^h} \end{bmatrix} \text{ — (1)}$$

~~Obviously~~ ~~Jacobian~~ Det of Jacobian matx is non zero.

Now, we can also consider; the reverse —

$$x^h = x^h(x^{h'}) \quad , \quad y^h = \frac{\partial x^h}{\partial x^{i'}} y^{i'}$$

Then Jacobian matx of transformation is given by

$$\frac{\partial (x^h, y^h)}{\partial (x^{h'}, y^{h'})} = \begin{bmatrix} \frac{\partial x^h}{\partial x^{h'}} & \frac{\partial x^h}{\partial y^{h'}} \\ \partial y^h / \partial x^{h'} & \frac{\partial y^h}{\partial y^{h'}} \end{bmatrix}$$

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P-t/

Since

$$\frac{\partial y^h}{\partial x^{h'}} = \frac{\partial}{\partial x^{h'}}\left(\frac{\partial x^h}{\partial x^{i'}} y^{i'}\right)$$

$$= \frac{\partial^2 x^h}{\partial x^{h'}\partial x^{i'}} y^{i'} + 0$$

$$\frac{\partial y^h}{\partial y^{h'}} = \frac{\partial}{\partial y^{h'}}\left(\frac{\partial x^h}{\partial x^{h'}} y^{h'}\right) = \frac{\partial x^h}{\partial x^{h'}} \cdot 1 + 0$$

$$= \frac{\partial x^h}{\partial x^{h'}}$$

Thus

$$\frac{\partial(x^h, y^h)}{\partial(x^{h'}, y^{h'})} = \begin{bmatrix} \frac{\partial x^h}{\partial x^{h'}} & 0 \\ \frac{\partial^2 x^h}{\partial x^{h'}\partial x^{i'}} y^{i'} & \frac{\partial x^h}{\partial x^{h'}} \end{bmatrix} \quad \text{— (2)}$$

Obviously det of (2) is non zero

Therefore we can say that Tangent Bundle is always orientable because its det. is non zero.

---

Vertical lift:

Let $M$ be a diff^ble Manifold of dim $n$.

& $f : M \to R$ (Real valued f^n)

Let $\pi : TM \to M$ (Projection Map)

Define $f^v = f \circ \pi : TM \to R$, called the vertical lift of $f$.

$$\begin{tikzcd} TM \arrow[d, "\pi"'] \arrow[dr, "f\circ\pi = f^v"] & \\ M \arrow[r, "f"'] & R \end{tikzcd}$$

Quest: For a $C^\infty$ f^n $f$ & $g$ on $M$, P.T.

(i) $(f+g)^v = f^v + g^v$

(ii) $(f \cdot g)^v = f^v g^v$

Pf. (1) $(f+g)^v = (f+g)\circ\pi = (f\circ\pi) + (g\circ\pi)$
$= f^v + g^v$

(2) $(f\cdot g)^v = (fg)\circ\pi = (f\circ\pi)(g\circ\pi)$
$= f^v g^v$

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P-7: Vertical lift of vector field:

Let $X \to$ vector field on $M$

Vertical lift of $X$ is denoted by $X^v$ & is defined as

$$X^v(\tau w) = (w(X))^v$$

where $\tau w$ is a $C^\infty$ $f^n$ in $TM$ & $w$ is 1-form on $M$.

+ $\tau: TM \to T_pM$ & $w: T_pM \to R$

i.e $\tau w: TM \to R$

Remark: $\boxed{X^v f^v = 0}$

Quest: PT

(1) $(X+Y)^v = X^v + Y^v$

(2) $[X^v, Y^v] = 0$

Pf: (1)
$$(X+Y)^v(\tau w) = (w(X+Y))^v$$
$$= (w(X) + w(Y))^v$$
$$= (w(X))^v + (w(Y))^v = X^v(\tau w) + Y^v(\tau w)$$
$$= (X^v + Y^v)(\tau w), \quad \forall \tau w$$

$$\Rightarrow (X+Y)^v = X^v + Y^v$$

(2)
$$[X^v, Y^v](\tau w) = X^v Y^v(\tau w) - Y^v X^v(\tau w)$$
$$= X^v(w(Y))^v - Y^v \cancel{X^v(w)}\,(w(X))^v$$
$$= 0 - 0$$
$$= 0 \quad (\because X^v f^v = 0)$$

---

Vertical lift of covariant tensor of rank '2'

Let $G$ be a covariant tensor of rank 2, with components $G_{ij}$

Then ~~$G_{ij}$~~ $G$ can be be locally expressed as

$$G = G_{ij}\, dx^i \otimes dx^j$$

Taking vertical lift on both sides—

$$G^v = (G_{ij})^v (dx^i)^v \otimes (dx^j)^v$$
$$= (G_{ij})^v\, dy^i \otimes dy^j$$

$\left(\because (dx^i)^v = dy^i \ \& \ \left(\frac{\partial}{\partial x^i}\right)^v = \frac{\partial}{\partial y^i}\right)$

P-8'

## Vertical lift of Contravariant Tensor of rank 2

Let $H \rightarrow$ Contravariant Tensor of rank 2 with comp. $H^{ij}$

Then
$$H = H^{ij}\left(\frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j}\right)$$

Taking vertical lift:
$$H^V = (H^{ij})^V \left(\frac{\partial}{\partial x^i}\right)^V \otimes \left(\frac{\partial}{\partial x^j}\right)^V$$

$$\boxed{H^V = (H^{ij})^V \frac{\partial}{\partial y^i} \otimes \frac{\partial}{\partial y^j}}$$

## Vertical lift of Tensor of type (1,1)

$$F = F_i^j \, dx^i \otimes \frac{\partial}{\partial x^j}$$

$$F^V = (F_i^j)^V (dx^i)^V \otimes \left(\frac{\partial}{\partial x^j}\right)^V$$

$$\boxed{F^V = (F_i^j)^V \, dy^i \otimes \frac{\partial}{\partial y^j}}$$