Riemann Zeta function

for any positive real r and a complex number z,
$r^z = \exp(z \log r)$ is well defined complex number. and if $z = x+iy$

$$|r^z| = |\exp\{(x+iy)\log r\}|$$
$$= |e^{x\log r} \cdot e^{iy\log r}|$$
$$= |e^{x\log r}|\cdot|e^{iy\log r}|$$
$$= |e^{x\log r}| \qquad \because |e^{iy\log r}| = 1$$
$$= r^x$$
$$= r^{Re(z)}$$

$\therefore$ if z is such that $Re(z) > 1$, then $\sum_{n=1}^{\infty} \frac{1}{n^z}$ Converges. (Proved in convergence of series)

Let D_1 be the half-plane, $D_1 = \{z \in \mathbb{C} : Re(z) > 1\}$.
It is an open set and for each $z \in D_1$,
$\sum_{n=1}^{\infty} \frac{1}{n^z}$ is a well-defined complex number.

The Riemann Zeta function is defined on D_1 by $\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}$

Note: for any $\delta > 0$ and $\forall z \in D_1$ such that $Re(z) \geq 1+\delta$

$$\sum_{n=1}^{\infty} \left|\frac{1}{n^z}\right| = \sum_{n=1}^{\infty} \left|\frac{1}{n^{Re(z)}}\right| \leq \sum_{n=1}^{\infty} \frac{1}{n^{1+\delta}}$$, which is convergent (as $\delta > 0$). So $\zeta(z)$ ~~is~~ Converges for $Re(z) > 1$.

Consider the function $g_n(z) = \frac{1}{n^{z-1}} - \frac{1}{(n+1)^{z-1}}$,

which is entire, with a zero at $z=1$.

Hence for each n, $\varphi_n(z)$ defined by

$$\varphi_n(z) = \frac{1}{z-1}\left(\frac{1}{n^{z-1}} - \frac{1}{(n+1)^{z-1}}\right),$$

which is also entire. We can write

$$\varphi_n(z) = \int_0^1 \frac{1}{(n+t)^z}\,dt$$

$$= \int_n^{n+1} \frac{1}{t^z}\,dt \qquad (1)$$

Similarly, for each n, the function $f_n(z)$ defined by

$$f_n(z) = \frac{1}{(n+1)^z} - \varphi_n(z)$$

$$= \frac{1}{(n+1)^z} - \int_0^1 \frac{1}{(n+t)^z}\,dt$$

$$= -z\int_0^1 \frac{t}{(n+t)^{z+1}}\,dt. \qquad (2)$$

which is also entire being the difference of two entire functions.

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$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}$$

$$= 1 + \frac{1}{2^z} + \frac{1}{3^z} + \cdots$$

$$= 1 + \sum_{n=1}^{\infty} \frac{1}{(n+1)^z}$$

$$= 1 + \sum_{n=1}^{\infty} \left\{ \int_0^1 \frac{dt}{(n+t)^z} + \frac{1}{(n+1)^z} - \int_0^1 \frac{dt}{(n+t)^z} \right\}$$

$$= 1 + \sum_{n=1}^{\infty} \int_0^1 \frac{dt}{(n+t)^z} - z \sum_{n=1}^{\infty} \int_0^1 \frac{t\,dt}{(n+t)^{z+1}} \quad \text{, (from (2))}$$

$$= 1 + \sum_{n=1}^{\infty} \int_n^{n+1} \frac{1}{t^z}\,dt - z \sum_{n=1}^{\infty} \int_0^1 \frac{t\,dt}{(n+t)^{z+1}}$$

(from (1))

$$\cancel{= 1 + \sum_{n=1}^{\infty}} = 1 + \int_1^{\infty} \frac{dt}{t^z} - z \sum_{n=1}^{\infty} \int_0^1 \frac{t\,dt}{(n+t)^{z+1}}$$

$$= 1 + \frac{1}{z-1} - z \sum_{n=1}^{\infty} \int_0^1 \frac{t\,dt}{(n+t)^{z+1}}$$

Hence $\zeta(z)$ is meromorphic on $\{z : Re(z) > 0\}$ with simple pole at $z = 1$.

Euler's Theorem Prove that for $Re(z) > 1$

$$\frac{1}{\zeta(z)} = \prod_{j=1}^{\infty}\left(1 - \frac{1}{p_j^z}\right)$$

Over all primes p_j.

Proof Since $\zeta(z) = 1 + \frac{1}{2^z} + \frac{1}{3^z} + \cdots + \frac{1}{n^z}$

(As 2 is the first prime)

$$\therefore \left(1 - \frac{1}{2^z}\right)\zeta(z) = \left(1 - \frac{1}{2^z}\right)\left(1 + \frac{1}{2^z} + \frac{1}{3^z} + \cdots\right)$$

$$= \left(1 + \frac{1}{2^z} + \frac{1}{3^z} + \cdots\right)$$

$$- \frac{1}{2^z}\left(1 + \frac{1}{2^z} + \frac{1}{3^z} + \cdots\right)$$

$$= 1 + \frac{1}{2^z} + \frac{1}{3^z} + \cdots - \frac{1}{2^z} - \frac{1}{4^z} - \frac{1}{6^z} - \cdots$$

(multiple of 2 cancell each other) $= 1 + \frac{1}{3^z} + \frac{1}{5^z} + \frac{1}{7^z} + \cdots$

3 is second prime

$$\therefore \left(1 - \frac{1}{3^z}\right)\left(1 - \frac{1}{2^z}\right)\zeta(z) = \left(1 - \frac{1}{3^z}\right)\left(1 + \frac{1}{3^z} + \frac{1}{5^z} + \frac{1}{7^z} + \cdots\right)$$

$$= \left(1 + \frac{1}{3^z} + \frac{1}{5^z} + \frac{1}{7^z} + \cdots\right)$$

$$- \frac{1}{3^z}\left(1 + \frac{1}{3^z} + \frac{1}{5^z} + \frac{1}{7^z} + \cdots\right)$$

$$= \left(1 + \frac{1}{3^z} + \frac{1}{5^z} + \frac{1}{7^z} + \cdots\right)$$

$$- \frac{1}{3^z} - \frac{1}{9^z} - \frac{1}{15^z} - \frac{1}{21^z} - \cdots$$

(multiple of 3 cancell each other) $= 1 + \frac{1}{5^z} + \frac{1}{7^z} + \frac{1}{11^z} + \cdots$

Proceeding thus, if $p_1, p_2, \dots, p_k$ are the first k primes, then we get

$$\prod_{j=1}^{k}\left(1-\frac{1}{p_j^{z}}\right)\zeta(z) = 1+\sum_{m}\frac{1}{m^{z}}$$

where m consists of those positive integers, which do not have $p_1, p_2, \dots, p_k$ as a factor.
Thus letting $k \to \infty$, we have

$$\prod_{j=1}^{\infty}\left(1-\frac{1}{p_j^{z}}\right)\zeta(z) = 1$$

or $$\boxed{\frac{1}{\zeta(z)} = \prod_{j=1}^{\infty}\left(1-\frac{1}{p_j^{z}}\right)}$$

Q Let $d(n)$ be the number of divisors of n, show that

$$\zeta^{2}(z) = \sum_{n=1}^{\infty}\frac{d(n)}{n^{z}}$$

Solⁿ $$\zeta(z) = 1+\frac{1}{2^{z}}+\frac{1}{3^{z}}+\frac{1}{4^{z}}+\dots$$

$$\zeta^{2}(z) = \left(1+\frac{1}{2^{z}}+\frac{1}{3^{z}}+\frac{1}{4^{z}}+\dots\right)\left(1+\frac{1}{2^{z}}+\frac{1}{3^{z}}+\frac{1}{4^{z}}+\dots\right)$$

$$= 1+\frac{2}{2^{z}}+\frac{2}{3^{z}}+\frac{3}{4^{z}}+\frac{2}{5^{z}}+\dots$$

$\because \frac{1}{2^{z}}\cdot\frac{1}{2^{z}}=\frac{1}{4^{z}}$

Since no. of divisor of 2 is 2
As 1 and 2 are the divisor of 2
$\therefore d(2) = 2$
Divisor of 3 are 1 and 3
$\therefore d(3) = 2$

Divisor of 4 are 1, 2 and 4

$\therefore d(4) = 3$

$\therefore \cancel{\zeta^2(z) = \sum_{n}}$

$$\zeta^2(z) = \frac{d(1)}{1^2} + \frac{d(2)}{2^2} + \frac{d(3)}{3^2} + \frac{d(4)}{4^2} + \cdots$$

or $$\zeta^2(z) = \sum_{n=1}^{\infty} \frac{d(n)}{n^2}.$$

Q If $\mu(n)$ denotes the Möbius function, where

$$\mu(n) = \begin{cases} (-1)^k, & \text{if } n \text{ is the product of } k \text{ distinct primes,} \\ 1, & \text{if } n = 1 \\ 0, & \text{if } n \text{ contains any prime factor to the power greater than one} \end{cases}$$

Prove that $\dfrac{1}{\zeta(z)} = \displaystyle\sum_{n=1}^{\infty} \frac{\mu(n)}{n^2}$.

Sol^n

$$\zeta(z) = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$$

$$\therefore \frac{1}{\zeta(z)} = \prod_{j=1}^{\infty}\left(1 - \frac{1}{p_j^2}\right) \quad \text{for all primes } p_j$$

$$= \left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{5^2}\right)\left(1 - \frac{1}{7^2}\right)\cdots$$

$$= 1 - \frac{1}{2^2} - \frac{1}{3^2} + 0.\frac{1}{4^2} - \frac{1}{5^2} + \frac{1}{6^2} - \frac{1}{7^2} + \cdots$$

$\because \mu(1) = 1$

$\mu(2) = (-1)^1 = -1$; 2 is a prime

$\mu(3) = (-1)^1 = -1$ 3. is a prime

$\because \frac{1}{2^2} \cdot \frac{1}{3^2} = \frac{1}{6^2}$

$\mu(4) = 0$, $\because$ $4 = 2^2$. (power greater than one)

$\mu(5) = (-1)^1 = -1$

$\mu(6) = (-1)^2 = 1$, product of two primes 2 & 3.

$\mu(7) = (-1)^1 = -1$

$$\therefore \frac{1}{\zeta(z)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^z}.$$

Problems 1 If $\sigma(n)$ be the sum of the divisors of n. Show that

$$\zeta(z)\,\zeta(z-1) = \sum_{n=1}^{\infty} \frac{\sigma(n)}{n^z}$$

Problem 2. Show that

$$\frac{\zeta(z-1)}{\zeta(z)} = \sum_{n=1}^{\infty} \frac{\phi(n)}{n^z},$$

where $\phi(n)$ is the number of integers less than n and relatively prime to n.

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