Department of Electronics and Communication Engineering
Faculty of Engineering & Technology
University of Lucknow

Notes of

Control System
EC-603

for

Electronics & Communication Engineering
6th sem(3rd year)

Topics

a) Concept of Stability
b) Routh- Hurwitz's stability criterion

2. Stability

2.1. Concept of stability

Stability is a very important characteristic of the transient performance of a system. Any working system is designed considering its stability. Therefore, all instruments are stable with in a boundary of parameter variations.

A linear time invariant (LTI) system is stable if the following two conditions are satisfied.

(i) **Notion-1:** When the system is excited by a bounded input, output is also bounded.

Proof:

A SISO system is given by

$$\frac{C(s)}{R(s)} = G(s) = \frac{b_0 s^m + b_1 s^{m-1} + \ldots + b_m}{a_0 s^n + a_1 s^{n-1} + \ldots + a_n} \tag{9.1}$$

So,

$$c(t) = \alpha^{-1}\left[G(s)R(s)\right] \tag{9.2}$$

Using convolution integral method

$$c(t) = \int_0^\infty g(\tau) r(t-\tau) d\tau \tag{9.3}$$

$g(\tau) = \alpha^{-1} G(s) =$ impulse response of the system

Taking absolute value in both sides,

$$\left|c(t)\right| = \left|\int_0^\infty g(\tau) r(t-\tau) d\tau\right| \tag{9.4}$$

Since, the absolute value of integral is not greater than the integral of absolute value of the integrand

$$\begin{aligned} &\left|c(t)\right| \le \int_0^\infty \left|g(\tau) r(t-\tau) d\tau\right| \\ &\Rightarrow \left|c(t)\right| \le \int_0^\infty \left|g(\tau) r(t-\tau)\right| d\tau \\ &\Rightarrow \left|c(t)\right| \le \int_0^\infty \left|g(\tau)\right| \left|r(t-\tau)\right| d\tau \end{aligned} \tag{9.5}$$

Let, r(t) and c(t) are bounded as follows.

$$\begin{aligned} &\left|r(t)\right| \le M_1 < \infty \\ &\left|c(t)\right| \le M_2 < \infty \end{aligned} \tag{9.6}$$

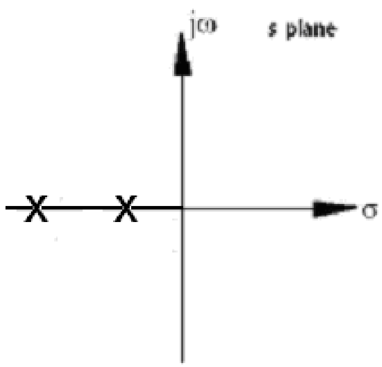
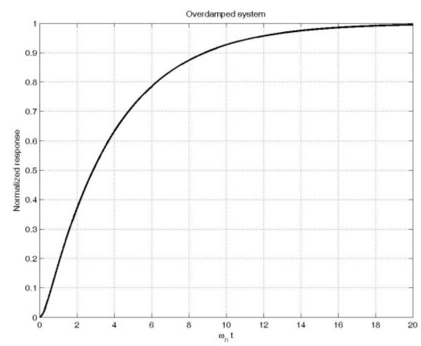
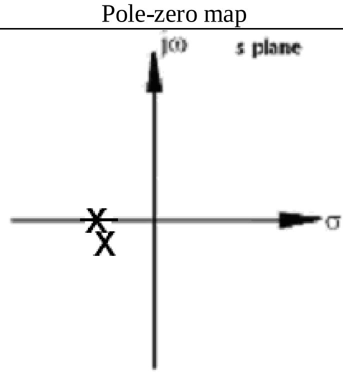
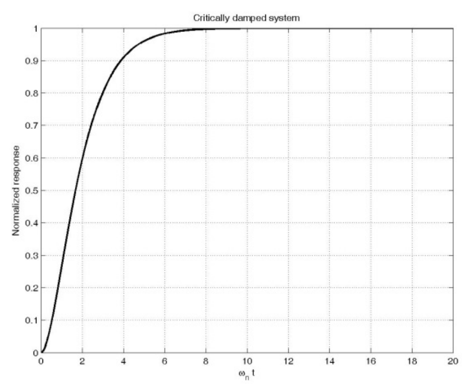
Then,

$$\left|c(t)\right| \le M_1 \int_0^{\infty} \left|g(\tau)\right| d\tau \le M_2 \tag{9.7}$$

Hence, first notion of stability is satisfied if $\int_0^{\infty} \left|g(\tau)\right| d\tau$ is finite or integrable.

(ii) **Notion-2:** In the absence of the input, the output tends towards zero irrespective of initial conditions. This type of stability is called asymptotic stability.

2.2. Effect of location of poles on stability

Pole-zero map	**Normalized response**
Over-damped close-loop poles	
	
Critically damped close-loop poles	
Pole-zero map	Normalized response
	
Under-dampedclose-loop poles	
Pole-zero map	Normalized response

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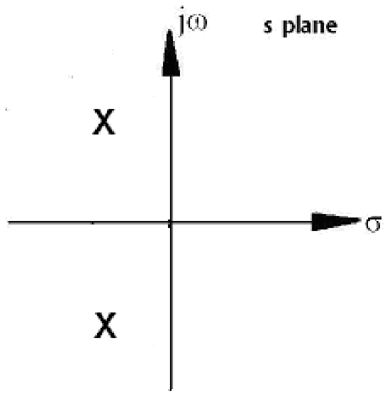
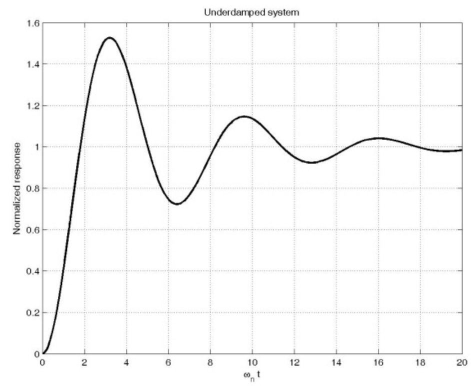
<tr><td>
</td><td>
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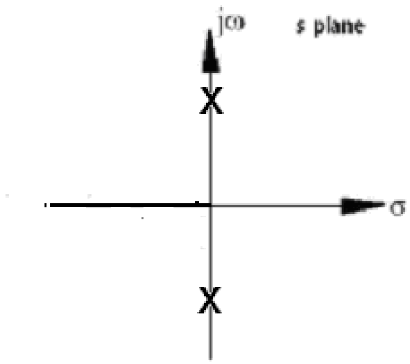
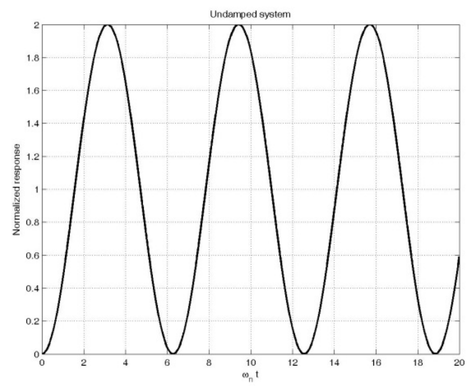
<tr><td colspan="2">Un-dampedclose-loop poles</td></tr>
<tr><td>Pole-zero map</td><td>Normalized response</td></tr>
<tr><td>
</td><td>
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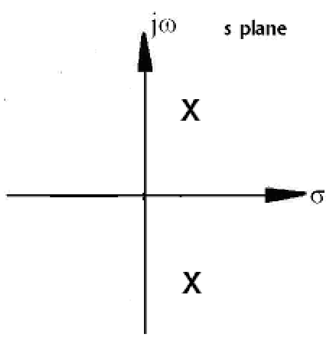
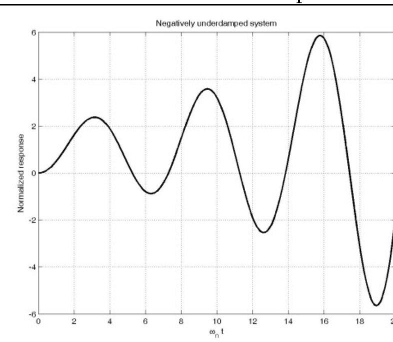
<tr><td>
</td><td>
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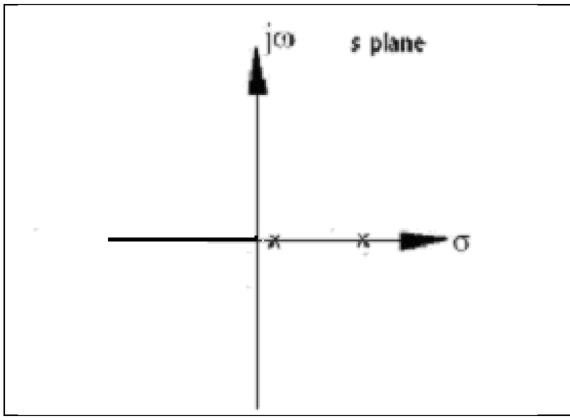

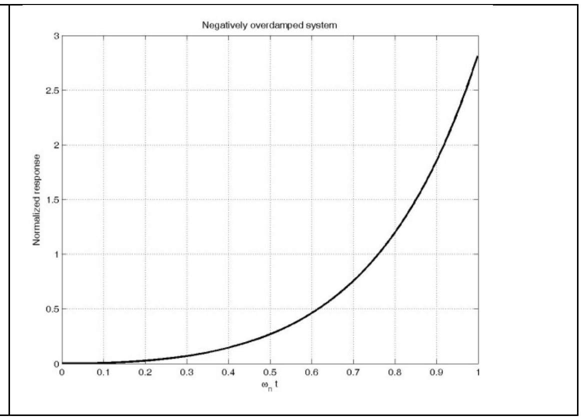

2.3. Closed-loop poles on the imaginary axis

Closed-loop can be located by replace the denominator of the close-loop response with s=jω.

Example:

1. Determine the close-loop poles on the imaginary axis of a system given below.

$$G(s) = \frac{K}{s(s+1)}$$

Solution:

Characteristics equation, $B(s) = s^2 + s + K = 0$

Replacing $s = jw$

$B(j\omega) = (j\omega)^2 + (j\omega) + K = 0$

$\Rightarrow (K - \omega^2) + j\omega = 0$

Comparing real and imaginary terms of L.H.S. with real and imaginary terms of R.H.S., we get

$\omega = \sqrt{K}$ and $\omega = 0$

Therefore, Closed-loop poles do not cross the imaginary axis.

2. Determinetheclose the imaginary axis of a system given below.
$B(s) = s^3 + 6s^2 + 8s + K = 0$.

Solution:

Characteristics equation,

$B(j\omega) = (j\omega)^3 + 6(j\omega)^2 + 8j\omega + K = 0$

$\Rightarrow (K - 6\omega^2) + j(8\omega - \omega^3) = 0$

Comparing real and imaginary terms of L.H.S. with real and imaginary terms of R.H.S., we get

$\omega = \pm\sqrt{8}$ rad/s and $K = 6\omega^2 = 48$

Therefore, Close-loop poles cross the imaginary axis for K>48.

2.4. Routh-Hurwitz's Stability Criterion

General form of characteristics equation,

$$B(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots a_1 s + a_0 = 0$$

$$\Rightarrow (s - r_1)(s - r_2) \cdots (s - r_n) = 0$$

Where, r_i = Roots of the characteristics equation

2.4.1. Necessary condition of stability:

Coefficients of the characteristic polynomial must be positive.

Example:

3. Consider a third order polynomial $B(s) = s^3 + 3s^2 + 16s + 130$. Although the coefficients of the above polynomial are positive, determine the roots and hence prove that the rule about coefficients being positive is only a necessary condition for the roots to be in the left s-plane.

Solution:

Characteristics equation, $B(s) = s^3 + 3s^2 + 16s + 130 = 0$

By using Newton-Raphson's method $r_1 = -5$ and $r_{2,3} = 1 \pm j5$

Therefore, from the above example, the condition that coefficients of a polynomial should be positive for all its roots to be in the left s-plane is only a necessary condition.

2.4.2. Sufficient condition of stability:

2.4.2.1. Method I (using determinants)

The coefficients of the characteristics equation are represented by determinant form as follows.

$$\Delta_n = \begin{vmatrix} a_{n-1} & a_{n-3} & a_{n-5} & \cdots \\ a_n & a_{n-2} & a_{n-4} & \cdots \\ 0 & a_{n-1} & a_{n-3} & \cdots \\ & & & \end{vmatrix} \tag{9.8}$$

Here, the determinant decreases by two along the row by one down the column. For stability, the following conditions must satisfy.

$$\Delta_1 = a_{n-1} > 0, \Delta_2 = \begin{vmatrix} a_{n-1} & a_{n-3} \\ a_n & a_{n-2} \end{vmatrix} > 0, \Delta_3 = \begin{vmatrix} a_{n-1} & a_{n-3} & a_{n-5} \\ a_n & a_{n-2} & a_{n-4} \\ 0 & a_{n-1} & a_{n-3} \end{vmatrix} > 0 \cdots \tag{9.9}$$

2.4.2.2. Method II (using arrays)

The coefficients of the characteristics equation are represented by array form as follows.

$$\begin{array}{c|ccc} s^n & a_n & a_{n-2} & a_{n-4} \\ s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} \\ s^{n-2} & b_{n-1} & b_{n-3} & b_{n-5} \\ s^{n-3} & c_{n-1} & c_{n-3} & c_{n-5} \\ \vdots & & & \end{array} \tag{9.10}$$

Where,

$$\begin{aligned} b_{n-1} &= \frac{(a_{n-1})(a_{n-2}) - a_n(a_{n-3})}{a_{n-1}} \\ b_{n-3} &= \frac{(a_{n-1})(a_{n-4}) - a_n(a_{n-5})}{a_{n-1}} \\ c_{n-1} &= \frac{(b_{n-1})(a_{n-3}) - a_{n-1}(b_{n-3})}{b_{n-1}} \end{aligned} \tag{9.11}$$

For stability, the following conditions must satisfy.
The number of roots of B(s) with positive real parts is equal to the number of sign changes a_n, a_{n-1}, b_{n-1}, c_{n-1}, etc.

Example:

4. Find stability of the following system given by $G(s) = \dfrac{K}{s(s+1)}$ and $H(s) = 1$ using Routh-Hurwitz stability criterion.

Solution:

In the system, $T(s) = \dfrac{G(s)}{1+G(s)H(s)} = \dfrac{\dfrac{K}{s(s+1)}}{1+\dfrac{K}{s(s+1)}} = \dfrac{K}{s^2+s+K}$

Method-I,

Characteristics equation, $B(s) = s^2 + s + K = 0$

Here, $\Delta_1 = 1$, $\Delta_2 = \begin{vmatrix} 1 & 0 \\ 1 & K \end{vmatrix} = K$

For stability, $\Delta_1 > 0$, $\Delta_2 > 0$

The system is always stable for K>0.

Method-II,

Characteristics equation, $B(s)=s^2+s+K=0$

Here, Routh array is

$$\begin{array}{c|cc} s^2 & 1 & K \\ s^1 & 1 & 0 \\ s^0 & K & \end{array}$$

There are no sign changes in first column elements of this array.Therefore, the system is always stable for K>0.

5. Find stability of the following system given by $G(s)=\dfrac{K}{s(s+2)(s+4)}$ and $H(s)=1$ using Routh-Hurwitz stability criterion.

Solution:

In the system, $\dfrac{C(s)}{R(s)}=\dfrac{G(s)}{1+G(s)H(s)}=\dfrac{\dfrac{K}{s(s+2)(s+4)}}{1+\dfrac{K}{s(s+2)(s+4)}}=\dfrac{K}{s^3+6s^2+8s+K}$

Method-I,

General form of characteristics equation, $B(s)=a_3s^3+a_2s^2+a_1s+a_0=0$

And in this system, characteristics equation is $B(s)=s^3+6s^2+8s+K=0$

Here,sufficient condition of stability suggests

$$\Delta_1=8>0,\Delta_2=\begin{vmatrix} 6 & K \\ 1 & 8 \end{vmatrix}=(48-K)>0,$$

$$\Delta_3=\begin{vmatrix} 6 & K & 0 \\ 1 & 8 & 0 \\ 0 & 6 & K \end{vmatrix}=K(48-K)>0$$

Therefore, the system is always stable for $K<48$.

Method-II,

Characteristics equation is $B(s)=s^3+6s^2+8s+K=0$

andRouth's array

$$\begin{array}{c|cc} s^3 & 1 & 8 \\ s^2 & 6 & K \\ s^1 & \dfrac{48-K}{6} & 0 \\ s^0 & K & \end{array}$$

There are no sign changes in first column elements of this array if $K<48$. Therefore, the system is always stable for $0<K<48$.

6. Find stability of the following system given by $B(s) = s^3 + 5s^2 + 10s + 3$ using Routh-Hurwitz stability criterion.

Solution:

In this problem, given Characteristics equation is $B(s) = s^3 + 5s^2 + 10s + 3 = 0$, andRouth's array is

$$\begin{array}{c|cc} s^3 & 1 & 10 \\ s^2 & 5 & 3 \\ s^1 & 9.4 & 0 \\ s^0 & 3 & \end{array}$$

There are no sign changes in first column elements of this array. Therefore, the system is always stable.

7. Find stability of the following system given by $B(s) = s^3 + 2s^2 + 3s + 10$ using Routh-Hurwitz stability criterion.

Solution:

In this problem, given characteristics equation is

$B(s) = s^3 + 2s^2 + 3s + 10 = 0$ and

Routh's array is

$$\begin{array}{c|cc} s^3 & 1 & 3 \\ s^2 & 2 & 10 \\ s^1 & -2 & 0 \\ s^0 & 10 & \end{array}$$

There are two sign changes in first column elements of this array. Therefore, the system is unstable.

8. Examine stability of the following system given by $s^5 + 2s^4 + 4s^3 + 8s^2 + 3s + 1$ using Routh-Hurwitz stability criterion.

Solution:

In this problem, Routh's array is

$$\begin{array}{c|ccc} s^5 & 1 & 4 & 3 \\ s^4 & 2 & 8 & 1 \\ s^3 & 0 & 2.5 & \\ s^2 & \infty & & \\ s^1 & & & \\ s^0 & & & \end{array}$$

Here, the criterion fails. To remove the above difficulty, the following two methods can be used.

Method-1

(i) Replace 0 by ε(very small number) and complete the array with ε.
(ii) Examine the sign change by taking $\varepsilon \to 0$

Now, Routh's array becomes

$$\begin{array}{c|ccc}
s^5 & 1 & 4 & 3 \\
s^4 & 2 & 8 & 1 \\
s^3 & \varepsilon & 2.5 & 0 \\
s^2 & \dfrac{5-8\varepsilon}{\varepsilon} & 1 & 0 \\
s^1 & \dfrac{2.5\left(\dfrac{5-8\varepsilon}{\varepsilon}\right)-\varepsilon}{\dfrac{5-8\varepsilon}{\varepsilon}} & & \\
s^0 & 1 & &
\end{array}$$

Now putting $\varepsilon \to 0$, Routh's array becomes

$$\begin{array}{c|ccc}
s^5 & 1 & 4 & 3 \\
s^4 & 2 & 8 & 1 \\
s^3 & \varepsilon & 2.5 & 0 \\
s^2 & \dfrac{5-8\varepsilon}{\varepsilon} & 1 & 0 \\
s^1 & \dfrac{2.5\left(\dfrac{5-8\varepsilon}{\varepsilon}\right)-\varepsilon}{\dfrac{5-8\varepsilon}{\varepsilon}} & & \\
s^0 & 1 & &
\end{array}$$

There are two sign changes in first column elements of this array. Therefore, the system is unstable.

Method-2

Replace s by $\frac{1}{Z}$. The system characteristic equation $s^5+2s^4+4s^3+8s^2+3s+1=0$ becomes

$$\frac{1}{Z^5}+\frac{2}{Z^4}+\frac{4}{Z^3}+\frac{8}{Z^2}+\frac{3}{Z}+1=0$$

$$\Rightarrow Z^5+3Z^4+8Z^3+4Z^2+2Z+1=0$$

Now, Routh's array becomes

$$\begin{array}{c|ccc}
s^5 & 1 & 8 & 2 \\
s^4 & 3 & 4 & 1 \\
s^3 & 6.67 & 1.67 & 0 \\
s^2 & 3.25 & 1 & 0 \\
s^1 & -0.385 & 0 & 0 \\
s^0 & 1 & 0 & 0
\end{array}$$

There are two sign changes in first column elements of this array. Therefore, the system is unstable.

9. Examine stability of the following system given by $s^5+2s^4+2s^3+4s^2+4s+8$ using Routh-Hurwitz stability criterion.

Solution:

In this problem, Routh's array is

$$\begin{array}{c|ccc} s^5 & 1 & 2 & 4 \\ s^4 & 2 & 4 & 8 \\ s^3 & 0 & 0 & 0 \\ s^2 & & & \\ s^1 & & & \\ s^0 & & & \end{array}$$

Here, the criterion fails. To remove the above difficulty, the following two methods can be used.

The auxillary equation is

$$A(s)=2s^4+4s^2+8$$

$$\Rightarrow \frac{dA(s)}{ds}=8s^3+8s$$

Now, the array is rewritten as follows.

$$\begin{array}{c|ccc} s^5 & 1 & 2 & 4 \\ s^4 & 2 & 4 & 8 \\ s^3 & 8 & 8 & 0 \\ s^2 & 2 & 8 & 0 \\ s^1 & -24 & 0 & \\ s^0 & 8 & & \end{array}$$

There are two sign changes in first column elements of this array. Therefore, the system is unstable.

10. Examine stability of the following system given by $s^4+5s^3+2s^2+3s+1=0$ using Routh-Hurwitz stability criterion. Find the number of roots in the right half of the s-plane.

Solution:

In this problem, Routh's array is

$$\begin{array}{c|ccc} s^4 & 1 & 2 & 2 \\ s^3 & 5 & 3 & 0 \\ s^2 & 1.4 & 2 & \\ s^1 & -4.14 & 0 & \\ s^0 & 2 & & \end{array}$$

There are two sign changes in first column elements of this array. Therefore, the system is unstable. There are two poles in the right half of the s-plane.

2.4.3. Advantages of Routh-Hurwitz stability

(i) Stability can be judged without solving the characteristic equation
(ii) Less calculation time
(iii) The number of roots in RHP can be found in case of unstable condition
(iv) Range of value of K for system stability can be calculated
(v) Intersection point with the jw-axis can be calculated
(vi) Frequency of oscillation at steady-state is calculated

2.4.4. Advantages of Routh-Hurwitz stability

(i) It is valid for only real coefficient of the characteristic equation
(ii) Unable to give exact locations of closed-loop poles
(iii) Does not suggest methods for stabilizing an unstable system
(iv) Applicable only to the linear system