CHAPTER 13

Fourier Series

Mathematicians of the eighteenth century, including Daniel Bernoulli and Leonard Euler, expressed the problem of the vibratory motion of a stretched string through partial differential equations that had no solutions in terms of "elementary functions." Their resolution of this difficulty was to introduce infinite series of sine and cosine functions that satisfied the equations. In the early nineteenth century, Joseph Fourier, while studying the problem of heat flow, developed a cohesive theory of such series. Consequently, they were named after him. Fourier series and Fourier integrals are investigated in this and the next chapter. As you explore the ideas, notice the similarities and differences with the chapters on infinite series and improper integrals.

PERIODIC FUNCTIONS

A function $f(x)$ is said to have a *period* T or to be *periodic* with period T if for all $x, f(x+T) = f(x)$, where T is a positive constant. The least value of $T > 0$ is called the *least period* or simply *the period* of $f(x)$.

EXAMPLE 1. The function $\sin x$ has periods $2\pi, 4\pi, 6\pi, \ldots,$ since $\sin(x+2\pi), \sin(x+4\pi), \sin(x+6\pi), \ldots$ all equal $\sin x$. However, 2π is the *least period* or *the period* of $\sin x$.

EXAMPLE 2. The period of $\sin nx$ or $\cos nx$, where n is a positive integer, is $2\pi/n$.

EXAMPLE 3. The period of $\tan x$ is π.

EXAMPLE 4. A constant has any positive number as period.

Other examples of periodic functions are shown in the graphs of Figures 13-1(a), (b), and (c) below.

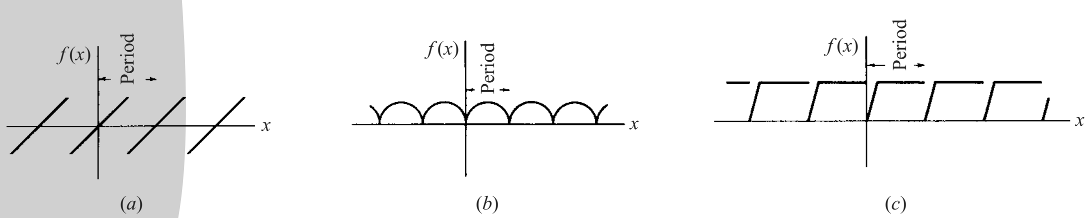

Fig. 13-1

FOURIER SERIES

Let $f(x)$ be defined in the interval $(-L, L)$ and outside of this interval by $f(x+2L) = f(x)$, i.e., $f(x)$ is $2L$-periodic. It is through this avenue that a new function on an infinite set of real numbers is created from the image on $(-L, L)$. The *Fourier series* or *Fourier expansion* corresponding to $f(x)$ is given by

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right) \tag{1}$$

where the *Fourier coefficients* a_n and b_n are

$$\begin{cases} a_n = \dfrac{1}{L}\displaystyle\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx \\ b_n = \dfrac{1}{L}\displaystyle\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx \end{cases} \qquad n = 0, 1, 2, \ldots \tag{2}$$

ORTHOGONALITY CONDITIONS FOR THE SINE AND COSINE FUNCTIONS

Notice that the Fourier coefficients are integrals. These are obtained by starting with the series, (*1*), and employing the following properties called orthogonality conditions:

$$\begin{aligned} &(a)\quad \int_{-L}^{L} \cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L}\,dx = 0 \text{ if } m \neq n \text{ and } L \text{ if } m = n \\ &(b)\quad \int_{-L}^{L} \sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}\,dx = 0 \text{ if } m \neq n \text{ and } L \text{ if } m = n \\ &(c)\quad \int_{-L}^{L} \sin\frac{m\pi x}{L}\cos\frac{n\pi x}{L}\,dx = 0. \text{ Where } m \text{ and } n \text{ can assume any positive integer values.} \end{aligned} \tag{3}$$

An explanation for calling these orthogonality conditions is given on Page 342. Their application in determining the Fourier coefficients is illustrated in the following pair of examples and then demonstrated in detail in Problem 13.4.

EXAMPLE 1. To determine the Fourier coefficient a_0, integrate both sides of the Fourier series (*1*), i.e.,

$$\int_{-L}^{L} f(x)\,dx = \int_{-L}^{L}\frac{a_0}{2}\,dx + \int_{-L}^{L}\sum_{n=1}^{\infty}\left\{a_n\cos\frac{n\pi x}{L} + b_n\sin\frac{n\pi x}{L}\right\}dx$$

Now $\int_{-L}^{L}\frac{a_0}{2}\,dx = a_0 L$, $\int_{-l}^{L}\sin\frac{n\pi x}{L}\,dx = 0$, $\int_{-L}^{L}\cos\frac{n\pi x}{L}\,dx = 0$, therefore, $a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx$

EXAMPLE 2. To determine a_1, multiply both sides of (*1*) by $\cos\frac{\pi x}{L}$ and then integrate. Using the orthogonality conditions $(3)_a$ and $(3)_c$, we obtain $a_1 = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{\pi x}{L}\,dx$. Now see Problem 13.4.

If $L = \pi$, the series (*1*) and the coefficients (*2*) or (*3*) are particularly simple. The function in this case has the period 2π.

DIRICHLET CONDITIONS

Suppose that

(1) $f(x)$ is defined except possibly at a finite number of points in $(-L, L)$

(2) $f(x)$ is periodic outside $(-L, L)$ with period $2L$

(3) $f(x)$ and $f'(x)$ are piecewise continuous in $(-L, L)$.

Then the series (*1*) with Fourier coefficients converges to

(a) $f(x)$ if x is a point of continuity

(b) $\dfrac{f(x+0)+f(x-0)}{2}$ if x is a point of discontinuity

Here $f(x+0)$ and $f(x-0)$ are the right- and left-hand limits of $f(x)$ at x and represent $\lim\limits_{\epsilon\to 0+} f(x+\epsilon)$ and $\lim\limits_{\epsilon\to 0+} f(x-\epsilon)$, respectively. For a proof see Problems 13.18 through 13.23.

The conditions (1), (2), and (3) imposed on $f(x)$ are *sufficient* but not necessary, and are generally satisfied in practice. There are at present no known necessary and sufficient conditions for convergence of Fourier series. It is of interest that continuity of $f(x)$ does not *alone* ensure convergence of a Fourier series.

ODD AND EVEN FUNCTIONS

A function $f(x)$ is called *odd* if $f(-x) = -f(x)$. Thus, $x^3, x^5 - 3x^3 + 2x, \sin x, \tan 3x$ are odd functions.

A function $f(x)$ is called *even* if $f(-x) = f(x)$. Thus, $x^4, 2x^6 - 4x^2 + 5, \cos x, e^x + e^{-x}$ are even functions.

The functions portrayed graphically in Figures 13-1(a) and 13-1(b) are odd and even respectively, but that of Fig. 13-1(c) is neither odd nor even.

In the Fourier series corresponding to an odd function, only sine terms can be present. In the Fourier series corresponding to an even function, only cosine terms (and possibly a constant which we shall consider a cosine term) can be present.

HALF RANGE FOURIER SINE OR COSINE SERIES

A half range Fourier sine or cosine series is a series in which only sine terms or only cosine terms are present, respectively. When a half range series corresponding to a given function is desired, the function is generally defined in the interval $(0, L)$ [which is half of the interval $(-L, L)$, thus accounting for the name *half range*] and then the function is specified as odd or even, so that it is clearly defined in the other half of the interval, namely, $(-L, 0)$. In such case, we have

$$\begin{cases} a_n = 0, \quad b_n = \dfrac{2}{L}\displaystyle\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx & \text{for } \textit{half range sine series} \\ b_n = 0, \quad a_n = \dfrac{2}{L}\displaystyle\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx & \text{for } \textit{half range cosine series} \end{cases} \tag{4}$$

PARSEVAL'S IDENTITY

If a_n and b_n are the Fourier coefficients corresponding to $f(x)$ and if $f(x)$ satisfies the Dirichlet conditions.

Then
$$\frac{1}{L}\int_{-L}^{L}\{f(x)\}^2\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) \tag{5}$$

(See Problem 13.13.)

Solved Problems

FOURIER SERIES

13.1. Graph each of the following functions.

$$(a)\ f(x) = \begin{cases} 3 & 0 < x < 5 \\ -3 & -5 < x < 0 \end{cases} \quad \text{Period} = 10$$

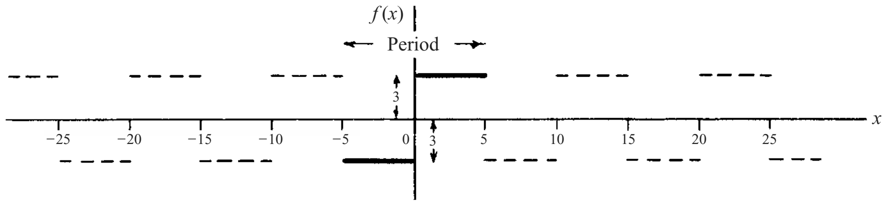

Fig. 13-3

Since the period is 10, that portion of the graph in $-5 < x < 5$ (indicated heavy in Fig. 13-3 above) is extended periodically outside this range (indicated dashed). Note that $f(x)$ is not defined at $x = 0, 5, -5, 10, -10, 15, -15$, and so on. These values are the *discontinuities* of $f(x)$.

$$(b)\ f(x) = \begin{cases} \sin x & 0 \leqq x \leqq \pi \\ 0 & \pi < x < 2\pi \end{cases} \quad \text{Period} = 2\pi$$

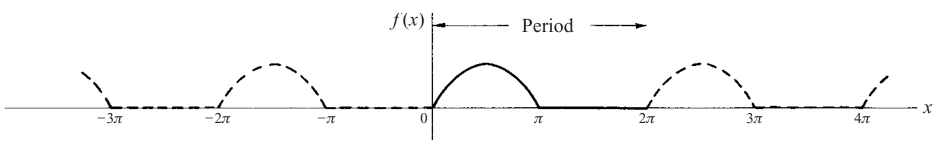

Fig. 13-4

Refer to Fig. 13-4 above. Note that $f(x)$ is defined for all x and is continuous everywhere.

$$(c)\ f(x) = \begin{cases} 0 & 0 \leqq x < 2 \\ 1 & 2 \leqq x < 4 \\ 0 & 4 \leqq x < 6 \end{cases} \quad \text{Period} = 6$$

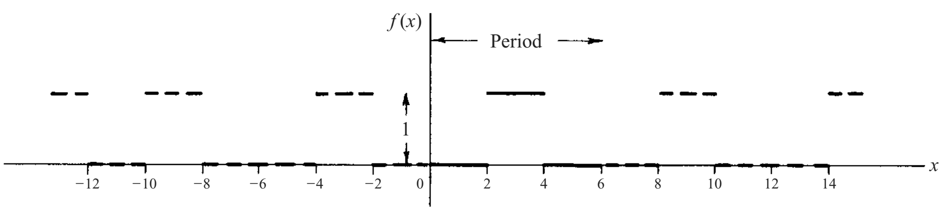

Fig. 13-5

Refer to Fig. 13-5 above. Note that $f(x)$ is defined for all x and is discontinuous at $x = \pm 2, \pm 4, \pm 8, \pm 10, \pm 14, \ldots$.

13.2. Prove $\int_{-L}^{L} \sin \frac{k\pi x}{L}\, dx = \int_{-L}^{L} \cos \frac{k\pi x}{L}\, dx = 0 \qquad \text{if } k = 1, 2, 3, \ldots.$

$$\int_{-L}^{L} \sin \frac{k\pi x}{L}\, dx = -\frac{L}{k\pi} \cos \frac{k\pi x}{L}\Big|_{-L}^{L} = -\frac{L}{k\pi} \cos k\pi + \frac{L}{k\pi} \cos(-k\pi) = 0$$

$$\int_{-L}^{L} \cos \frac{k\pi x}{L}\, dx = \frac{L}{k\pi} \sin \frac{k\pi x}{L}\Big|_{-L}^{L} = \frac{L}{k\pi} \sin k\pi - \frac{L}{k\pi} \sin(-k\pi) = 0$$

13.3. Prove (*a*) $\int_{-L}^{L} \cos \frac{m\pi x}{L} \cos \frac{n\pi x}{L}\, dx = \int_{-L}^{L} \sin \frac{m\pi x}{L} \sin \frac{n\pi x}{L}\, dx = \begin{cases} 0 & m \neq n \\ L & m = n \end{cases}$

(*b*) $\int_{-L}^{L} \sin \frac{m\pi x}{L} \cos \frac{n\pi x}{L}\, dx = 0$

where m and n can assume any of the values $1, 2, 3, \ldots$.

(*a*) From trigonometry: $\cos A \cos B = \frac{1}{2}\{\cos(A - B) + \cos(A + B)\}$, $\sin A \sin B = \frac{1}{2}\{\cos(A - B) - \cos(A + B)\}$.

Then, if $m \neq n$, by Problem 13.2,

$$\int_{-L}^{L} \cos \frac{m\pi x}{L} \cos \frac{n\pi x}{L}\, dx = \frac{1}{2}\int_{-L}^{L} \left\{\cos \frac{(m-n)\pi x}{L} + \cos \frac{(m+n)\pi x}{L}\right\} dx = 0$$

Similarly, if $m \neq n$,

$$\int_{-L}^{L} \sin \frac{m\pi x}{L} \sin \frac{n\pi x}{L}\, dx = \frac{1}{2}\int_{-L}^{L} \left\{\cos \frac{(m-n)\pi x}{L} - \cos \frac{(m+n)\pi x}{L}\right\} dx = 0$$

If $m = n$, we have

$$\int_{-L}^{L} \cos \frac{m\pi x}{L} \cos \frac{n\pi x}{L}\, dx = \frac{1}{2}\int_{-L}^{L} \left(1 + \cos \frac{2n\pi x}{L}\right) dx = L$$

$$\int_{-L}^{L} \sin \frac{m\pi x}{L} \sin \frac{n\pi x}{L}\, dx = \frac{1}{2}\int_{-L}^{L} \left(1 - \cos \frac{2n\pi x}{L}\right) dx = L$$

Note that if $m = n$ these integrals are equal to $2L$ and 0 respectively.

(*b*) We have $\sin A \cos B = \frac{1}{2}\{\sin(A - B) + \sin(A + B)\}$. Then by Problem 13.2, if $m \neq n$,

$$\int_{-L}^{L} \sin \frac{m\pi x}{L} \cos \frac{n\pi x}{L}\, dx = \frac{1}{2}\int_{-L}^{L} \left\{\sin \frac{(m-n)\pi x}{L} + \sin \frac{(m+n)\pi x}{L}\right\} dx = 0$$

If $m = n$,

$$\int_{-L}^{L} \sin \frac{m\pi x}{L} \cos \frac{n\pi x}{L}\, dx = \frac{1}{2}\int_{-L}^{L} \sin \frac{2n\pi x}{L}\, dx = 0$$

The results of parts (*a*) and (*b*) remain valid even when the limits of integration $-L, L$ are replaced by $c, c + 2L$, respectively.

13.4. If the series $A + \sum_{n=1}^{\infty}\left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L}\right)$ converges uniformly to $f(x)$ in $(-L, L)$, show that for $n = 1, 2, 3, \ldots,$

(*a*) $a_n = \frac{1}{L}\int_{-L}^{L} f(x) \cos \frac{n\pi x}{L}\, dx$, (*b*) $b_n = \frac{1}{L}\int_{-L}^{L} f(x) \sin \frac{n\pi x}{L}\, dx$, (*c*) $A = \frac{a_0}{2}$.

(*a*) Multiplying

$$f(x) = A + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right) \tag{1}$$

by $\cos\dfrac{m\pi x}{L}$ and integrating from $-L$ to L, using Problem 13.3, we have

$$\begin{aligned}\int_{-L}^{L} f(x)\cos\frac{m\pi x}{L}\,dx &= A\int_{-L}^{L}\cos\frac{m\pi x}{L}\,dx \\ &\quad + \sum_{n=1}^{\infty}\left\{a_n\int_{-L}^{L}\cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L}\,dx + b_n\int_{-L}^{L}\cos\frac{m\pi x}{L}\sin\frac{n\pi x}{L}\,dx\right\} \\ &= a_m L \qquad \text{if } m \neq 0\end{aligned}$$

Thus $$a_m = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{m\pi x}{L}\,dx \qquad \text{if } m = 1, 2, 3, \ldots$$

(*b*) Multiplying (*1*) by $\sin\dfrac{m\pi x}{L}$ and integrating from $-L$ to L, using Problem 13.3, we have

$$\begin{aligned}\int_{-L}^{L} f(x)\sin\frac{m\pi x}{L}\,dx &= A\int_{-L}^{L}\sin\frac{m\pi x}{L}\,dx \\ &\quad + \sum_{n=1}^{\infty}\left\{a_n\int_{-L}^{L}\sin\frac{m\pi x}{L}\cos\frac{n\pi x}{L}\,dx + b_n\int_{-L}^{L}\sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}\,dx\right\} \\ &= b_m L\end{aligned}$$

Thus $$b_m = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{m\pi x}{L}\,dx \qquad \text{if } m = 1, 2, 3, \ldots$$

(*c*) Integrating of (*1*) from $-L$ to L, using Problem 13.2, gives

$$\int_{-L}^{L} f(x)\,dx = 2AL \qquad \text{or} \qquad A = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$$

Putting $m = 0$ in the result of part (*a*), we find $a_0 = \dfrac{1}{L}\displaystyle\int_{-L}^{L} f(x)\,dx$ and so $A = \dfrac{a_0}{2}$.

The above results also hold when the integration limits $-L, L$ are replaced by $c, c+2L$.

Note that in all parts above, interchange of summation and integration is valid because the series is *assumed* to converge uniformly to $f(x)$ in $(-L, L)$. Even when this assumption is not warranted, the coefficients a_m and b_m as obtained above are called *Fourier coefficients* corresponding to $f(x)$, and the corresponding series with these values of a_m and b_m is called the *Fourier series* corresponding to $f(x)$. An important problem in this case is to investigate conditions under which this series actually converges to $f(x)$. Sufficient conditions for this convergence are the *Dirichlet conditions* established in Problems 13.18 through 13.23.

13.5. (*a*) Find the Fourier coefficients corresponding to the function

$$f(x) = \begin{cases} 0 & -5 < x < 0 \\ 3 & 0 < x < 5 \end{cases} \qquad \text{Period} = 10$$

(*b*) Write the corresponding Fourier series.

(*c*) How should $f(x)$ be defined at $x = -5$, $x = 0$, and $x = 5$ in order that the Fourier series will converge to $f(x)$ for $-5 \leqq x \leqq 5$?

The graph of $f(x)$ is shown in Fig. 13-6.

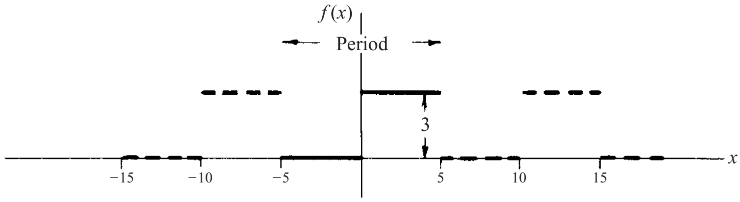

Fig. 13-6

(*a*) Period $= 2L = 10$ and $L = 5$. Choose the interval c to $c + 2L$ as -5 to 5, so that $c = -5$. Then

$$\begin{aligned} a_n &= \frac{1}{L}\int_c^{c+2L} f(x)\cos\frac{n\pi x}{L}\,dx = \frac{1}{5}\int_{-5}^{5} f(x)\cos\frac{n\pi x}{5}\,dx \\ &= \frac{1}{5}\left\{\int_{-5}^{0}(0)\cos\frac{n\pi x}{5}\,dx + \int_0^5 (3)\cos\frac{n\pi x}{5}\,dx\right\} = \frac{3}{5}\int_0^5 \cos\frac{n\pi x}{5}\,dx \\ &= \frac{3}{5}\left(\frac{5}{n\pi}\sin\frac{n\pi x}{5}\right)\Bigg|_0^5 = 0 \qquad \text{if } n \neq 0 \end{aligned}$$

If $n = 0$, $a_n = a_0 = \dfrac{3}{5}\displaystyle\int_0^5 \cos\frac{0\pi x}{5}\,dx = \frac{3}{5}\int_0^5 dx = 3.$

$$\begin{aligned} b_n &= \frac{1}{L}\int_c^{c+2L} f(x)\sin\frac{n\pi x}{L}\,dx = \frac{1}{5}\int_{-5}^{5} f(x)\sin\frac{n\pi x}{5}\,dx \\ &= \frac{1}{5}\left\{\int_{-5}^{0}(0)\sin\frac{n\pi x}{5}\,dx + \int_0^5 (3)\sin\frac{n\pi x}{5}\,dx\right\} = \frac{3}{5}\int_0^5 \sin\frac{n\pi x}{5}\,dx \\ &= \frac{3}{5}\left(-\frac{5}{n\pi}\cos\frac{n\pi x}{5}\right)\Bigg|_0^5 = \frac{3(1-\cos n\pi)}{n\pi} \end{aligned}$$

(*b*) The corresponding Fourier series is

$$\begin{aligned} \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L} + b_n\sin\frac{n\pi x}{L}\right) &= \frac{3}{2} + \sum_{n=1}^{\infty}\frac{3(1-\cos n\pi)}{n\pi}\sin\frac{n\pi x}{5} \\ &= \frac{3}{2} + \frac{6}{\pi}\left(\sin\frac{\pi x}{5} + \frac{1}{3}\sin\frac{3\pi x}{5} + \frac{1}{5}\sin\frac{5\pi x}{5} + \cdots\right) \end{aligned}$$

(*c*) Since $f(x)$ satisfies the Dirichlet conditions, we can say that the series converges to $f(x)$ at all points of continuity and to $\dfrac{f(x+0)+f(x-0)}{2}$ at points of discontinuity. At $x = -5$, 0, and 5, which are points of discontinuity, the series converges to $(3+0)/2 = 3/2$ as seen from the graph. If we redefine $f(x)$ as follows,

$$f(x) = \begin{cases} 3/2 & x = -5 \\ 0 & -5 < x < 0 \\ 3/2 & x = 0 \\ 3 & 0 < x < 5 \\ 3/2 & x = 5 \end{cases} \qquad \text{Period} = 10$$

then the series will converge to $f(x)$ for $-5 \leqq x \leqq 5$.

13.6. Expand $f(x) = x^2$, $0 < x < 2\pi$ in a Fourier series if (*a*) the period is 2π, (*b*) the period is not specified.

(*a*) The graph of $f(x)$ with period 2π is shown in Fig. 13-7 below.

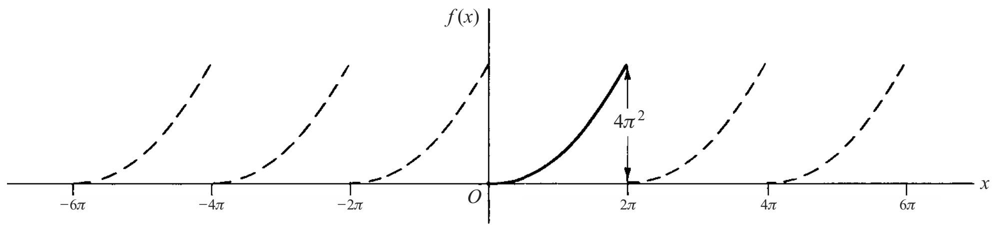

Fig. 13-7

Period $= 2L = 2\pi$ and $L = \pi$. Choosing $c = 0$, we have

$$a_n = \frac{1}{L}\int_c^{c+2L} f(x)\cos\frac{n\pi x}{L}\,dx = \frac{1}{\pi}\int_0^{2\pi} x^2\cos nx\,dx$$
$$= \frac{1}{\pi}\left\{(x^2)\left(\frac{\sin nx}{n}\right) - (2x)\left(\frac{-\cos nx}{n^2}\right) + 2\left(\frac{-\sin nx}{n^3}\right)\right\}\Bigg|_0^{2\pi} = \frac{4}{n^2}, \qquad n \neq 0$$

If $n = 0$, $a_0 = \dfrac{1}{\pi}\displaystyle\int_0^{2\pi} x^2\,dx = \dfrac{8\pi^2}{3}$.

$$b_n = \frac{1}{L}\int_c^{c+2L} f(x)\sin\frac{n\pi x}{L}\,dx = \frac{1}{\pi}\int_0^{2\pi} x^2\sin nx\,dx$$
$$= \frac{1}{\pi}\left\{(x^2)\left(-\frac{\cos nx}{n}\right) - (2x)\left(-\frac{\sin nx}{n^2}\right) + (2)\left(\frac{\cos nx}{n^3}\right)\right\}\Bigg|_0^{2\pi} = \frac{-4\pi}{n}$$

Then $f(x) = x^2 = \dfrac{4\pi^2}{3} + \displaystyle\sum_{n=1}^{\infty}\left(\frac{4}{n^2}\cos nx - \frac{4\pi}{n}\sin nx\right)$.

This is valid for $0 < x < 2\pi$. At $x = 0$ and $x = 2\pi$ the series converges to $2\pi^2$.

(*b*) If the period is not specified, the Fourier series cannot be determined uniquely in general.

13.7. Using the results of Problem 13.6, prove that $\dfrac{1}{1^2} + \dfrac{1}{2^2} + \dfrac{1}{3^2} + \cdots = \dfrac{\pi^2}{6}$.

At $x = 0$ the Fourier series of Problem 13.6 reduces to $\dfrac{4\pi^2}{3} + \displaystyle\sum_{n=1}^{\infty}\frac{4}{n^2}$.

By the Dirichlet conditions, the series converges at $x = 0$ to $\frac{1}{2}(0 + 4\pi^2) = 2\pi^2$.

Then $\dfrac{4\pi^2}{3} + \displaystyle\sum_{n=1}^{\infty}\frac{4}{n^2} = 2\pi^2$, and so $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$.

ODD AND EVEN FUNCTIONS, HALF RANGE FOURIER SERIES

13.8. Classify each of the following functions according as they are even, odd, or neither even nor odd.

(*a*) $f(x) = \begin{cases} 2 & 0 < x < 3 \\ -2 & -3 < x < 0 \end{cases}$ Period $= 6$

From Fig. 13-8 below it is seen that $f(-x) = -f(x)$, so that the function is odd.

(*b*) $f(x) = \begin{cases} \cos x & 0 < x < \pi \\ 0 & \pi < x < 2\pi \end{cases}$ Period $= 2\pi$

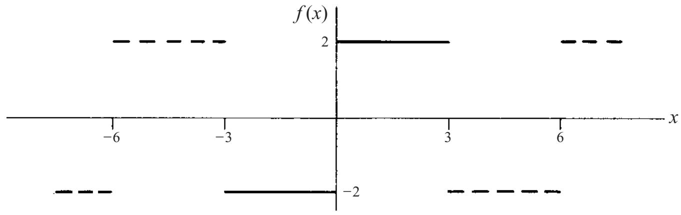

Fig. 13-8

From Fig. 13-9 below it is seen that the function is neither even nor odd.

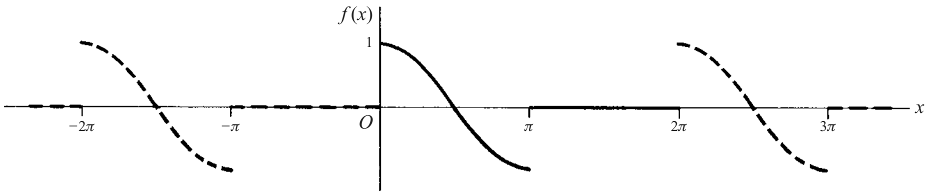

Fig. 13-9

(c) $f(x) = x(10 - x), 0 < x < 10$, Period $= 10$.

From Fig. 13-10 below the function is seen to be even.

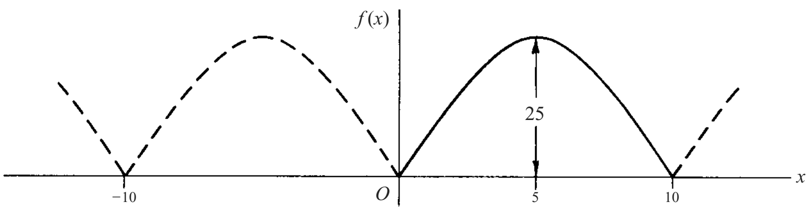

Fig. 13-10

13.9. Show that an even function can have no sine terms in its Fourier expansion.

Method 1: No sine terms appear if $b_n = 0, n = 1, 2, 3, \ldots$. To show this, let us write

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx = \frac{1}{L}\int_{-L}^{0} f(x)\sin\frac{n\pi x}{L}\,dx + \frac{1}{L}\int_{0}^{L} f(x)\sin\frac{n\pi x}{L}\,dx \tag{1}$$

If we make the transformation $x = -u$ in the first integral on the right of (*1*), we obtain

$$\begin{aligned}\frac{1}{L}\int_{-L}^{0} f(x)\sin\frac{n\pi x}{L}\,dx &= \frac{1}{L}\int_{0}^{L} f(-u)\sin\left(-\frac{n\pi u}{L}\right)du = -\frac{1}{L}\int_{0}^{L} f(-u)\sin\frac{n\pi u}{L}\,du \\ &= -\frac{1}{L}\int_{0}^{L} f(u)\sin\frac{n\pi u}{L}\,du = -\frac{1}{L}\int_{0}^{L} f(x)\sin\frac{n\pi x}{L}\,dx\end{aligned} \tag{2}$$

where we have used the fact that for an even function $f(-u) = f(u)$ and in the last step that the dummy variable of integration u can be replaced by any other symbol, in particular x. Thus, from (*1*), using (*2*), we have

$$b_n = -\frac{1}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx + \frac{1}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx = 0$$

Method 2: Assume $$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L} + b_n\sin\frac{n\pi x}{L}\right).$$

Then $$f(-x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L} - b_N\sin\frac{n\pi x}{L}\right).$$

If $f(x)$ is even, $f(-x) = f(x)$. Hence,

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L} + b_n\sin\frac{n\pi x}{L}\right) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L} - b_n\sin\frac{n\pi x}{L}\right)$$

and so $$\sum_{n=1}^{\infty} b_n\sin\frac{n\pi x}{L} = 0, \qquad \text{i.e., } f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\frac{n\pi x}{L}$$

and no sine terms appear.

In a similar manner we can show that an odd function has no cosine terms (or constant term) in its Fourier expansion.

13.10. If $f(x)$ is even, show that (*a*) $a_n = \dfrac{2}{L}\displaystyle\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx$, (*b*) $b_n = 0$.

(*a*) $$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx = \frac{1}{L}\int_{-L}^{0} f(x)\cos\frac{n\pi x}{L}\,dx + \frac{1}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx$$

Letting $x = -u$,

$$\frac{1}{L}\int_{-L}^{0} f(x)\cos\frac{n\pi x}{L}\,dx = \frac{1}{L}\int_0^L f(-u)\cos\left(\frac{-n\pi u}{L}\right)du = \frac{1}{L}\int_0^L f(u)\cos\frac{n\pi u}{L}\,du$$

since by definition of an even function $f(-u) = f(u)$. Then

$$a_n = \frac{1}{L}\int_0^L f(u)\cos\frac{n\pi u}{L}\,du + \frac{1}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx$$

(*b*) This follows by Method 1 of Problem 13.9.

13.11. Expand $f(x) = \sin x, 0 < x < \pi$, in a Fourier cosine series.

A Fourier series consisting of cosine terms alone is obtained only for an even function. Hence, we extend the definition of $f(x)$ so that it becomes even (dashed part of Fig. 13-11 below). With this extension, $f(x)$ is then defined in an interval of length 2π. Taking the period as 2π, we have $2L = 2\pi$ so that $L = \pi$.

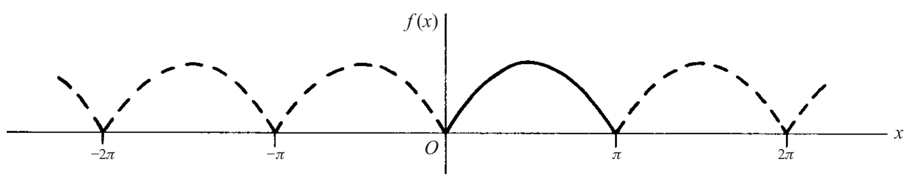

Fig. 13-11

By Problem 13.10, $b_n = 0$ and

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx = \frac{2}{\pi}\int_0^{\pi}\sin x\cos nx\,dx$$

$$= \frac{1}{\pi}\int_0^\pi \{\sin(x+nx)+\sin(x-nx)\} = \frac{1}{\pi}\left\{-\frac{\cos(n+1)x}{n+1}+\frac{\cos(n-1)x}{n-1}\right\}\Bigg|_0^\pi$$
$$= \frac{1}{\pi}\left\{\frac{1-\cos(n+1)\pi}{n+1}+\frac{\cos(n-1)\pi-1}{n-1}\right\} = \frac{1}{\pi}\left\{\frac{1+\cos n\pi}{n+1}-\frac{1+\cos n\pi}{n-1}\right\}$$
$$= \frac{-2(1+\cos n\pi)}{\pi(n^2-1)} \qquad \text{if } n \neq 1.$$

For $n = 1$, $\quad a_1 = \dfrac{2}{\pi}\displaystyle\int_0^\pi \sin x \cos x\, dx = \dfrac{2}{\pi}\dfrac{\sin^2 x}{2}\bigg|_0^\pi = 0.$

For $n = 0$, $\quad a_0 = \dfrac{2}{\pi}\displaystyle\int_0^\pi \sin x\, dx = \dfrac{2}{\pi}(-\cos x)\bigg|_0^\pi = \dfrac{4}{\pi}.$

Then
$$f(x) = \frac{2}{\pi} - \frac{2}{\pi}\sum_{n=2}^{\infty}\frac{(1+\cos n\pi)}{n^2-1}\cos nx$$
$$= \frac{2}{\pi} - \frac{4}{\pi}\left(\frac{\cos 2x}{2^2-1}+\frac{\cos 4x}{4^2-1}+\frac{\cos 6x}{6^2-1}+\cdots\right)$$

13.12. Expand $f(x) = x, 0 < x < 2$, in a half range (*a*) sine series, (*b*) cosine series.

(*a*) Extend the definition of the given function to that of the odd function of period 4 shown in Fig. 13-12 below. This is sometimes called the *odd extension* of $f(x)$. Then $2L = 4, L = 2$.

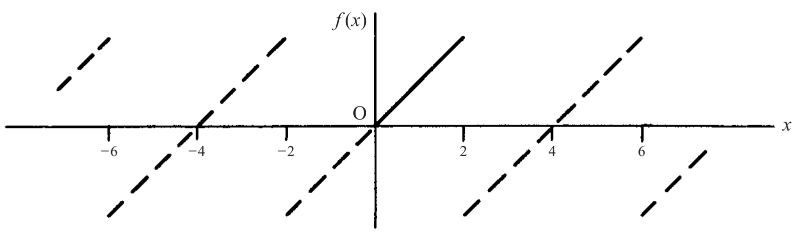

Fig. 13-12

Thus $a_n = 0$ and

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx = \frac{2}{2}\int_0^2 x\sin\frac{n\pi x}{2}\,dx$$
$$= \left\{(x)\left(\frac{-2}{n\pi}\cos\frac{n\pi x}{2}\right)-(1)\left(\frac{-4}{n^2\pi^2}\sin\frac{n\pi x}{2}\right)\right\}\Bigg|_0^2 = \frac{-4}{n\pi}\cos n\pi$$

Then
$$f(x) = \sum_{n=1}^{\infty}\frac{-4}{n\pi}\cos n\pi \sin\frac{n\pi x}{2}$$
$$= \frac{4}{\pi}\left(\sin\frac{\pi x}{2}-\frac{1}{2}\sin\frac{2\pi x}{2}+\frac{1}{3}\sin\frac{3\pi x}{2}-\cdots\right)$$

(*b*) Extend the definition of $f(x)$ to that of the even function of period 4 shown in Fig. 13-13 below. This is the *even extension* of $f(x)$. Then $2L = 4, L = 2$.

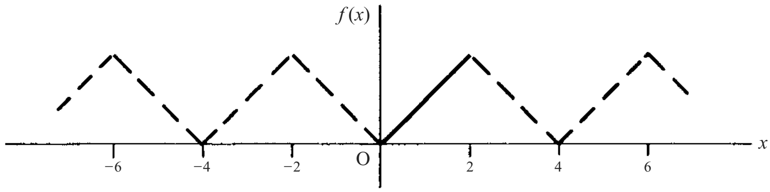

Fig. 13-13

Thus $b_n = 0$,

$$\begin{aligned} a_n &= \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx = \frac{2}{2}\int_0^2 x\cos\frac{n\pi x}{2}\,dx \\ &= \left\{(x)\left(\frac{2}{n\pi}\sin\frac{n\pi x}{2}\right) - (1)\left(\frac{-4}{n^2\pi^2}\,cos\,\frac{n\pi x}{2}\right)\right\}\Bigg|_0^2 \\ &= \frac{4}{n^2\pi^2}(\cos n\pi - 1) \qquad \text{If } n \neq 0 \end{aligned}$$

If $\;n = 0,\, a_0 = \displaystyle\int_0^2 x\,dx = 2.$

Then
$$\begin{aligned} f(x) &= 1 + \sum_{n=1}^{\infty}\frac{4}{n^2\pi^2}(\cos n\pi - 1)\cos\frac{n\pi x}{2} \\ &= 1 - \frac{8}{\pi^2}\left(\cos\frac{\pi x}{2} + \frac{1}{3^2}\cos\frac{3\pi x}{2} + \frac{1}{5^2}\cos\frac{5\pi x}{2} + \cdots\right) \end{aligned}$$

It should be noted that the given function $f(x) = x$, $0 < x < 2$, is represented *equally well* by the two *different* series in (*a*) and (*b*).

PARSEVAL'S IDENTITY

13.13. Assuming that the Fourier series corresponding to $f(x)$ converges uniformly to $f(x)$ in $(-L, L)$, prove Parseval's identity

$$\frac{1}{L}\int_{-L}^{L}\{f(x)\}^2\,dx = \frac{a_0^2}{2} + \Sigma(a_n^2 + b_n^2)$$

where the integral is assumed to exist.

If $f(x) = \dfrac{a_0}{2} + \displaystyle\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L} + b_n\sin\frac{n\pi x}{L}\right)$, then multiplying by $f(x)$ and integrating term by term from $-L$ to L (which is justified since the series is uniformly convergent) we obtain

$$\begin{aligned} \int_{-L}^{L}\{f(x)\}^2\,dx &= \frac{a_0}{2}\int_{-L}^{L} f(x)\,dx + \sum_{n=1}^{\infty}\left\{a_n\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx + b_n\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx\right\} \\ &= \frac{a_0^2}{2}L + L\sum_{n=1}^{\infty}(a_n^2 + b_n^2) \qquad (1) \end{aligned}$$

where we have used the results

$$\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx = La_n, \qquad \int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx = Lb_n, \qquad \int_{-L}^{L} f(x)\,dx = La_0 \qquad (2)$$

obtained from the Fourier coefficients.

The required result follows on dividing both sides of (*1*) by L. Parseval's identity is valid under less restrictive conditions than that imposed here.

13.14. (*a*) Write Parseval's identity corresponding to the Fourier series of Problem 13.12(*b*).

(*b*) Determine from (*a*) the sum S of the series $\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\cdots+\frac{1}{n^4}+\cdots$.

(*a*) Here $L=2$, $a_0=2$, $a_n=\frac{4}{n^2\pi^2}(\cos n\pi-1)$, $n\neq 0$, $b_n=0$.

Then Parseval's identity becomes

$$\frac{1}{2}\int_{-2}^{2}\{f(x)\}^2\,dx=\frac{1}{2}\int_{-2}^{2}x^2\,dx=\frac{(2)^2}{2}+\sum_{n=1}^{\infty}\frac{16}{n^4\pi^4}(\cos n\pi-1)^2$$

or $\frac{8}{3}=2+\frac{64}{\pi^4}\left(\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+\cdots\right)$, i.e., $\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+\cdots=\frac{\pi^4}{96.}$

(*b*)
$$\begin{aligned} S=\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\cdots &= \left(\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+\cdots\right)+\left(\frac{1}{2^4}+\frac{1}{4^4}+\frac{1}{6^4}+\cdots\right)\\ &=\left(\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+\cdots\right)+\frac{1}{2^4}\left(\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\cdots\right)\\ &=\frac{\pi^4}{96}+\frac{S}{16},\quad \text{from which } S=\frac{\pi^4}{90}\end{aligned}$$

13.15. Prove that for all positive integers M,

$$\frac{a_0^2}{2}+\sum_{n=1}^{M}(a_n^2+b_n^2)\leqq\frac{1}{L}\int_{-L}^{L}\{f(x)\}^2\,dx$$

where a_n and b_n are the Fourier coefficients corresponding to $f(x)$, and $f(x)$ is assumed piecewise continuous in $(-L, L)$.

Let
$$S_M(x)=\frac{a_0}{2}+\sum_{n=1}^{M}\left(a_n\cos\frac{n\pi x}{L}+b_n\sin\frac{n\pi x}{L}\right)\tag{1}$$

For $M=1,2,3,\ldots$ this is the sequence of partial sums of the Fourier series corresponding to $f(x)$.

We have

$$\int_{-L}^{L}\{f(x)-S_M(x)\}^2\,dx\geqq 0\tag{2}$$

since the integrand is non-negative. Expanding the integrand, we obtain

$$2\int_{-L}^{L}f(x)\,S_M(x)\,dx-\int_{-L}^{L}S_M^2(x)\,dx\leqq\int_{-L}^{L}\{f(x)\}^2\,dx\tag{3}$$

Multiplying both sides of (*1*) by $2f(x)$ and integrating from $-L$ to L, using equations (*2*) of Problem 13.13, gives

$$2\int_{-L}^{L}f(x)\,S_M(x)\,dx=2L\left\{\frac{a_0^2}{2}+\sum_{n=1}^{M}(a_n^2+b_n^2)\right\}\tag{4}$$

Also, squaring (*1*) and integrating from $-L$ to L, using Problem 13.3, we find

$$\int_{-L}^{L}S_M^2(x)\,dx=L\left\{\frac{a_0^2}{2}+\sum_{n=1}^{M}(a_n^2+b_n^2)\right\}\tag{5}$$

Substitution of (*4*) and (*5*) into (*3*) and dividing by L yields the required result.

Taking the limit as $M \to \infty$, we obtain *Bessel's inequality*

$$\frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) \leqq \frac{1}{L}\int_{-L}^{L}\{f(x)\}^2\,dx \tag{6}$$

If the equality holds, we have Parseval's identity (Problem 13.13).

We can think of $S_M(x)$ as representing an *approximation* to $f(x)$, while the left-hand side of (*2*), divided by $2L$, represents the *mean square error* of the approximation. Parseval's identity indicates that as $M \to \infty$ the mean square error approaches zero, while Bessels' inequality indicates the possibility that this mean square error does not approach zero.

The results are connected with the idea of *completeness* of an orthonormal set. If, for example, we were to leave out one or more terms in a Fourier series (say $\cos 4\pi x/L$, for example), we could never get the mean square error to approach zero no matter how many terms we took. For an analogy with three-dimensional vectors, see Problem 13.60.

DIFFERENTIATION AND INTEGRATION OF FOURIER SERIES

13.16. (*a*) Find a Fourier series for $f(x) = x^2$, $0 < x < 2$, by integrating the series of Problem 13.12(*a*).
(*b*) Use (*a*) to evaluate the series $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2}$.

(*a*) From Problem 13.12(*a*),

$$x = \frac{4}{\pi}\left(\sin\frac{\pi x}{2} - \frac{1}{2}\sin\frac{2\pi x}{2} + \frac{1}{3}\sin\frac{3\pi x}{2} - \cdots\right) \tag{1}$$

Integrating both sides from 0 to x (applying the theorem of Page 339) and multiplying by 2, we find

$$x^2 = C - \frac{16}{\pi^2}\left(\cos\frac{\pi x}{2} - \frac{1}{2^2}\cos\frac{2\pi x}{2} + \frac{1}{3^2}\cos\frac{3\pi x}{2} - \cdots\right) \tag{2}$$

where $C = \dfrac{16}{\pi^2}\left(1 - \dfrac{1}{2^2} + \dfrac{1}{3^2} - \dfrac{1}{4^2} + \cdots\right)$.

(*b*) To determine C in another way, note that (*2*) represents the Fourier cosine series for x^2 in $0 < x < 2$. Then since $L = 2$ in this case,

$$C = \frac{a_0}{2} = \frac{1}{L}\int_0^L f(x) = \frac{1}{2}\int_0^2 x^2\,dx = \frac{4}{3}$$

Then from the value of C in (*a*), we have

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2} = 1 - \frac{1}{2^2} + \frac{1}{3^2} = \frac{1}{4^2} + \cdots = \frac{\pi^2}{16}\cdot\frac{4}{3} = \frac{\pi^2}{12}$$

13.17. Show that term by term differentiation of the series in Problem 13.12(*a*) is not valid.

Term by term differentiation yields $2\left(\cos\dfrac{\pi x}{2} - \cos\dfrac{2\pi x}{2} + \cos\dfrac{3\pi x}{2} - \cdots\right)$.

Since the nth term of this series does not approach 0, the series does not converge for any value of x.