"GRAPH Theory and Applications"

→ Unit 1 :- Introduction :-

- Finite and infinite graph
- Incidence and degree.
- Isolated and Pendent vertex, null graph
- Isomorphism
- Subgraph, walks, paths and circuits
- Edge connectivity (Not covered before MST)
- Computer representation of graph.
- Digraph (Not covered Before MST)

→ What is a Graph?

- Definition :- A Graph (or Linear graph) $G = (V, E)$ consists of objects $V = \{v_1, v_2, ---\}$ called vertices and another set $E = \{e_1, e_2, ---\}$ whose elements are called edges s.t. each edge e_k is identified with an unordered pair (v_i, v_j) of vertices.

- The vertices v_i, v_j associated with edge e_k are called the end vertices of e_k.

- Graph is represented by means of a diagram such that its vertices are represented by points and each edge as a line segment or curve joining its end vertices.

Example:-

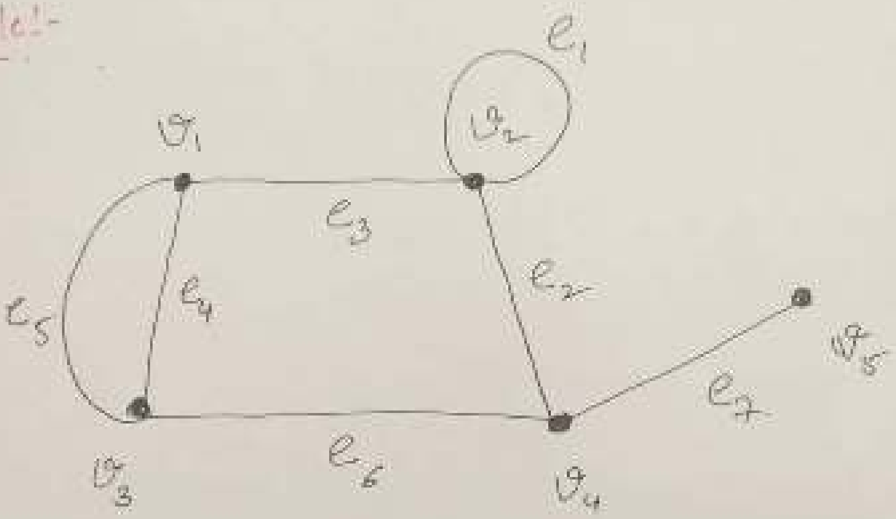


Fig. 1

→ (v_1, v_2) are vertices associated with e_3

- Self Loop:- Edge associated with same vertex as both its end vertices is called self loop.

→ e_1 is a self loop with vertex v_2 as both its ~~et~~ end vertices.

- Parallel edges:- Two or more edges associated same pair of end vertices are called parallel edges.

→ Example:- e_4 and e_5 are parallel vertices associated with (v_1, v_3).

- Simple Graph:- A graph that has neither self loops nor parallel edges is called a simple graph.

Example of simple graph

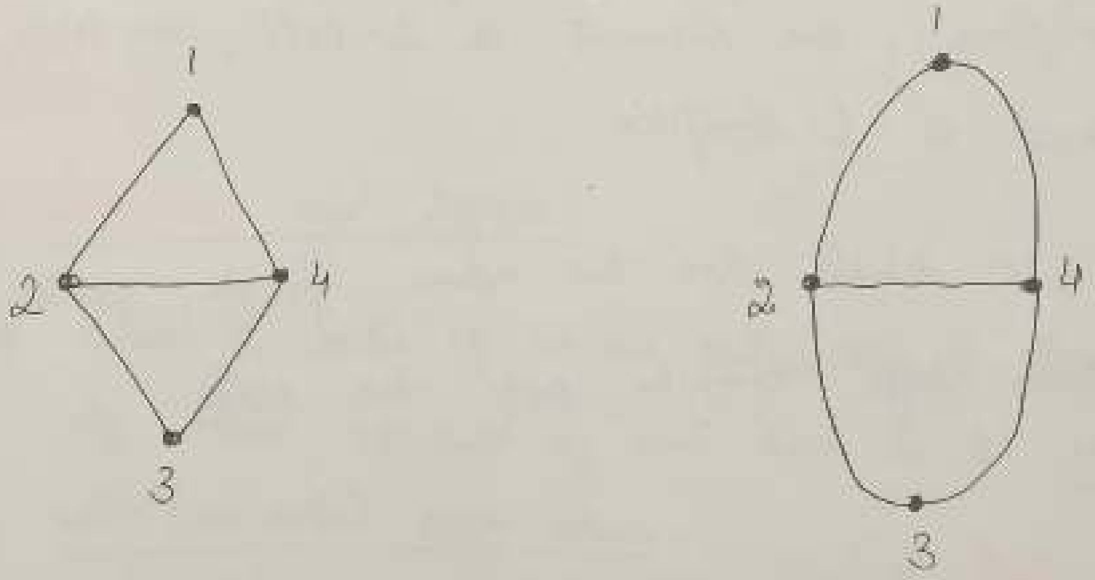


Figure 2

→ In drawing a graph it is immaterial whether the lines are drawn straight or curved, long or short, what is important is the incidence between the edges and vertices.

→ Both the graphs in figure 2 are ~~same~~ same.

• → A graph is also called linear complex or 1- complex or a one-dimensional complex.

→ A vertex is also referred to as a node, a junction, a point, 0- cell and 0- simplex.

→ An edge is also referred as a branch, a line, an element, a 1- cell, an arc, and a 1- simplex.

→ We shall use the name graph, vertex and edge through out the notes.

→ Finite and Infinite Graph:-

→ • A graph with a finite number of vertices as well as a finite number of edges is called a finite graph, otherwise an infinite graph.

• figure 1 and 2 are examples of finite Graph.

→ Example of infinite graphs :-

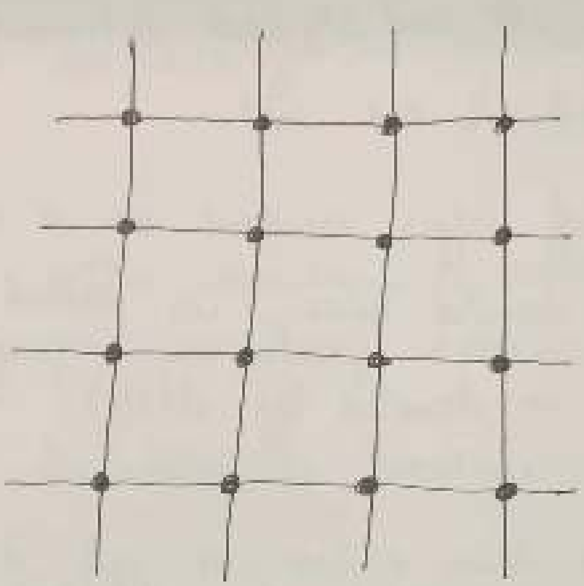


figure 3
Portion of an infinite graph.

→ Incidence and Degree :-

- When a vertex v_i is an end vertex of some edge e_j, then v_i and e_j are said to be incident with (on or to) each other.

→ In figure 1, e_3, e_4, e_5 is incident on v_1

- Two non-parallel edges are said to be adjacent if they are incident on a common vertex.

→ In figure 2, e_3 and e_4 are adjacent, e_2 and e_6 are adjacent.

- Two vertices are said to be adjacent if they are the end vertices of the same edge.

→ In figure 1, v_1 and v_2 are adjacent vertices whereas v_1 and v_4 are not.

- The number of edges incident on a vertex v_i, with self-loop counted twice, is called the <u>degree</u>, and is denoted by $d(v_i)$.

→ <u>In figure 1,</u>

$$d(v_1) = 3, \quad d(v_2) = 4, \quad d(v_3) = 3,$$
$$d(v_4) = 3, \quad d(v_5) = 1.$$

→ The degree of a vertex is some times also referred to as its valency.

- For a graph G with n vertices, ~~and~~ $\{v_1, v_2, -, v_n\}$ and m edges, $\{e_1, e_2, -, e_m\}$

$$\sum_{i=1}^{n} d(v_i) = 2m$$

is known as the <u>Handshakes theorem</u>.

The above ~~eta~~ results follows from the fact each edge contributes two degrees (i.e. both its end vertices) hence the sum of ~~dg~~ degree of all vertices is twice of the number of ~~edg~~ edges in G.

→ **Theorem** :- The number of vertices of odd degree in a graph is always even.

Proof :- According to previous result, if a graph has n- vertices and m- edges

$$\{v_1, v_2, \ldots, v_n\} \qquad \{e_1, e_2, \ldots, e_m\}$$

then $\sum_{i=1}^{n} d(v_i) = 2m$ which is an even number.

(1)

Also, $$\sum_{i=1}^{n} d(v_i) = \sum_{\substack{\text{even}\\ \text{degree}}} d(v_i) + \sum_{\substack{\text{odd}\\ \text{degree}}} d(v_i)$$

first term is also even. $\Rightarrow \sum_{\substack{\text{odd}\\ \text{degree}}} d(v_i) = \text{even}$

This is possible only if ~~$d(v_i)$ set~~ number of v_i with odd degree is even. Hence the result.

Isolated Vertex, Pendant Vertex and Null graph

→ A vertex having no incident edge is called an isolated vertex.

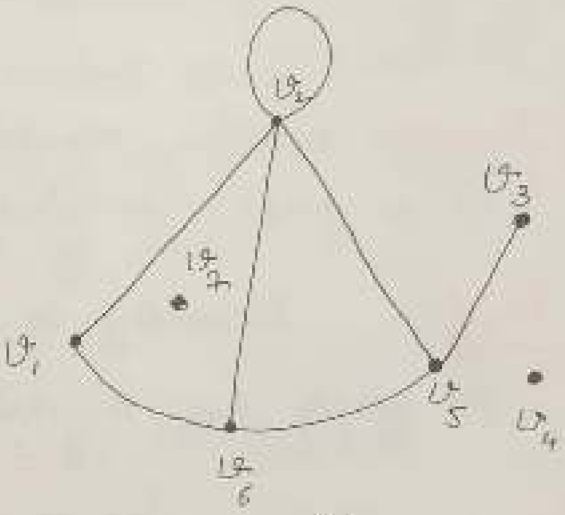


Figure-4

- v_4 and v_7 are isolated vertex.

→ A vertex with degree one is called Pendant ~~graph~~ vertex or end vertex.

- v_3 is a pendant vertex.

→ Two adjacent edges are said to be in series if their common vertex is of degree two.

- Two ~~vertices~~ edges incident on v_1 are in series.

→ A graph $G = (V, E)$ with no edges is called null graph.

- Figure 5 is an example of null graph with 7-vertices.
- Note that there should be atleast one vertex in a null graph.

Figure 5

Applications of Graph:-

Königsberg Bridge problem:-

→ This is one the best known problems of graph theory. This problem was first solved by Euler in 1736 by means of graph and hence the theory of Graphs began.

→ Problem:- Two islands C and D, formed by the Pregel river in Königsberg (now in west Soviet Russia) were connected to each other and ~~to~~ to the banks A and B with seven ~~brig~~ bridges as shown in figure 6.

The problem is to start from any of the four land areas, walk over each of the seven bridges exactly once and return to the starting point (without swimming across the river).

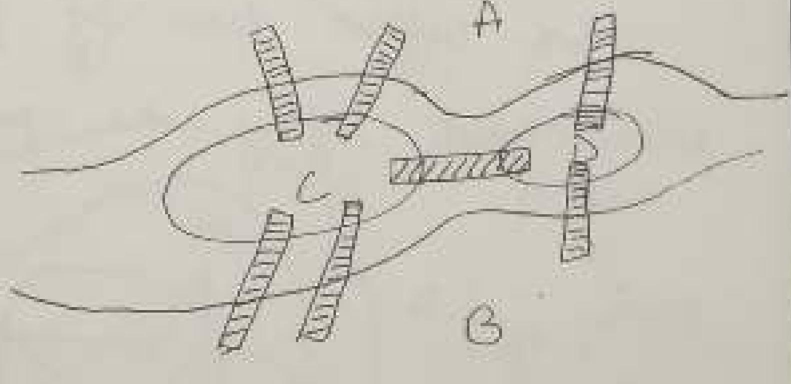
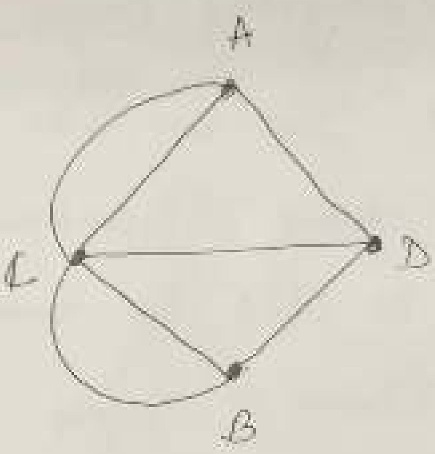


Figure-6

→ <u>Solution</u>:- Euler represented this situation by means of a graph

- Each city in a vertex.
- Each bridge is represented by an edge.



- Shown in figure 7.

Figure-7

Euler showed that the solution of this problem doesnot exists. (We shall prove this when we discuss Euler Graphs).

→ Similarly we shall be able prove that we cannot draw the following figure without lifting the pen on ~~the~~ a paper

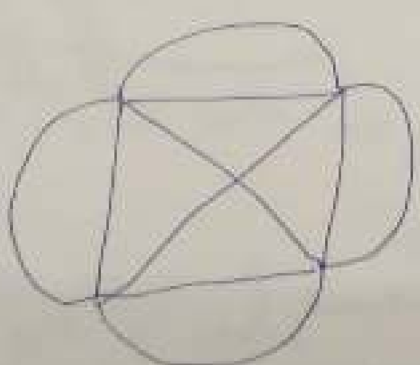
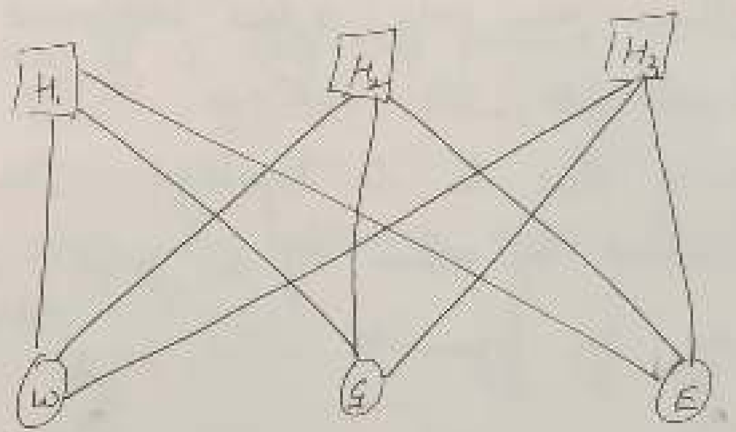


Figure-8

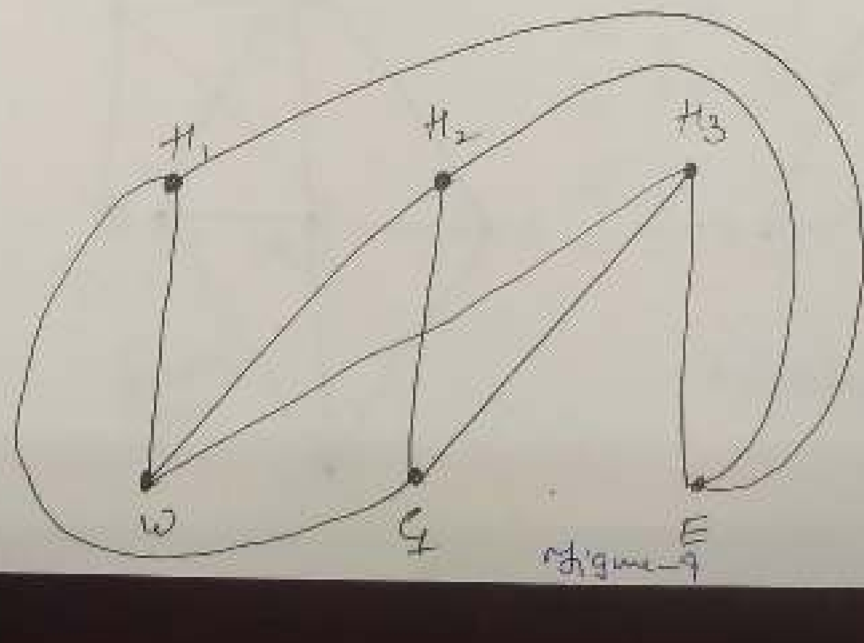
Utilities problem:-

Problem:-
There are three houses H_1, H_2 and H_3 each to be connected to three utilities - Water (W), gas (G) and Electricity (E). by means of ~~[illegible]~~ Conduits. Is it possible to make such connections without any cross overs of the conduits?

Soln:-
The answer to this ~~[illegible]~~ problem is No.
(We shall prove ~~it~~ later).



Three utilities problem.



Edge joining H_3 and W crosses over.

Graphical representation

figure-9

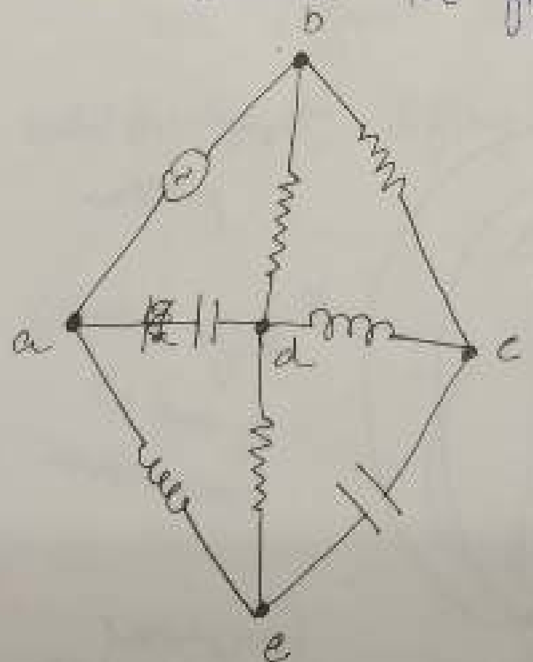
Electrical Network Properties:-

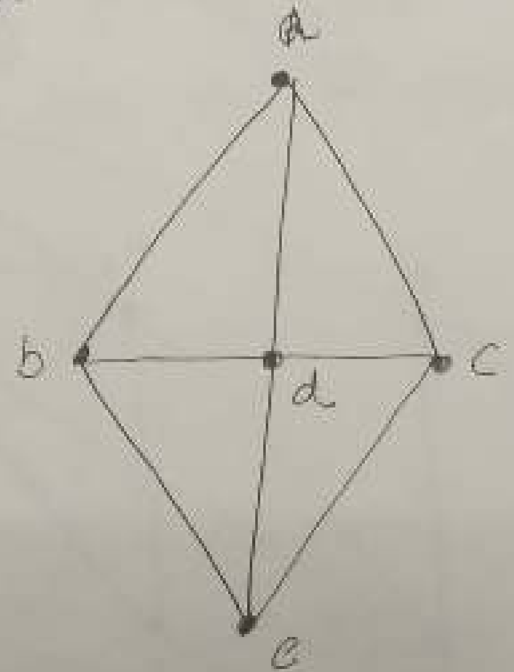
→ Properties of any Electrical network depends on two things:-

(1) The nature and value of elements forming the network example resistors, inductors, capacitors batteries and so forth.

(2) The way these elements are connected i.e. its topology.

→ Since there are only a few different electrical elements, the variations in the network is mainly due to its topology. The Topology can be studied by means of a graph. Example is shown in figure-10.





→ Representation of a Graph:-

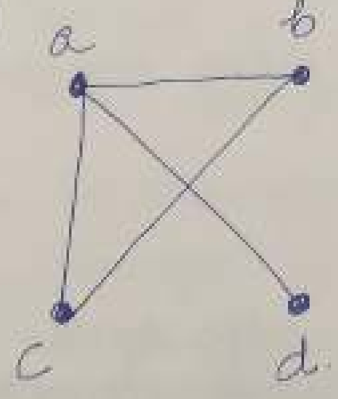
-: Adjacency Matrix:-

- Let $G = (V, E)$ be a ~~Simple~~ Graph, Suppose that vertices of G are listed as $1, 2, \ldots, n$. Then the adjacency matrix A of G is defined as an $n \times n$ matrix, defined such that

- If there is an edge b/w v_i & v_j then $a_{ij} = 1$ otherwise zero.
- If there are more than edges b/w v_i & v_j (Say k) then $a_{ij} = k$
- If there is a self-loop present on the ith vertex then $a_{ii} = 1$ (one self-loop counted one's).

Example:- Write the adjacency matrix for the following graph:-

(1)-

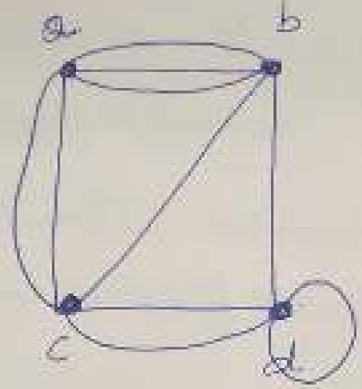


Graph

$$
\begin{array}{c|cccc}
 & a & b & c & d \\
\hline
a & 0 & 1 & 1 & 1 \\
b & 1 & 0 & 1 & 0 \\
c & 1 & 1 & 0 & 0 \\
d & 1 & 0 & 0 & 0
\end{array}
$$

Adjacency Matrix

(2)



Graph

$$\begin{array}{c|cccc} & a & b & c & d \\ \hline a & 0 & 3 & 2 & 0 \\ b & 1 & 0 & 1 & 1 \\ c & 2 & 1 & 0 & 2 \\ d & 0 & 1 & 2 & 1 \end{array}$$

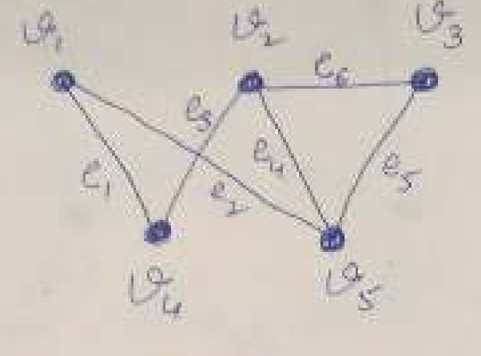
Adjacency Matrix

→ **Incidence Matrix:-**

- Another common way to represent graphs is to use incidence matrices.
- Let $G=(V,E)$ be an undirected graph with $1, 2, 3, \text{——}, n$ vertices and $e_1, e_2, \text{——}, e_m$ ~~to~~ edges.
- Then the incidence matrix w.r.t. this ordering of V and E in the $n \times m$ matrix $M = [m_{ij}]_{n\times m}$, where

$$m_{ij} = \begin{cases} 1 \; ; & \text{when edge } e_j \text{ is incident with vertex } i. \\ 0 \; ; & \text{Otherwise.} \end{cases}$$

→ **Example:-** Represent the following graph with an incidence matrix:-



Graph

$$\begin{array}{c|cccccc} & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \\ \hline v_1 & 1 & 1 & 0 & 0 & 0 & 0 \\ v_2 & 0 & 0 & 1 & 1 & 0 & 1 \\ v_3 & 0 & 0 & 0 & 0 & 1 & 1 \\ v_4 & 1 & 0 & 1 & 0 & 0 & 0 \\ v_5 & 0 & 1 & 0 & 1 & 1 & 0 \end{array}$$

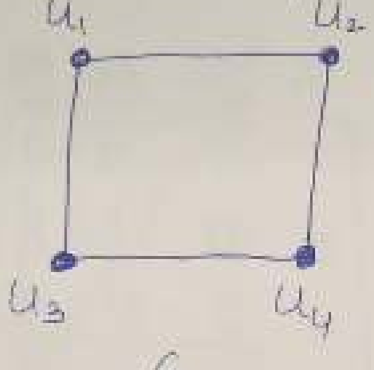
Incidence Matrix

→ Isomorphism of Graphs:-

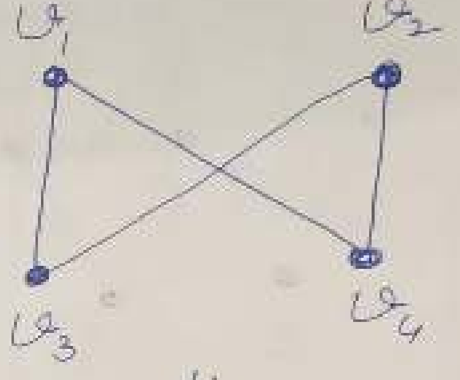
→ **Definition:-** The simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are isomorphic, if there is a one-to-one and onto function f from V_1 to V_2 with the property that a & b are adjacent in G_1 iff $f(a)$ and $f(b)$ are adjacent in G_2, for all a & b in V_1. Such a function f is called and isomorphism.

- In other words, when two simple graphs are isomorphic, there is a one-to-one correspondence b/w vertices of the two graphs that preserves the adjacency relationship.

Example:- Show that the following two graphs are isomorphic.



G



H

– Let us define the function f as.

$$f(u_1) = v_1$$
$$f(u_2) = v_4$$
$$f(u_3) = v_3$$
$$f(u_4) = v_2.$$

– Write the adjacency of G:

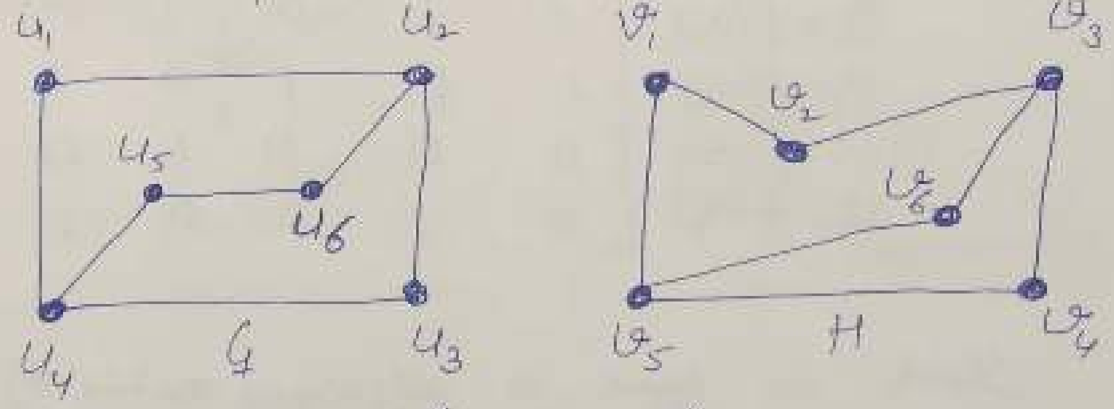
$$\begin{array}{c|c} & \begin{matrix} u_1 & u_2 & u_3 & u_4 \end{matrix} \\ \begin{matrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{matrix} & \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

– Write the ~~adjency~~ adjacency of matrix of H s.t. ith row & ith column of it corresponds to $f(u_i)$:-

$$
\begin{array}{c|cccc}
 & \overset{v_1}{f(u_1)} & \overset{v_4}{f(u_2)} & \overset{v_3}{f(u_3)} & \overset{v_2}{f(u_4)} \\
\hline
v_1 = f(u_1) & 0 & 1 & 1 & 0 \\
v_4 = f(u_2) & 1 & 0 & 0 & 1 \\
v_3 = f(u_3) & 1 & 0 & 0 & 1 \\
v_2 = f(u_4) & 0 & 1 & 1 & 0
\end{array}
$$

– Since both the matrices are same, hence G & H are isomorphic to each other.

<u>Example</u> Show that following graphs are isomorphic:-



– Define the function f s.t.

$f(u_1) = v_6$ $\quad$ $f(u_4) = v_5$

$f(u_2) = v_3$ $\quad$ $f(u_5) = v_1$

$f(u_3) = v_4$ $\quad$ $f(u_6) = v_2$

(<u>Hint</u>:- function is written s.t. we try to preserve the adjacency relation of vertices ~~is preserved~~)

- Adjacency matrix of G.

$$\begin{array}{c|cccccc} & u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \\ \hline u_1 & 0 & 1 & 0 & 1 & 0 & 0 \\ u_2 & 1 & 0 & 1 & 0 & 0 & 1 \\ u_3 & 0 & 1 & 0 & 1 & 0 & 0 \\ u_4 & 1 & 0 & 1 & 0 & 1 & 0 \\ u_5 & 0 & 0 & 0 & 1 & 0 & 1 \\ u_6 & 0 & 1 & 0 & 0 & 1 & 0 \end{array}$$

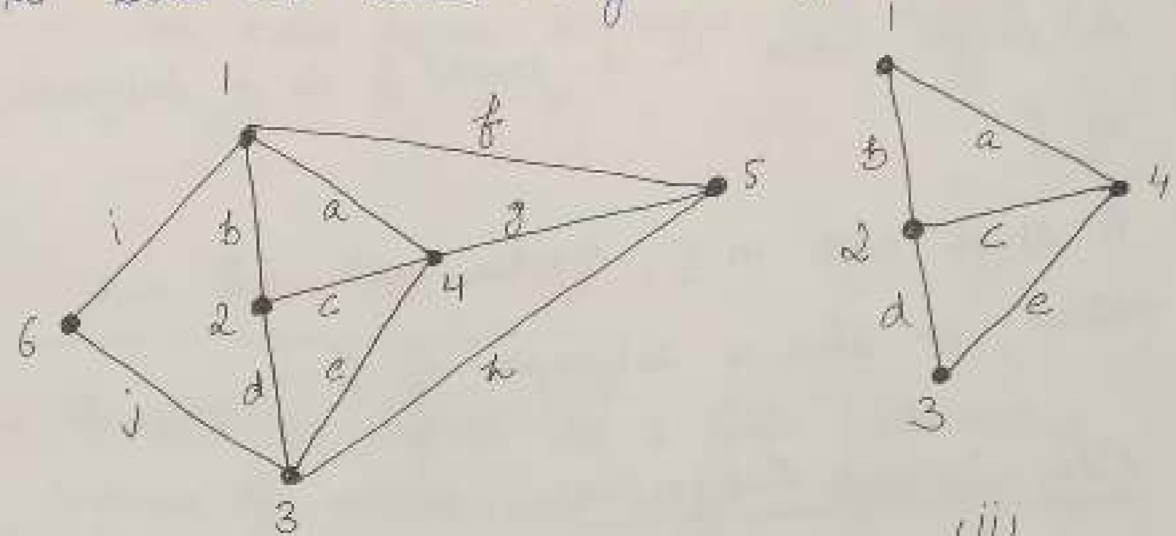
- Adjacency matrix of H

$$\begin{array}{c|cccccc} & \overset{v_6}{f(u_1)} & \overset{v_3}{f(u_2)} & \overset{v_4}{f(u_3)} & \overset{v_5}{f(u_4)} & \overset{v_1}{f(u_5)} & \overset{v_2}{f(u_6)} \\ \hline v_6 = f(u_1) & 0 & 1 & 0 & 1 & 0 & 0 \\ v_3 = f(u_2) & 1 & 0 & 1 & 0 & 0 & 1 \\ v_4 = f(u_3) & 0 & 1 & 0 & 1 & 0 & 0 \\ v_5 = f(u_4) & 1 & 0 & 1 & 0 & 1 & 0 \\ v_1 = f(u_5) & 0 & 0 & 0 & 1 & 0 & 1 \\ v_2 = f(u_6) & 0 & 1 & 0 & 0 & 1 & 0 \end{array}$$

Which is same as adjacency matrix of G
hence G is isomorphic to H.

Subgraph:-

→ A graph 'g' is said to be a subgraph of a graph G if all the vertices and all the edges of g are in G, and each edge of g has the same end vertices in g as in G.



(i)

(ii)

Figure (1)

- (ii) is a Subgraph of (i)

- A Subgraph can be thought of as being contained in (or a part of) another graph. The Symbol from set theory, $g \subset G$, is used in stating "g is a subgraph of G."

→ The following observations can be made immediately :-

1) Every graph is its own Subgraph.

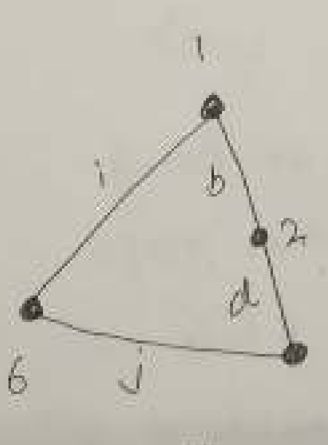
2) A Subgraph of a subgraph of G is a Subgraph of G.

3) A Single vertex in a graph G is a subgraph of G.

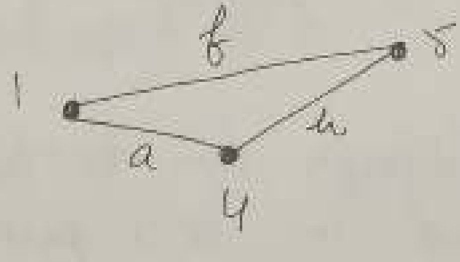
4). A single edge in G, together with its end vertices, is also a subgraph of G.

→ Edge-Disjoint Subgraphs :-

Two Subgraphs g_1 and g_1 of a graph G are said to be edge disjoint if g_1 + g_2 do not have any edges in common.



(iii)



(iv)

Figure (2)

• (iii) & (iv) are the edge-disjoint subgraphs of (i) in figure (1).

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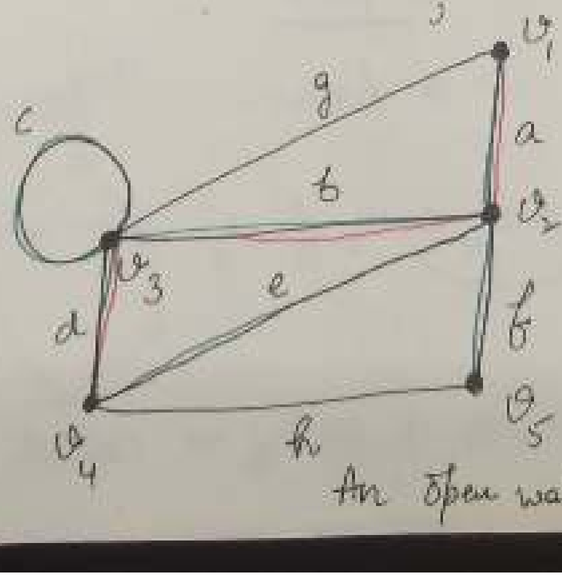
- Note that although edge-disjoint graphs donot have any edge in common, they may have vertices in common.
- Sub-graphs that donot even have vertices in common are said to be vertex disjoint.
  - Are vertex disjoint subgraphs edge disjoint?

    Yes.

Walks, Paths and Circuits :-

→ A walk is defined as a finite alternating sequence of vertices and edges, beginning and ending with vertices, such that each edge is incident with the vertices preceding and following it.

- No edge ~~appeared~~ appears more than once in a walk. A vertex, however, may appear more than once.



An open walk (with green), A path of length 3.

- $v_1 a v_2 b v_3 c v_3 d v_4 e v_2 f v_3$ is a walk shown with green line.
- A walk is also referred to as ~~an~~ an edge train or a chain.
- The set of vertices and edges constituting a given walk in a graph G is clearly a subgraph of G.
- Vertices with which a walk begins and ends are called its terminal vertices. v_1 and v_5 are the terminal vertices.
- If the walk begin and end at the same vertex, such a walk is called a closed walk.
- A walk walk that is not closed is called an open walk.

→ An open walk in which no vertex appears more than once is called a path.

- $v_1 a v_2 b v_3 d v_4$ is a path.
- $v_1 a v_2 b v_3 c v_3 d v_4 e v_2 f v_3$ is ~~so~~ not a path.

→ The number of edges in a path is called the length of a path.

- An edge which is not a self-loop is a path of length one.
- $v_1\, a\, v_2\, b\, v_3\, d\, v_4$ is a path of length 3.
- Self loop can be included in a walk but not in a path.
- The terminal vertices of a path are of degree 1 and the rest of the vertices are of degree 2.
  - This degree, of course, is counted only w.r.t the edges included in the path and not the entire graph in which the path may be contained.

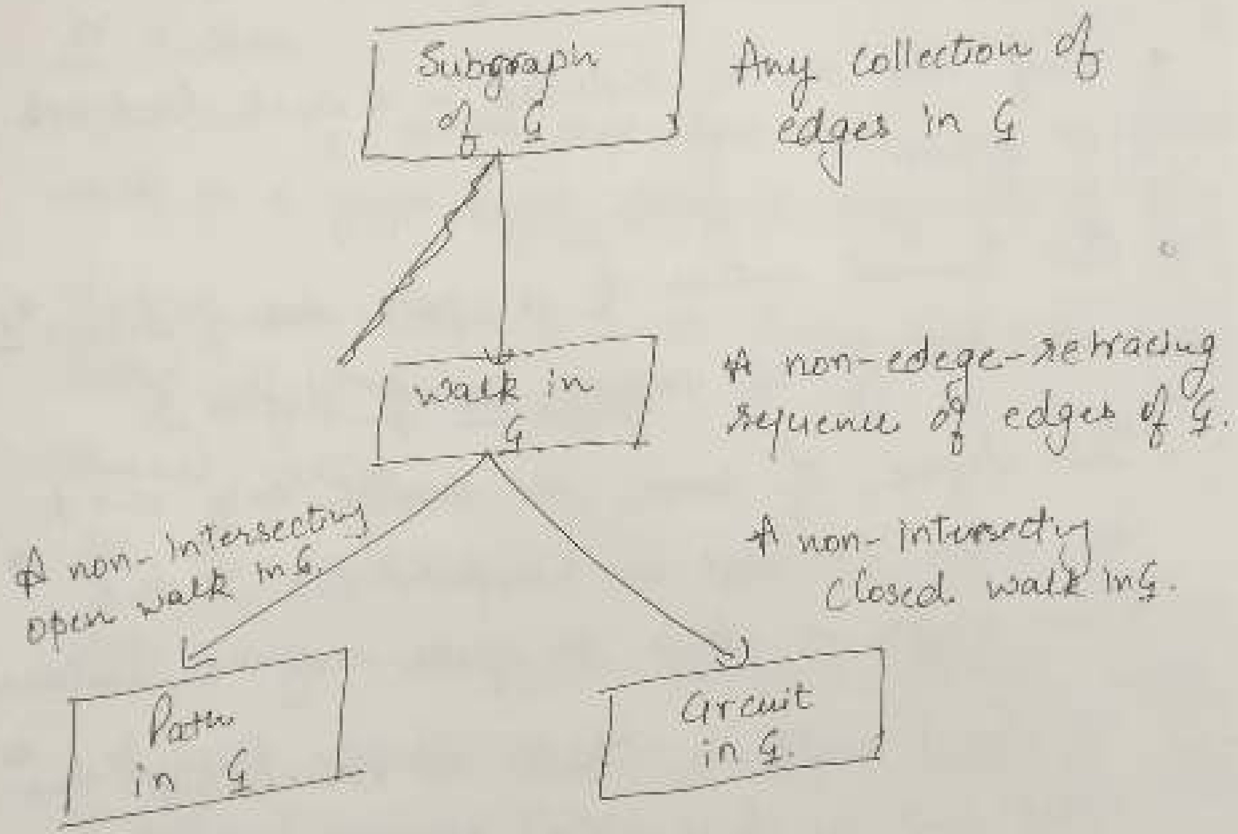
→ A closed walk in which no vertex (except the initial and the final vertex) appears more than once is called a <u>circuit</u>

- $v_2\, b\, v_3\, d\, v_4\, e\, v_2$ is a circuit.

<u>Examples</u>

- Every vertex in a circuit is of degree 2.

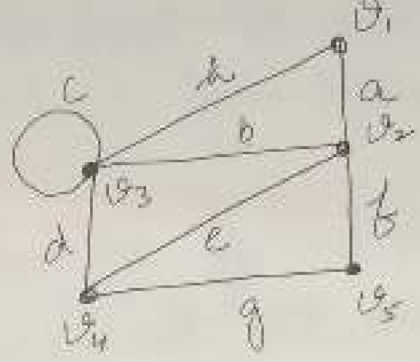
→ A circuit is also called a cycle, elementary cycle, circular path and polygon.



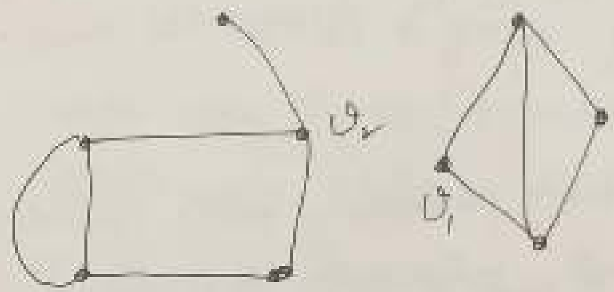
Connected Graphs, Disconnected Graphs and Components:-

Definition:- A Graph G is said to be connected if there is atleast one path between every pair of vertices in G. Otherwise G is disconnected.

- A null graph of more than one vertex is disconnected.



Example of Connected graph.



Example of disconnected graph.

v_1 and v_2 are not connected to each other.

→ A disconnected graph consists of two or more connected graphs. Each of these connected subgraphs is called a component.

→ Another way of looking at components is:- In the above drawn example consider the vertex v_1, since the graph is disconnected, not all the vertices of G are joined by path to v_1. Vertex v_1 and all the vertices of G that have

Paths to v_i, together with all the edges incident on them, form a component.

Theorem A Graph is disconnected iff its vertex set V can be partitioned into two nonempty, disjoint Subsets V_1 and V_2 s.t. there exists no edge in G whose one end vertex is in Subset V_1 and the other in Subset V_2.

Proof:- • Suppose that such partitioning exists. Consider two arbitrary vertices a and b in G, s.t $a \in V_1$ and $b \in V_2$. No path can exist b/w a & b.

• Otherwise there would be at least one edge whose end vertex would be in V_1 and the other in V_2

• Hence G is not connected.

Converse, Let G be a disconnected Graph

• Let a be any vertex in G. Let V_1 be the set of all vertices that are joined by path to a.

• Since G is disconnected V_1 doesnot include all the vertices of G.

• The remaining vertices form nonempty set V_2

- By construction No vertex in V_1 is joined to any in V_2 by an edge. Hence the partition

Theorem:- If a graph has exactly two vertices of odd degree, there must be a path joining these two vertices.

Proof:- • Let G be a graph with all the vertices with even degree (even vertices) except v_1 & v_2 which are odd.

- If G is connected then there is nothing to prove.
- If G is disconnected then we can divide G in two or more components.
- v_1 & v_2 should belong to the same component as each component is a graph in itself and a graph must have even no. of vertices with odd degree (proved earlier).

hence there is a path joining v_1 & v_2.

Theorem:- A simple graph (ie. a graph without parallel edges or self loops) with n-vertices and k components can have atmost $(n-k)(n-k+1)/2$ edges.

- Before proving the above thm, we shall prove the following inequality:- for $n_1, n_2, \ldots, n_k \geq 1$ s.t. $n_1 + n_2 + \cdots + n_k = n$

$$\sum n_i^2 \leq n^2 - (k-1)(2n-k)$$

Now, $\sum_{i=1}^{k} n_i = n \Rightarrow \sum_{i=1}^{k} (n_i - 1) = n - k$

Squaring both sides, $\left(\sum_{i=1}^{k}(n_i - 1)\right)^2 = (n-k)^2$

$$\Rightarrow \sum_{i=1}^{k} (n_i - 1)^2 + \text{non-negative cross terms} = n^2 + k^2 - 2nk$$

$$(\because n_i \geq 1 \ \forall i)$$

$$\Rightarrow \sum_{i=1}^{k}(n_i^2 - 2n_i) + k + \text{non-neg terms} = n^2 + k^2 - 2nk$$

$$\Rightarrow \sum_{i=1}^{k} n_i^2 + \text{non-neg. terms} = n^2 + k^2 - 2nk - k + 2n$$

$$\Rightarrow \sum_{i=1}^{k} n_i^2 + \text{non-neg. terms} = n^2 - (k-1)(2n-k)$$

$$\Rightarrow \sum_{i=1}^{k} n_i^2 \leq n^2 - (k-1)(2n-k) \longrightarrow (*)$$

- Coming back to the proof of the thm:-

Let $n_1, n_2, \ldots n_k$ be the no. of vertices in k-components of the graph G.

Thus, $n_1 + n_2 + \cdots + n_k = n$ & $n_i \geq 1 \; \forall i$

The max no. of edges in the ith component of G is $\frac{1}{2} n_i (n_i - 1)$

$\therefore$ the max. no. of edges in G is,

$$\frac{1}{2} \sum_{i=1}^{k} n_i (n_i - 1) = \frac{1}{2} \left[\sum_{i=1}^{k} n_i^2 - \sum_{i=1}^{k} n_i \right]$$

$$\leq \frac{1}{2} \left[n^2 - (k-1)(2n-k) - n \right]$$

$\Rightarrow$ Max. No. of edges $\leq \frac{1}{2}(n-k)(n-k+1)$

Operations on Graphs:-

- Since the graphs are defined mathematically as set of vertices and edges, it is natural to define operations b/w graphs.

- Union of two graphs:-

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs, then the union of two graphs $G = G_1 \cup G_2$ is defined as a graph whose vertex set $V_3 = V_1 \cup V_2$ and Edge set is $E_1 \cup E_2$.

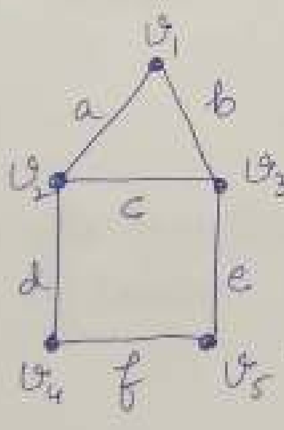
- Intersection of two Graphs:-

$G_4 = G_1 \cap G_2$ is defined as a graph containing only those edges & vertices which are common in G_1 & G_2 both

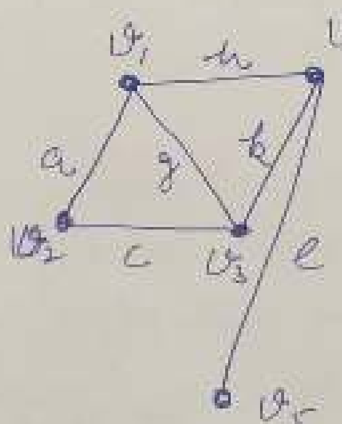
- Ring Sum of two graphs:-

$G_5 = G_1 \oplus G_2$ ($\oplus$ denotes the ring sum) is a graph consisting of the vertex set $V_1 \cup V_2$ and of edges that are either in G_1 or G_2, but not in both.

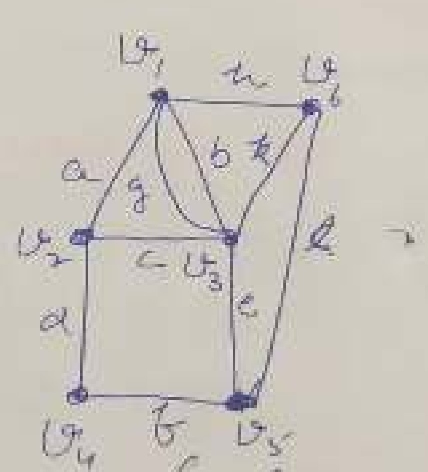
Example



G_1

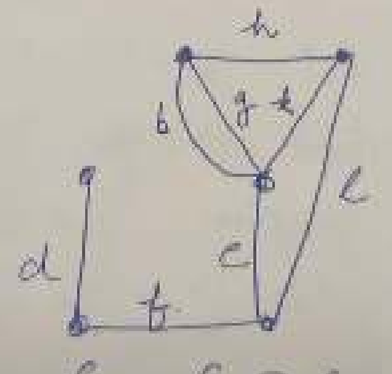


G_2



$G_3 = G_1 \cup G_2$

$G_4 = G_1 \cap G_2$



$G_5 = G_1 \oplus G_2$

- ~~At~~ These definitions of Union, Intersection, ringsum ~~are~~ can be extended to any finite no. of graphs in a similar manner

- From the above definations it is simple to deduce that

$$G_1 \cup G_2 = G_2 \cup G_1$$
$$G_1 \cap G_2 = G_2 \cap G_1$$
$$G_1 \oplus G_2 = G_2 \oplus G_1$$

- Also if G_1 & G_2 are edge disjoint then $G_1 \cap G_2$ is a null graph and

$$G_1 \cup G_2 = G_1 \oplus G_2$$

- If G_1 & G_2 are vertex disjoint then $G_1 \cap G_2$ is an empty graph.

- Also $G \cap G = G = G \cup G$ and

$G \oplus G =$ null graph

- If g is a subgraph of G, then $G \oplus g$ is that subgraph of G that remains after all the edges in g have been removed from G.

$\therefore$ $G \oplus g$ can also be ~~writ~~ written as $G - g$, ~~as~~ $g \subseteq G$.

- ∴ of this property $G \oplus g$ is also called complement of g in G.

- **Decomposition:-** A graph G is said to have been decomposed into two subgraphs g_1 & g_2 if

$$g_1 \cup g_2 = G \quad \& \quad g_1 \cap g_2 = \text{a null graph.}$$

ie • every edge of G occurs in either g_1 ★ or g_2 but not in both.

- Some vertices, however may occur in both g_1 & g_2.

- A graph containing m-edges $[e_1, e_2, \ldots, e_m]$ can be decomposed in $2^{m-1} - 1$ different ways into pairs of subgraphs g_1 & g_2.

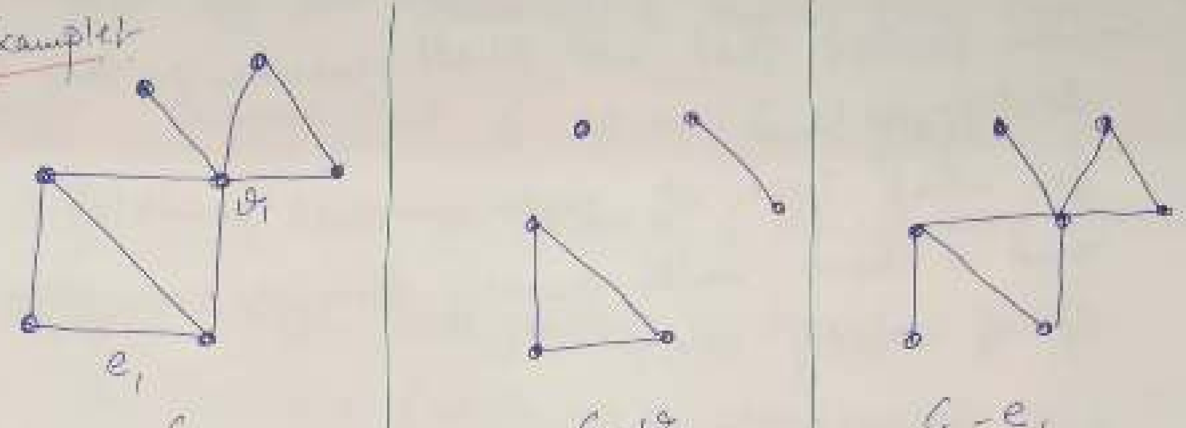
(Proof is left as an exercise).

- **Deletion:-** Deletion of an vertex:-
  - If v_1 is a vertex of graph G, then $G - v_1$ denotes a subgraph of G obtained by deleting v_1 (and the edges incident on it) from G.

Deletion of an edge:-

- If e_1 is an edge of G then $G - e_1$ is the graph obtained by edge e_1 from the graph (not the vertex on which it is incident) $\therefore$ $G - e_1 = G \oplus e_1$

Example:-



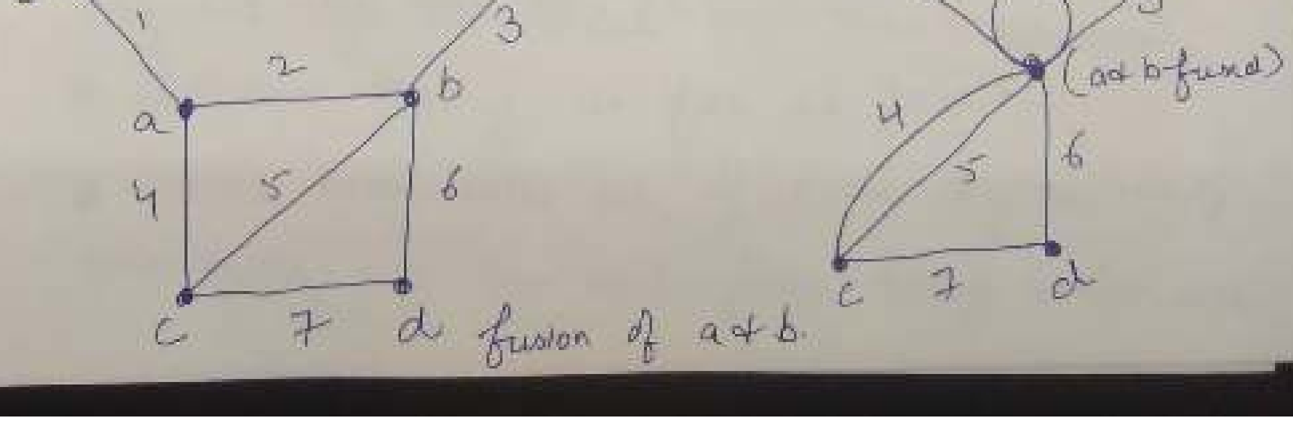
G $\quad$ $G - v_1$ $\quad$ $G - e_1$

- Fusion :-

- A pair of vertices a, b in a graph are said to be fused (merged or identified) if the two vertices are replaced by a single new vertex, s.t. every edge that was incident on either a or b or on both is incident on the new vertex.

- Fusion preserves the no. of edges & decrease the no. of vertecies by one.

Example:-



fusion of a & b.

Euler Graph :-

- The question that we shall answer in this section is :-
In what type of graph G is it possible to find a closed walk running through every edge of G exactly one?

Such a walk is now called an Euler line and a graph that consists of an Euler line is called Euler Graph.

- Walk is always connected. Since Euler line contains all the edges of a graph, an Euler graph is always connected, except for any isolated vertices the graph may have.
Since isolated vertices do not contribute anything extra to the understanding of our topic, we can assume that Euler graphs do not have any isolated vertices and are therefore connected.

Theorem:- A given connected graph G is an Euler graph iff all vertices of G are of even degree.

Proof:- Suppose that G is an Euler graph.

To show:- all the vertices of G are of even degree

~~G~~-G is an Euler graph, it therefore contains an Euler line (which is a closed walk).

- In tracing this walk we observe that every time the walk meets a vertex v it goes through two new edges incident on v — with one we entered v and with one we exited from v

- This is true for all vertices. Thus ~~[illegible]~~ all the vertices G has every vertex of even degree

Converse:- All vertices of G are of even degree. and G is connected

To show:-

G is an Euler graph.

- • Construct a walk starting at an arbitrary vertex v and going through the edges of G s.t. no edge is traced more than once.

• Continue tracing as far as possible

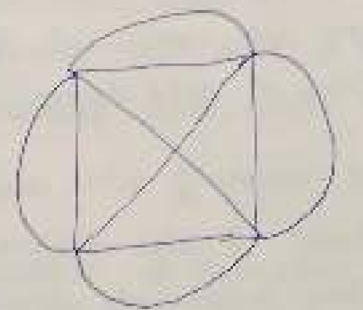
• Since every vertex is of even degree is of even degree, we can exit from every vertex we enter, the tracing cannot stop at any vertex but v.

- And since v is also of even degree, we shall eventually reach v when the tracing comes to an end.
- If this closed walk h, just traced includes all the edges of G, G is an Euler Graph.
- If not, we remove from G all the edges in h and obtain a subgraph h' of G formed by the remaining edges.
- Since G & h have all vertices of even degree, so is h'.
- Moreover, h' must touch h at least at one vertex a, because G is connected.
- Starting from a, we can again construct a new walk in graph h'. Since all the vertices of h' are of even degree, this walk in h' must terminate at vertex a, but this walk in h' can be combined with h to form a new walk, which starts and ends at vertex v and has more edges than h.
- This process can be repeated until we obtain a closed walk that traverses all the edges of G. Thus G is an Euler Graph.

→ Konigsberg Bridge Problem:-

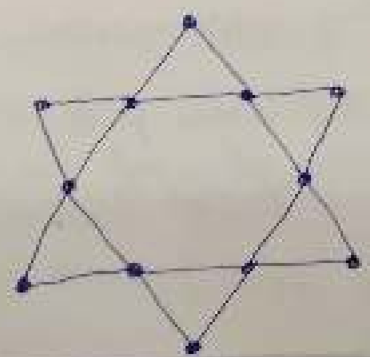
Now looking at the graph of the Konisberg bridges ~~problem~~ we find that not all its vertices are of even degree. Hence, it is not an Euler graph. Thus it is not possible to walk over each of the seven bridges exactly once and return to the starting point.

→ More Examples of Euler's Graph:-



– We cannot trace this figure covering all the edges without ~~rept~~ repeating the edges, and ~~removing~~ picking up the pen.

– Soln: not possible as the all the vertices are not of even degree and soln of this is an Euler's Graph



– This is an Euler's graph as all vertices are of even degree

- **Definition:-** An open walk that includes (or traces or covers) all edges of a graph without retracing any edge is called a <u>Unicursal line or an open Euler line</u>

- A (connected) graph that has a unicursal line will be called a unicursal graph.
- It is clear that by adding an edge b/w the initial and final vertices of unicursal line we shall get an Euler line.
- Thus a connected graph is unicursal iff it has exactly two vertices of odd degree.

Theorem:- In a connected graph G with exactly $2k$ odd vertices, there exist k edge-disjoint subgraphs s.t they together contain all edges of G and that each is a unicursal graph.

Proof:- - Let the odd vertices of the given graph G be named $v_1, v_2, \ldots, v_k$ & $w_1, w_2, \ldots, w_k$ in any arbitrary order.

- Add k edges to G b/w the vertex pairs $(v_1, w_1), (v_2, w_2), \ldots, (v_k, w_k)$ to form a new graph G'.

- Since every vertex in G' is of even degree, G' consists of an Euler line f.
- Now if we remove from f the k edges we just added (no two of these edges are incident on the same vertex), f will be ~~spit~~ split into k walks, each of which is a unicursal line;
- The second removal will split that into two unicursal lines, and each successive removal will split a unicursal line into two unicursal lines, until there are k of them.

Hence the theorem

Theorem:- A connected graph G is an Euler graph iff it can be decomposed into circuits;

Proof:- Suppose that graph G can be decomposed into circuits, i.e., G is a union of edge-disjoint circuits.

- Since the degree of every vertex in a circuit is two, the degree of every vertex in G is even.
- Hence G is an Euler graph.

Converse:- - Let G be an Euler graph.

- Consider a vertex v_1, there are at least two edges incident at v_1. Let one of these be b/w v_1 & v_2.

- Since v_2 also has even degree, it must have atleast ~~one~~ another edge say b/w v_2 & v_3.

- Proceeding in this fashion, we eventually arrive at a vertex that has previously been traversed, thus forming a circuit Γ.

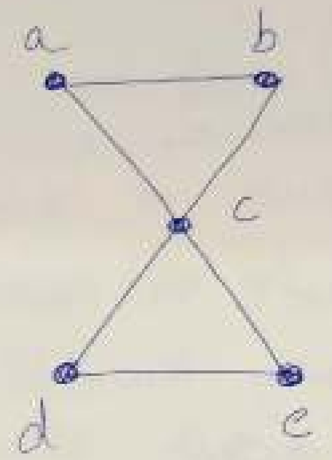
- Let us remove Γ from G. All vertices in the remaining graph (not necessarily connected) must also be of even degree.

- Remove another circuit in ~~the~~ similar manner as above from remaining graph.

- Continue this process untill no edge is left and we have decomposed the graph into edge disjoint circuits. Hence the theorem.

→ Arbitrarily traceable Graphs:-

Consider the following Euler's Graph:-



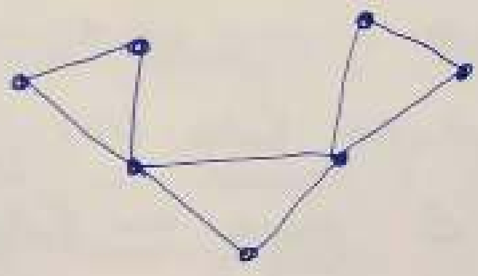
- Trace the graph starting from a, onto the path a b c, from c now we have ~~two~~ three choices, go towards a or e or d.
  - abca is not an Euler line
  - abcd or abce gives an Euler line when further traced.
- Thus starting from a, we cannot trace the entire Euler line simply by moving along any edge (choosen arbitrarly if we have two options) that has not already been transversed.
- The question here is:- What property vertex v must have in Euler graph, have such an Euler line is always obtained when one follows any walk from vertex v according to the single rule that whenever one arrives at a vertex one shall select any edge (which has not been previously

traversed)?

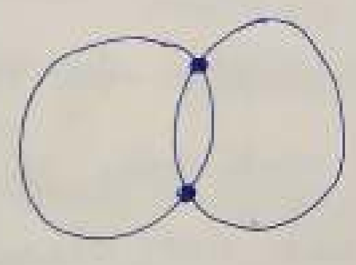
Such a graph is called an arbitrarily traceable graph from vertex v.

- For example, vertex C in previous example makes it an arbitrarily traceable graph. But not from any other vertex.

Example:-



Euler graph, but not arbitrarily traceable from any vertex.

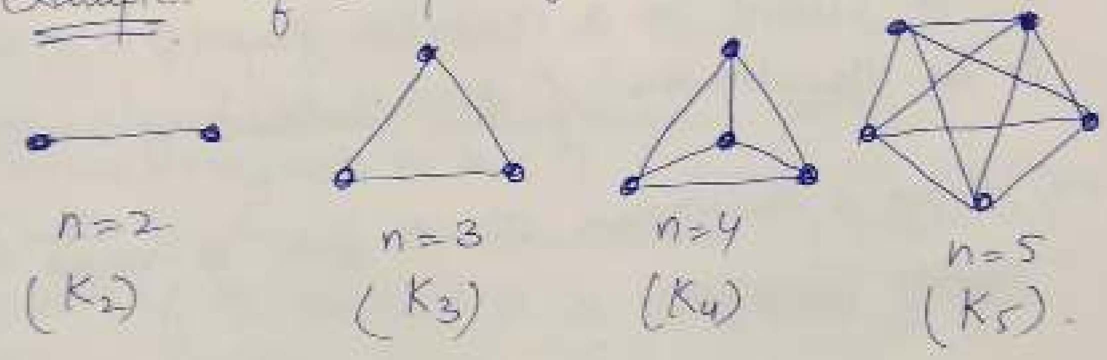


Arbitrarily traceable graph from all vertices.

Theorem:- (only statement) An Euler graph G is arbitrarily traceable from vertex v in G iff every ~~vertex~~ circuit in G contains v.

→ Definition:- Complete Graph (K_n).

- A simple graph in which there exists an edge b/w every pair of vertices is called a complete graph.
- It is denoted by K_n, where n is the no. of vertices in the simple graph.
- Example:- of complete graph

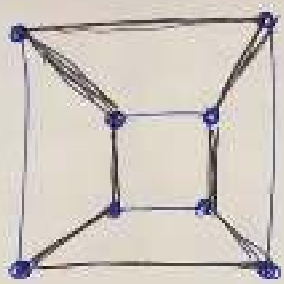


- Since every vertex is joined with every other vertex through one edge, the degree of every vertex is $(n-1)$ in a complete graph G of n-vertices.
- The total no. of edges in G is $\frac{n(n-1)}{2}$.

→ Hamiltonian Paths and Circuits:-

Definition:- Hamiltonian circuit in a connected graph is defined as a closed walk that traverses every vertex of G exactly once, except of course the Starting vertex at which the walk also Terminates

Example:-



Edges in dark form the Hamiltonian Circuit

→ More formally the definition of Hamiltonian graphs is a circuit in a connected graph G is said to be Hamiltonian if it includes every vertex of G

→ Hence a Hamiltonian circuit with n ~~edges~~ vertices has exactly n-edges.

→ Next question at hand is :-
What is a necessary and sufficient condition for a connected graph G to have a Hamiltonian circuit?

This problem is still an unsolved for an arbitrary Graph, however we have certain special cases in which such conditions (results) exists.

Note:- Finding ~~an~~ a Hamiltonian circuit is more complex than finding an Euler line and their resemblance is deceptive.

→ Definition:- If we remove one edge from a Hamiltonian circuit, we are left with a path and such path is called Hamiltonian path.

- Every graph that has a Hamiltonian ~~path~~ circuit also has a Hamiltonian path, Converse may not be true? (Counter Ex. is an exercise)
- The length of a Hamiltonian path (if it exists) in a connected graph of n-vertices is (n-1).

→ No. of Hamiltonian Circuits in a Graph:-

- If an Hamiltonian circuit exists in a graph, there can be more than one Hamiltonian graphs. Of particular interests are Edge-disjoint Hamiltonian circuits.

- The exact no. of edge-disjoint Hamiltonian circuits (or paths) in a graph in general is an unsolved problem. However the result exists for ~~com~~ complete graph with odd no. of vertices.

Theorem:- In a complete graph with n vertices there are $(n-1)/2$ edge-disjoint Hamiltonian circuits, if n is an odd number with $n \geq 3$.

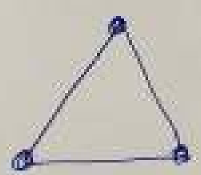
Proof:- A complete graph with n-vertices has $n(n-1)/2$ edges

- Therefore the no. of edge disjoint Hamiltonian ~~g~~ circuits cannot exceed $(n-1)/2$ as each Hamiltonian circuit has n-edges.

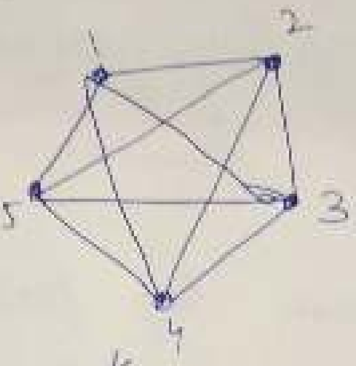
(for there we have not assumed that n is odd)

- Now we shall show that for $n \geq 3$, odd, there exactly $(n-1)/2$ of these. we shall show this ~~for~~ by principle of mathematical induction.

for $n=3$.

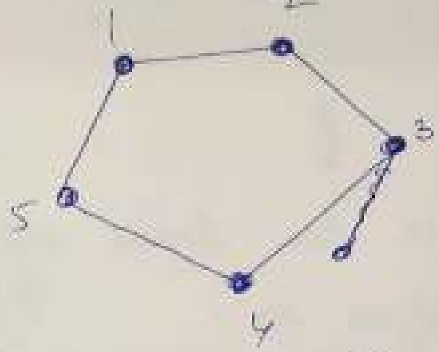
K_3 is  and there is one Hamiltonian circuit in it i.e $\frac{(3-1)}{2} = 1$

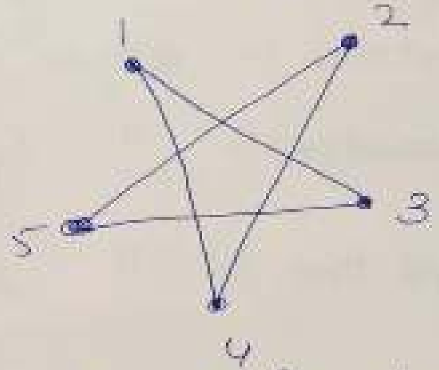
$\underline{n=5}$



K_5

Hamiltoin Circuits
(edge-disjoint)





→ There are 2-edge-disjoint Hamiltonian circuits, (ie $\frac{5-1}{2} = 2$) which is

$\left(\frac{n-1}{2}\right)$

- Let the result be true for $n = k$ (odd).

We shall prove the result for $n = k+2$ (next odd no.)

<u>Note</u>:- In drawing edge-disjoint Hamiltonian circuits for $n=5$, we first took the edges joing adjacent ~~of~~ vertices, then for second we skipped one vertex and took the

edges joining ~~at~~ alternative vertices.

→ Our assumption:-
for $n=k$, there are $\frac{k-1}{2}$ edge-disjoint Hamiltonian graphs

&

We can draw all of these by picking edges as:-

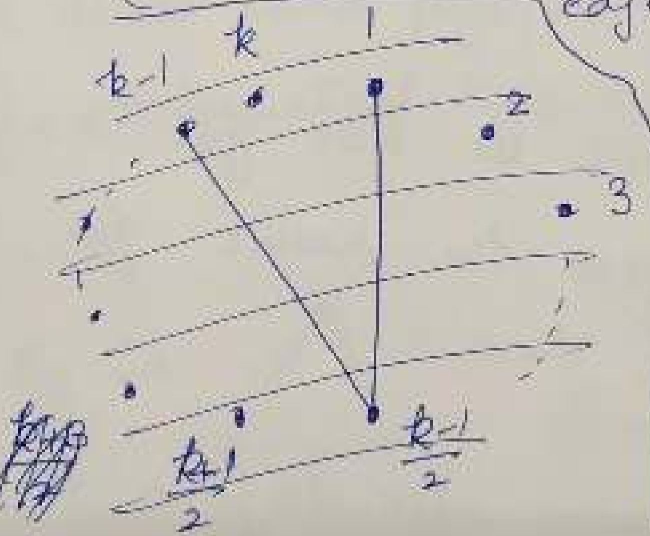
first one :- edges joining adjacent vertices

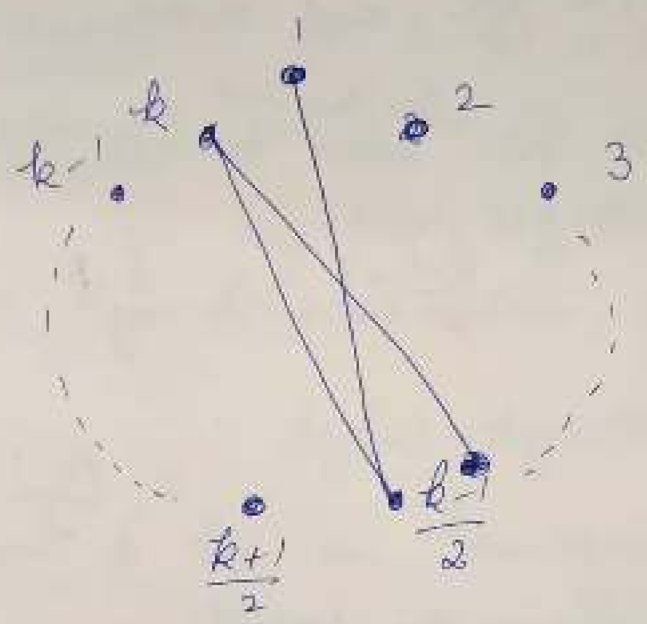
second one :- " " vertices by leaving one vertex

third one :- " " vertices by leaving two vertices inbetween.

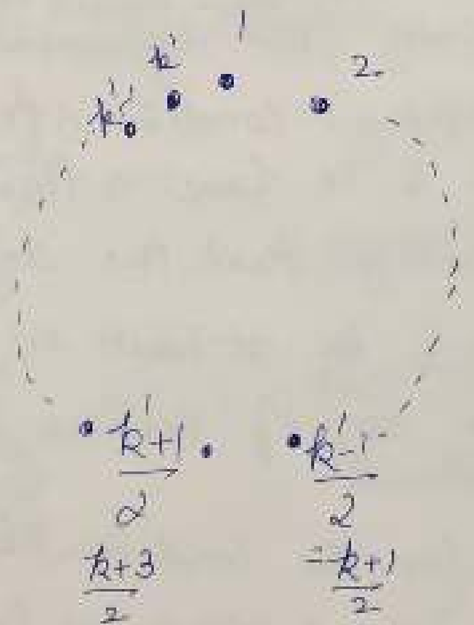
⋮

$\left(\frac{k-1}{2}\right)^{th}$ one :- We join the vertex 1 with $(k-1)/2$ then we skip $(k-3)/2$ vertices and take the edge joining $\left(\frac{k-1}{2}\right)$ to (k) and so-on forming an Hamiltonian circuit, as depicted in the figure below:-





for $n = k+2 = k'$ (say)



- Drawing $\frac{k-1}{2}$ circuits in the similar manner (since we have assumed that as before, ~~if we see~~ there exists) it is easier to see that we will have one more ~~circuit~~ edge disjoint Hamiltonian circuit possible ie beginning with edge joining 1 & $\frac{k+1}{2}$ and so-on.

hence the total no. ~~now~~ for this case becomes

$$\frac{k-1}{2}+1 = \frac{k+1}{2} = \frac{(k+2)-1}{2}$$

$$= \frac{n-1}{2}$$

And hence the result.

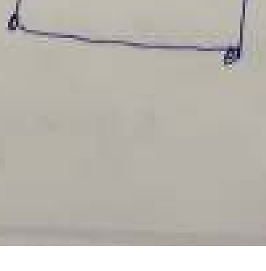
→ Below we give some results which can be used to show if a given graph has an Hamiltonian circuit or not.

<u>Theorem</u>:- (By G.A. Dirac!)
(only statement)

A sufficient (but by no means necessary) condition for a simple graph G to have a Hamiltonian circuit is that the degree of every vertex in G be at least $n/2$, where n is the no. of vertices in G.

<u>Counter Example</u>:- (for the condition not being necessary)

Circuit with five vertices is an Hamiltonian graph but degree of every vertex is ~~two~~



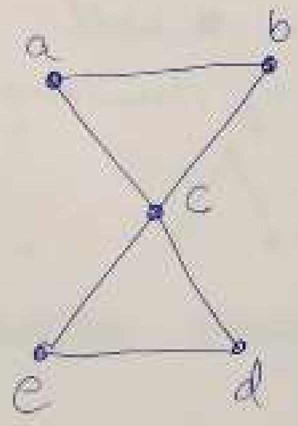
$2 \not> 5/2$ (ie $n/2$)

Ore's Theorem:- (only statement) Let G be a simple graph with no. of vertices, $n \geq 1$
For any pair u & v of two ~~vertices~~ non-adjacent vertices, if

$$deg(u) + deg(v) \geq n$$

then G has an Hamiltonian Circuit (or is an Hamiltonian Graph).

Counter Example of necessary:-



$n = 5 \geq 1$

$deg(a) + deg(e) = 4 \not\geq 5$

but this is an Hamiltonian Graph

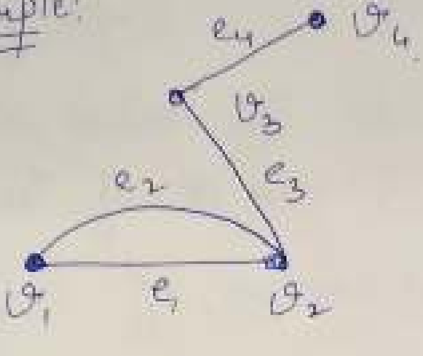
→ **Closure of a graph G:-** (denoted by $[G]$ or $cl(G)$)

Let G be a graph with n-vertices. The $cl(G)$ is the graph obtained by adding edges b/w non-adjacent vertices (u & v) for which

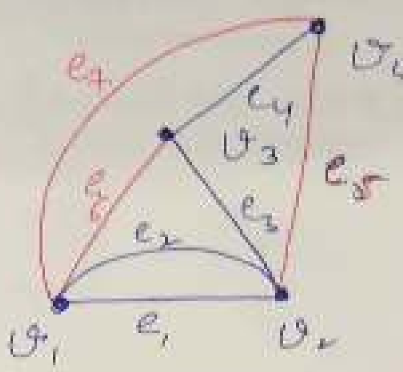
$$deg(u) + d(v) \geq n$$

until this can no longer be done.

Example:



G

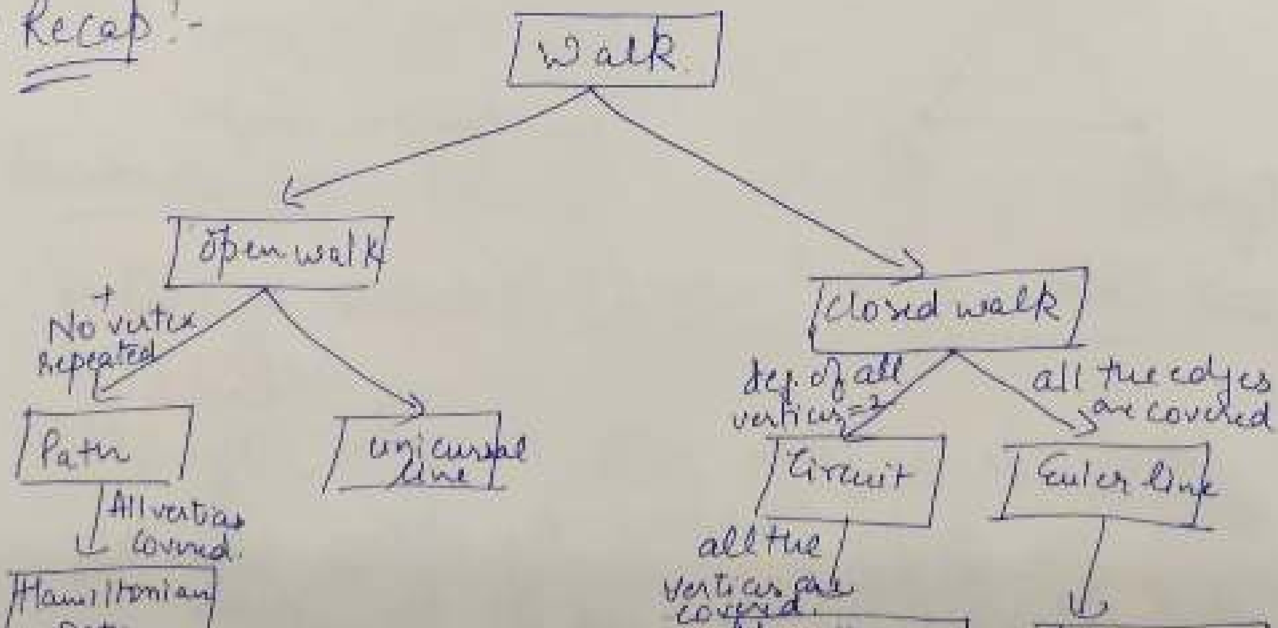


Cl(G).

Theorem:- (only statement) A graph G is Hamiltonian iff its closure ie ~~of~~ Cl(G) is Hamiltonian.

→ Travelling Salesman problem is a well known example of application of Hamiltonian graph.

→ Recap:-



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