4.4 OSCILLATORY EQUATIONS

In this section, we shall study the equation

$$u'' + a(t)u = 0, \tag{4.4.1}$$

where $a(t)$ is a real-valued and continuous function on $t_0 \leq t < \infty$. If all the nontrivial solutions of (4.4.1) have an infinite number of zeros on $t_0 \leq t < \infty$, then (4.4.1) is referred to as an oscillatory equation and these nontrivial solutions are termed oscillatory solutions. For any equation, if some solutions are oscillatory and the remaining are nonoscillatory, then the equation is called nonoscillatory.

Example 4.4.1 (i) Consider the equation $u'' + m^2u = 0$, where m is a real constant. This is an oscillatory equation because all its nontrivial solutions $\cos mt$, $\sin mt$ are oscillatory.

(ii) The equation

$$u''' - u'' + u' - u = 0 \tag{4.4.2}$$

is nonoscillatory because one of its nontrivial solutions, namely, e^t, is not oscillatory.

To derive some of the oscillatory properties of the solutions of second order linear differential equations, we need the following basic result which is a special case of the *Sturm comparison theorem.*

Theorem 4.4.1 If all the nontrivial solutions of (4.4.1) are oscillatory, $b(t)$ is continuous, and $b(t) \geq a(t)$, $t_0 \leq t < \infty$, then all the nontrivial solutions of

$$v'' + b(t)v = 0 \tag{4.4.3}$$

are oscillatory. On the other hand, if some nontrivial solutions of equation (4.4.3) are nonoscillatory and $b(t) \geq a(t)$, then some nontrivial solutions of (4.4.1) must be nonoscillatory.

Proof Let $u(t)$ and $v(t)$ be the nontrivial solutions of (4.4.1) and (4.4.3), respectively. Multiplying (4.4.3) by u, (4.4.1) by v, and subtracting, we get

$$uv'' - vu'' + (b(t) - a(t))uv = 0,$$

that is,

$$d(uv' - vu') + (b(t) - a(t))uv = 0. \tag{4.4.4}$$

Let t_1 and t_2 be any two consecutive zeros of $u(t)$ and assume that $t_0 \leq t_1 < t_2$ and that $u(t) \geq 0$ on the interval $t_1 \leq t \leq t_2$ (see Fig. 4.4.1). By integrating equation (4.4.4) from t_1 to t_2, we obtain

$$u(t_2)v'(t_2) - v(t_2)u'(t_2) - u(t_1)v'(t_1) + v(t_1)u'(t_1)$$
$$+ \int_{t_1}^{t_2} [b(s) - a(s)]u(s)v(s)\, ds = 0. \tag{4.4.5}$$

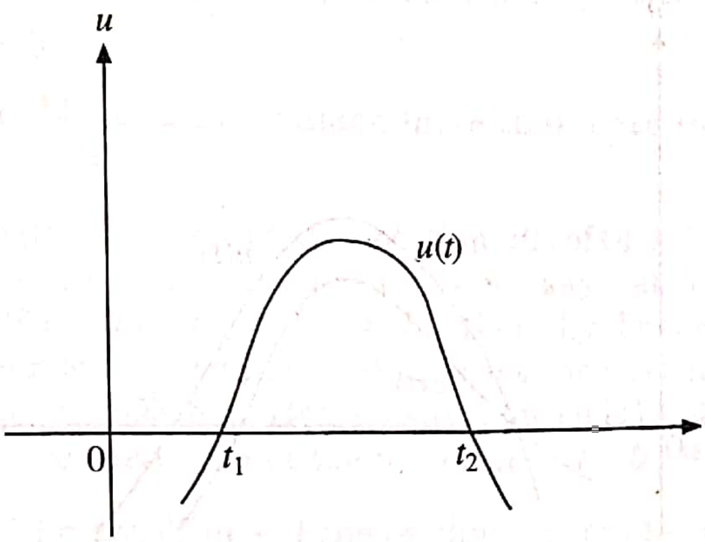

Fig. 4.4.1 Graph of function u.

Since t_1 and t_2 are two consecutive zeros of $u(t)$, we have $u(t_1) = 0$, $u(t_2) = 0$ with $u'(t_1) > 0$, $u'(t_2) < 0$. Therefore, from equation (4.4.5), we obtain

$$v(t_1)u'(t_1) - v(t_2)u'(t_2) + \int_{t_1}^{t_2} [b(s) - a(s)]u(s)v(s)\, ds = 0. \tag{4.4.6}$$

We claim that $v(t)$ has a zero on $[t_1, t_2]$. Suppose this is not true. Then, $v(t)$ does not change its sign on $[t_1, t_2]$. Since $u'(t_1) > 0$, $u'(t_2) < 0$ and $u(t)$, $b(t) - a(t)$ are nonnegative on $t_1 \le t \le t_2$, equation (4.4.6) leads to a contradiction. Hence, $v(t)$ changes its sign in the interval $[t_1, t_2]$. This implies that $v(t)$ has a zero in the interval $[t_1, t_2]$. This shows that between any two consecutive zeros of $u(t)$ there is a zero of $v(t)$. The second part of our assertion follows if an argument similar to that for the first part is used. ∎

Remark 4.4.1 From the proof just given, we can infer that, if the solutions $u(t)$ and $v(t)$ of (4.4.1) and (4.4.3), respectively, have a common zero at $t = t_1$, the solution $v(t)$ must have a zero in the interval $t_1 < t < t_2$ (see Fig. 4.4.2).

We now prove this statement. Since $v(t_1) = 0$, from (4.4.6), we have

$$v(t_2)u'(t_2) = \int_{t_1}^{t_2} [b(s) - a(s)]u(s)v(s)\, ds.$$

If $v(t)$ is positive (or negative) for all t in the interval $t_1 < t < t_2$, then this equation leads to a contradiction because $b(t) - a(t) \ge 0$, $u(t) \ge 0$ on $[t_1, t_2]$, and $u'(t_2) < 0$. Hence, $v(t)$ must change its sign on $t_1 < t < t_2$. Thus, the result follows.

Corollary 4.4.1 The nontrivial solutions of

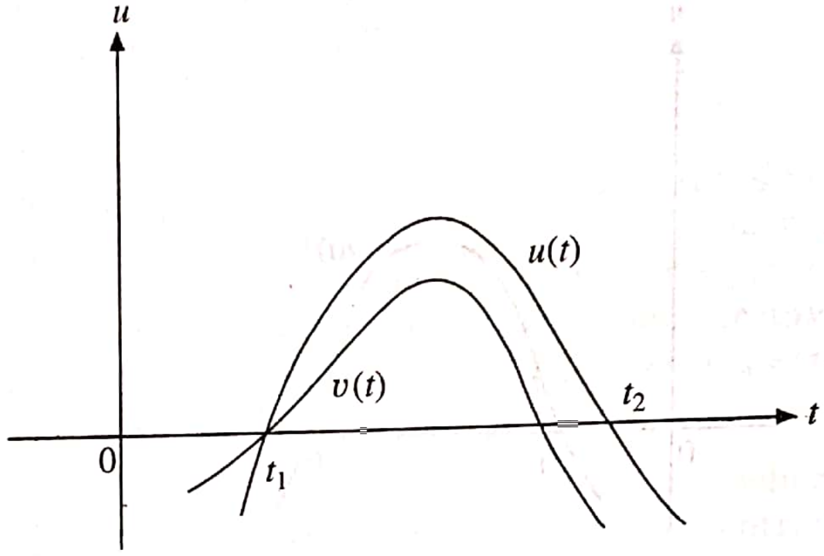

Fig. 4.4.2 Graph of functions u and v.

$$u'' + (1 + \phi(t))u = 0,$$

where $\phi(t) \to 0$ as $t \Rightarrow \infty$, are oscillatory.

Proof Since $\lim_{t \to \infty} \phi(t) = 0$, for sufficiently large t_0, we have

$$|\phi(t)| \leq \varepsilon \quad \text{for } t \geq t_0,$$

that is, $-\varepsilon \leq \phi(t) \leq \varepsilon$. Therefore,

$$1 - \varepsilon \leq 1 + \phi(t) \leq 1 + \varepsilon.$$

Choose $\varepsilon = \frac{1}{2}$. Then, we have

$$1 + \phi(t) \geq \tfrac{1}{2} \quad \text{for } t \geq t_0.$$

Since all the nontrivial solutions of $u'' + (1/2)u = 0$ are oscillatory, the result follows from Theorem 4.4.1. ∎

Corollary 4.4.2 The nontrivial solutions of

$$u'' + \phi(t)u = 0$$

are oscillatory if $\phi(t) \geq m^2 > 0$ for all t.

Proof Since all the nontrivial solutions of $u'' + m^2u = 0$ are oscillatory, the result follows from Theorem 4.4.1. ∎

Corollary 4.4.3 If $\lim_{t \to \infty} a(t) = \infty$ monotonically, then all the nontrivial solutions of (4.4.1) are oscillatory.

Proof From the hypothesis, it is clear that $a(t) > \varepsilon > 0$ for all t greater than some t_0. Since all the nontrivial solutions of $u'' + \varepsilon u = 0$, $\varepsilon > 0$, are oscillatory, the result follows from Theorem 4.4.1. ∎

Corollary 4.4.4 Any nontrivial solution of the equation

$$u'' + k(t)u = 0 \tag{4.4.7}$$

in the interval $a < t < b$ cannot vanish more than once in this interval if $k(t) \leq 0$ for all $t \in (a, b)$.

Proof Suppose there is a nontrivial solution $u(t)$ of (4.4.7) which vanishes more than once in the interval $a < t < b$, say, at $t = t_1$ and $t = t_2$ $(a < t_1 < t_2 < b)$. If $0 \geq k(t)$ for all $t \in (a, b)$, then, by Theorem 4.4.1, every nontrivial solution of $u'' = 0$ will vanish at least once in the closed interval $t_1 \leq t \leq t_2$. This is impossible. Hence, any nontrivial solution of (4.4.7) cannot vanish more than once in the interval $a < t < b$. ▮

Remark 4.4.2 The amplitude of the oscillations of (4.4.1) under certain conditions on $a(t)$ will never increase, as the following theorem shows. By Rolls' theorem, it is clear that between any two consecutive zeros of a solution there exists a zero of its derivative.

Theorem 4.4.2 Suppose $a(t)$ is continuously differentiable and $a(t) > 0$, $a'(t) \geq 0$ on $0 \leq t < \infty$. Then, if $u(t)$ is a nontrivial solution of (4.4.1) and t_1 and t_2 are two consecutive zeros of its derivative,

$$|u(t_2)| \leq |u(t_1)|. \tag{4.4.8}$$

Proof Multiplying (4.4.1) by $2u'(t)$ and integrating from t_1 to t_2, we get

$$u'^2(t_2) - u'^2(t_1) + 2\int_{t_1}^{t_2} a(t)u'(t)u(t)\,dt = 0.$$

Since t_1 and t_2 are two consecutive zeros of $u'(t)$, we have

$$\int_{t_1}^{t_2} a(t)u(t)u'(t)\,dt = 0.$$

Integrating this equation by parts, we obtain

$$a(t_2)u^2(t_2) - a(t_1)u^2(t_1) = \int_{t_1}^{t_2} u^2(t)a'(t)\,dt. \tag{4.4.9}$$

Since $u'(t)$ does not change its sign in $t_1 \leq t \leq t_2$, the solution $u(t)$ is strictly monotonic in this interval. When $a'(t) \equiv 0$ for $t_1 \leq t \leq t_2$, inequality (4.4.8) clearly holds. Therefore, we may assume that $a'(t) \not\equiv 0$ for $t_1 \leq t \leq t_2$. We now claim that (4.4.8) is true. Suppose this is not so. Then, we have

$$u^2(t_2) > u^2(t_1). \tag{4.4.10}$$

Also,

$$\int_{t_1}^{t_2} a'(t)u^2(t)\,dt < (a(t_2) - a(t_1))u^2(t_2).$$

Therefore, from relation (4.4.9), we get

$$a(t_1)(u^2(t_2) - u^2(t_1)) < 0.$$

This contradicts inequality (4.4.10). Thus, we conclude that $u^2(t_2) \leq u^2(t_1)$, and hence the result follows. ∎

Remark 4.4.3 The equation $u'' + (1/(4t^2))u = 0$ is nonoscillatory since its nontrivial solution $u(t) = t^{1/2}$ is nonoscillatory. But the following theorem shows that the equation

$$u'' + \frac{1+\varepsilon}{4t^2}u = 0, \qquad \varepsilon > 0,$$

is oscillatory.

Theorem 4.4.3 If $a(t) \geq (1+\varepsilon)/(4t^2)$, $\varepsilon > 0$, for all $t \geq t_0$, then all the nontrivial solutions of (4.4.1) are oscillatory.

Proof We know that all the nontrivial solutions of the equation

$$u'' + m^2 u = 0, \qquad m^2 > 0, \tag{4.4.11}$$

where $m > 0$ is a real constant, are oscillatory. By letting $t_1 = e^t$ in (4.4.11), we obtain

$$t_1^2\frac{d^2u}{dt_1^2} + t_1\frac{du}{dt_1} + m^2 u = 0.$$

The substitution $u = v/\sqrt{t_1}$ reduces this equation to

$$\frac{d^2v}{dt_1^2} + \frac{1+4m^2}{4t_1^2}v = 0.$$

This equation is oscillatory because the transformations we have just considered do not affect the zeros of (4.4.11). Therefore, by Theorem 4.4.1, all the nontrivial solutions of (4.4.1) are oscillatory if

$$a(t) \geq \frac{1+4m^2}{4t^2} = \frac{1+\varepsilon}{4t^2}, \qquad \varepsilon > 0. \ ∎$$

So far we have considered the cases where, in (4.4.1), $a(t) \to \infty$ as $t \to \infty$ and $a(t) \to \alpha^2 \neq 0$ as $t \to \infty$. We shall now discuss the case where $a(t) \to 0$ as $t \Rightarrow \infty$ in (4.4.1).

Since a nontrivial solution $t^{3/2}$ of $u'' - 3/(4t^2))u = 0$, $t > 0$, is unbounded as $t \to \infty$, we may want to know whether a nontrivial solution of (4.4.1) asymptotically approaches a nontrivial solution of $u'' = 0$ if $a(t) \to 0$ as $t \to \infty$. This is indeed so under a certain condition on $a(t)$, as will be evident from the following result.

Theorem 4.4.4 If $\int_1^\infty t|a(t)|\,dt < \infty$, then, for any solution $u_1(t)$ of (4.4.1), $\lim_{t\to\infty} u_1'(t)$ exists and the general solution $u(t)$ of (4.4.1) is asymptotic to $K_1 t + K_2$ for some constants K_1 and K_2, not both zero.

Proof Let $u_1(t)$ be any solution of (4.4.1) such that $u_1'(t_0) = 1$, $t_0 \in [1, \infty)$. From (4.4.1), we have

$$u_1'' = -a(t)u_1.$$

Integrating this equation twice between t_0 and t and using the order of integration, we obtain

$$u_1(t) = c_1 + t - \int_{t_0}^{t} (t - s)a(s)u_1(s)\, ds,$$

where c_1 depends upon t_0 and $u_1(t_0)$. Therefore, since $t_0 \geq 1$, we get, for $t \geq t_0$,

$$|u_1(t)| \leq (|c_1| + 1)t + t\int_{t_0}^{t} |a(s)|\,|u_1(s)|\, ds.$$

Hence, for $t \geq t_0$,

$$\frac{|u_1(t)|}{t} \leq (|c_1| + 1) + \int_{t_0}^{t} s|a(s)|\frac{|u_1(s)|}{s}\, ds.$$

Thus, it follows, from Theorem 1.5.6, that

$$\frac{|u_1(t)|}{t} \leq (|c_1| + 1) \exp \Big[\int_{t_0}^{t} s|a(s)|\, ds\Big].$$

Therefore,

$$\frac{|u_1(t)|}{t} \leq (|c_1| + 1) \exp \Big[\int_{t_0}^{\infty} s|a(s)|\, ds\Big] = c_2 < \infty.$$

Since c_1 depends upon t_0 and c_2 upon c_1, we can choose t_0 such that

$$1 - c_2\int_{t_0}^{\infty} t|a(t)|\, dt > 0.$$

This implies

$$\int_{t_0}^{t} |a(s)|\,|u_1(s)|\, ds \leq c_2\int_{t_0}^{t} s|a(s)|\, ds$$
$$< 1 \quad \text{for all } t \geq t_0.$$

Further, we have

$$u_1'(t) = 1 - \int_{t_0}^{t} a(s)u_1(s)\, ds.$$

This relation and the inequality preceding it imply that $\lim_{t \to \infty} u_1'(t)$ exists and has a nonzero limiting value. Hence, $u_1(t)$ is asymptotic to $K_1 t$ with $K_1 \neq 0$ as $t \to \infty$. Moreover, since

$$u_2(t) = u_1(t) \int_t^\infty \frac{1}{u_1^2(s)}\, ds$$

is another linearly independent solution of (4.4.1), it follows that $u_2(t)$ is asymptotic to 1 as $t \to \infty$. Hence, the general solution $u(t)$ of (4.4.1) is asymptotic to $K_1 t + K_2$ as $t \to \infty$. ∎

NUMBER OF ZEROS

We shall now take up the problem of determining the number of zeros of a nontrivial solution of the general second order differential equation

$$(p(t)u')' + q(t)u = 0, \tag{4.4.12}$$

where the functions $p(t)$ and $q(t)$ are continuous on some interval $a \le t \le b$.

Before we can obtain some of the main results, we need to know the following transformation.

Lemma 4.4.1 (Prüfer's transformation) Let $u(t)$ be a nontrivial solution of (4.4.12) existing on the interval $a \le t \le b$. Then, the transformation

$$\rho = (u^2 + p^2 u'^2)^{1/2} > 0, \qquad \phi = \tan^{-1}\left(\frac{u}{pu'}\right) \tag{4.4.13}$$

reduces (4.4.12) to

$$\phi' = \frac{1}{p(t)} \cos^2 \phi + q(t) \sin^2 \phi, \tag{4.4.14}$$

$$\rho' = -\left(q(t) - \frac{1}{p(t)}\right)\rho \sin\phi \cos\phi. \tag{4.4.15}$$

Proof From relations (4.4.13), it can be easily shown that

$$u = \rho \sin \phi, \qquad pu' = \rho \cos \phi. \tag{4.4.16}$$

Differentiating (4.4.13) with respect to t and using relations (4.4.16), we obtain the differential equations (4.4.14) and (4.4.15). ∎

Remark 4.4.4 Equation (4.4.14) has only one unknown function, namely, ϕ. By solving (4.4.14) and substituting its solution ϕ in (4.4.15), we can easily determine the function $\rho(t)$.

Transformation (4.4.13) is particularly useful in studying the zeros of the nontrivial solution $u(t)$ of (4.4.12) since $u(\tilde{t}) = 0$ for some $\tilde{t} \in [a, b]$ if and only if $\phi(\tilde{t}) \equiv 0 \bmod \pi$.

Theorem 4.4.5 Let the coefficient functions $p(t) > 0$ and $q(t)$ in (4.4.12) be continuous on $a \le t \le b$ and let $u(t)$ be a nontrivial solution of (4.4.12). Suppose $u(t)$ has exactly n (≥ 1) zeros at $t = t_1, t_2, \ldots, t_n$ ($t_1 < t_2 < \ldots < t_n$) on $[a, b]$. If $\phi(t)$ is a function defined by (4.4.13), then $\phi(t_k) = k\pi$ and

$$\phi(t) > k\pi \quad \text{for } t_k < t \le b$$

$$< k\pi \quad \text{for } a \le t < t_k,$$

where $k = 1, 2, \ldots, n$.

Proof Since $t = t_1, t_2, \ldots, t_n$ are the zeros of $u(t)$, it follows, from the second relation of (4.4.13), that $\phi(t) \equiv 0 \bmod \pi$ at $t = t_k$ $(k = 1, 2, \ldots, n)$. Thus, for these values of t, from (4.4.14), we have $\phi' = 1/p(t) > 0$. From the continuity of ϕ, this implies that $\phi(t)$ is increasing in some neighbourhood of the points $t = t_k$ $(k = 1, 2, \ldots, n)$. Hence, if $\phi(\tilde{t}) \ge n\pi$ for some $\tilde{t} \in [a, b]$, it follows that $\phi(t) > n\pi$ for all $t \in (\tilde{t}, b]$. If $\phi(\tilde{t}) \le n\pi$, then $\phi(t) < n\pi$ for all $t \in [a, \tilde{t})$. This gives the result. ∎

In what follows, we shall discuss the system of differential equations

$$(p_1(t)u')' + q_1(t)u = 0, \tag{4.4.17}$$

$$(p_2(t)u')' + q_2(t)u = 0, \tag{4.4.18}$$

where the functions $p_1(t)$, $p_2(t) > 0$, $q_1(t)$, and $q_2(t)$ are continuous on the interval $a \le t \le b$. If the inequalities

$$p_1(t) \ge p_2(t) > 0, \qquad q_1(t) \le q_2(t) \tag{4.4.19}$$

hold for all $t \in [a, b]$, then (4.4.18) is called a *Sturm majorant* of (4.4.17) on the interval $[a, b]$ and (4.4.17) a *Sturm minorant* of (4.4.18) on $[a, b]$. In addition to (4.4.19), if any one of the two strict inequalities $p_1(t) > p_2(t) > 0$, $q(t) \ne 0$, and $q_1(t) < q_2(t)$ holds at some point $t \in [a, b]$, then (4.4.18) is called a *strict Sturm majorant* of (4.4.17) on $[a, b]$ and (4.4.17) a *strict Sturm minorant* of (4.4.18).

The following result compares the number of zeros of the nontrivial solutions of (4.4.17) and (4.4.18).

Theorem 4.4.6 (Sturm's comparison theorem) Assume that

(i) $p_1(t)$, $p_2(t)$, $q_1(t)$, and $q_2(t)$ are continuous on the interval $a \le t \le b$;

(ii) $u_1(t)$ and $u_2(t)$ are the nontrivial solutions of (4.4.17) and (4.4.18), respectively;

(iii) (4.4.18) is a Sturm majorant of (4.4.17) on $[a, b]$;

(iv) the inequality

$$\frac{p_2(a)u_2'(a)}{u_2(a)} \le \frac{p_1(a)u_1'(a)}{u_1(a)} \tag{4.4.20}$$

holds [if $u_1(a) = 0$ (or $u_2(a) = 0$), then the left (or right) expression of (4.4.20) is defined as $+\infty$]; and

(v) $u_1(t)$ has exactly n (≥ 1) zeros at $t = t_1, t_2, \ldots, t_n$ ($t_1 < t_2 < \ldots < t_n$) on $(a, b]$.

Then, the solution $u_2(t)$ has at least n-zeros on $(a, t_n]$.

Proof Define a pair of continuous functions $\phi_1(t)$, $\phi_2(t)$ on the interval $a \leq t \leq b$ by the relations

$$\phi_1(t) = \tan^{-1}\left(\frac{u_1(t)}{p_1(t)u_1'(t)}\right), \tag{4.4.21}$$

$$\phi_2(t) = \tan^{-1}\left(\frac{u_2(t)}{p_2(t)u_2'(t)}\right) \tag{4.4.22}$$

for $0 \leq \phi_i < \pi$, $i = 1, 2$. Thus, from (4.4.20), it is clear that

$$0 \leq \phi_1(a) \leq \phi_2(a) < \pi. \tag{4.4.23}$$

Using the Prüfer transformation (4.4.13), it can be easily shown that

$$\phi_1' = \frac{1}{p_1(t)} \cos^2 \phi_1 + q_1(t) \sin^2 \phi_1, \tag{4.4.24}$$

$$\phi_2' = \frac{1}{p_2(t)} \cos^2 \phi_2 + q_2(t) \sin^2 \phi_2. \tag{4.4.25}$$

Now, set

$$f_i(t, \phi) \equiv \frac{1}{p_i(t)} \cos^2 \phi + q_i(t) \sin^2 \phi, \qquad i = 1, 2. \tag{4.4.26}$$

From the smooth properties on f_1 and f_2, it follows that the solutions $\phi_1(t)$ of equation (4.4.24) through $(a, \phi_1(a))$ and $\phi_2(t)$ of equation (4.4.25) through $(a, \phi_2(a))$ exist, and are unique, on the interval $a \leq t \leq b$. Assumption (iii) and relations (4.4.26) yield

$$f_1(t, \phi) \leq f_2(t, \phi) \quad \text{for } t \in [a, b] \text{ and all } \phi.$$

Hence, from inequality (4.4.23) and the theorem on differential inequality (see Section 1.5), we obtain

$$\phi_1(t) \leq \phi_2(t) \quad \text{for } t \in [a, b]. \tag{4.4.27}$$

Therefore, $\phi_1(t_n) = n\pi$ implies $\phi_2(t_n) \geq n\pi$. Now, the application of Theorem 4.4.5 yields the result. ∎

Corollary 4.4.5 In addition to the assumptions of Theorem 4.4.6, if (i) the strict inequality in (4.4.20) holds or (ii) (4.4.18) is a strict Sturm majorant of (4.4.17) on $[a, t_n]$, then the solution $u_2(t)$ has at least n-zeros on (a, t_n).

Proof of (i) Suppose the strict inequality in (4.4.20) holds. Then, we have

$\phi_1(a) < \phi_2(a)$. Let $\hat{\phi}_2(t)$ be any solution of (4.4.25) with the initial condition $\hat{\phi}_2(a) = \phi_1(a)$. Then, $\hat{\phi}_2(a) < \phi_2(a)$, and hence, from the uniqueness of the solutions of (4.4.25), it follows that

$$\hat{\phi}_2(t) < \phi_2(t) \quad \text{for } t \in [a, b].$$

Thus, the analogue of inequality (4.4.27) yields

$$\phi_1(t) \le \hat{\phi}_2(t) < \phi_2(t),$$

and therefore $\phi_2(t_n) > n\pi$. Hence, $u_2(t)$ has n-zeros on (a, t_n).

Proof of (ii) Suppose (4.4.18) is a strict Sturm majorant of (4.4.17). Now, in (4.4.20) either the strict inequality or equality holds. The strict inequality has already been covered [see proof of (i)]. We now consider the case when the equality in (4.4.20) holds and, at some point $t \in [a, t_n]$, either

$$p_1(t) > p_2(t) > 0, \qquad q(t) \ne 0,$$

or

$$q_1(t) < q_2(t)$$

also holds. From (4.4.25), we have

$$\phi_2' = \frac{1}{p_1(t)} \cos^2 \phi_2 + q_1(t) \sin^2 \phi_2 + \mu(t),$$

where

$$\mu(t) = \left(\frac{1}{p_2(t)} - \frac{1}{p_1(t)}\right) \cos^2 \phi_2 + (q_2(t) - q_1(t)) \sin^2 \phi_2.$$

Clearly, $\mu(t) \ge 0$. We claim that $u_2(t)$ has n-zeros on (a, t_n). Suppose this is not true. Then, from our preceding discussion, it follows that $\phi_1(t) = \phi_2(t)$ for $t \in [a, t_n]$. Therefore, $\phi_1'(t) = \phi_2'(t)$, and hence $\mu(t) = 0$ for $t \in [a, t_n]$. This implies

$$\left(\frac{1}{p_2} - \frac{1}{p_1}\right) \cos^2 \phi_2 + (q_2 - q_1) \sin^2 \phi_2 = 0.$$

Since $\sin \phi_2(t) = 0$ only at the zeros of $u_2(t)$, it follows that $q_1(t) = q_2(t)$ for $t \in [a, t_n]$, and also that

$$\left(\frac{1}{p_2} - \frac{1}{p_1}\right) \cos^2 \phi_2 = 0.$$

Therefore, $(1/p_2 - 1/p_1) > 0$ at some $t \in [a, t_n]$ implies $\cos^2 \phi_2(t) = 0$, that is, $u_2' = 0$. Moreover, if $q_1(t) < q_2(t)$ does not hold at any point $t \in [a, t_n]$, then $p_1(t) > p_2(t) > 0$, $q_2(t) \ne 0$, holds at some t and, hence, holds on some subinterval of $[a, t_n]$. But $u_2' = 0$, and consequently $(p_2 u_2')' = 0$, on this subinterval. This clearly contradicts the fact that $q_2(t) \neq 0$ on this interval. Thus, our claim is true. Hence, the assertion of the corollary follows. ∎

Corollary 4.4.6 (Sturm's separation theorem) Let assumptions (i), (ii), and (iii) of Theorem 4.4.6 hold. Then, if $u_1(t)$ vanishes at a pair of points t_1 and t_2 ($t_2 > t_1$) of $[a, b]$, $u_2(t)$ has at least one zero on $[t_1, t_2]$.

The proof of the corollary is direct and left as an exercise.

In particular, if $p_1 \equiv p_2$ and $q_1 \equiv q_2$, and $u_1(t)$ and $u_2(t)$ are two linearly independent solutions of (4.4.17) [≡ (4.4.18)], then we have the following assertion.

Corollary 4.4.7 The zeros of two linearly independent solutions $u_1(t)$ and $u_2(t)$ of (4.4.17) interlace, that is, between two consecutive zeros of one solution there lies a zero of the other solution.

Proof Let t_1 and t_2 be the two consecutive zeros of $u_1(t)$. Since $u_1(t)$ and $u_2(t)$ are the solutions of (4.4.17), we have

$$(p_1u_1')' + q_1u_1 = 0, \qquad (p_1u_2')' + q_1u_2 = 0.$$

Multiplying the first equation by u_2 and the second by u_1 and then subtracting, we get

$$(p_1(u_1'u_2 - u_1u_2'))' = 0.$$

Integrating this equation from t_1 to t_2, we obtain

$$(p_1(u_1'u_2 - u_1u_2'))\Big|_{t_1}^{t_2} = 0.$$

But $u_1(t_1) = 0$ and $u_1(t_2) = 0$. Therefore,

$$p_1(t_2)u_1'(t_2)u_2(t_2) = p_1(t_1)u_1'(t_1)u_2(t_1).$$

Since t_1 and t_2 are the two consecutive zeros of $u_1(t)$, $u_1'(t_1)$ and $u_1'(t_2)$ have opposite signs. Hence, from $p_1(t) > 0$ for $t \in [a, b]$, it follows that $u_2(t_1)$ and $u_2(t_2)$ must have opposite signs. Therefore, $u_2(t)$ must vanish at least once between t_1 and t_2. Finally, by interchanging the roles of u_1 and u_2, we see that their zeros interlace. ∎