# The Gamma function

Consider the function $\phi$ defined by the infinite product

$$\phi(z) = \prod_{j=1}^{\infty} (1 + z/j) \exp(-z/j) \qquad \text{--- (1)}$$

which is entire function with simple zeros at $-1, -2, -3, \ldots$

The function $\psi(z) = z\phi(z)$ is also entire with simple zeros at $z = 0, -1, -2, -3, \ldots$.

Let $\phi_k$ denote the finite product

$$\phi_k(z) = \prod_{j=1}^{k} (1 + z/j) \exp(-z/j) \qquad \text{--- (2)}$$

It is an entire function for each $k = 1, 2, \ldots$ and $\lim_{k \to \infty} \phi_k(z) = \phi(z)$.

Taking log on both sides in (2), we have

$$\log \phi_k(z) = \sum_{j=1}^{k} \log(1 + z/j) - \sum_{j=1}^{k} \frac{z}{j}$$

Differentiate it, we have

$$\frac{\phi_k'(z)}{\phi_k(z)} = \sum_{j=1}^{k} \frac{1}{1+z/j}\left(\frac{1}{j}\right) - \sum_{j=1}^{k} \frac{1}{j}$$

$$= \sum_{j=1}^{k} \left\{ \frac{1}{z+j} - \frac{1}{j} \right\} \qquad \text{--- (3)}$$

Similarly $$\frac{\phi_k'(z-1)}{\phi_k(z-1)} = \sum_{j=1}^{k} \left\{ \frac{1}{z-1+j} - \frac{1}{j} \right\} \qquad \text{--- (4)}$$

So that

$$\frac{\varphi_k'(z-1)}{\varphi_k(z-1)} - \frac{\varphi_k'(z)}{\varphi_k(z)} \qquad \{\text{from (3) \& (4)}\}$$

$$= \sum_{j=1}^{k} \left\{ \frac{1}{z-1+j} - \frac{1}{j} - \frac{1}{z+j} + \frac{1}{j} \right\}$$

$$= \sum_{j=1}^{k} \left\{ \frac{1}{z-1+j} - \frac{1}{z+j} \right\}$$

$$= \frac{1}{z} - \frac{1}{z+k}$$

i.e $$\boxed{\frac{\varphi_k'(z-1)}{\varphi_k(z-1)} - \frac{\varphi_k'(z)}{\varphi_k(z)} = \frac{1}{z} - \frac{1}{z+k}}$$

**Proposition:** Let $\varphi$ be the function such that

$$\varphi(z) = \prod_{j=1}^{\infty} (1 + z/_j)\, e^{-z/j}.$$

Then there exist a constant $\gamma$ such that for any $z$

$$\varphi(z-1) = \exp(\gamma).\, z\, \varphi(z).$$

**Proof:** The function $\psi_1(z) = z\varphi(z)$ is entire with simple zeros at $0, -1, -2, -3, \ldots$. Also the function $\psi_2(z) = \varphi(z-1)$ is entire with the same simple zeros. Hence by Weierstrass's factorisation theorem, there exist an entire function $\gamma(z)$ such that

$$\varphi(z-1) = \exp(\gamma(z))\, z\, \varphi(z). \quad \text{——} (1)$$

We have to show that $\gamma$ is a constant function. for this, ~~if~~ it suffices to show that

~~$\gamma'(0)$~~ $\gamma' = 0$

~~Differentiating both sides (1)~~

Taking log in (1), we get

$$\log \varphi(z-1) = \gamma(z) + \log z + \log \varphi(z)$$

Differentiate it, we have

$$\frac{\varphi'(z-1)}{\varphi(z-1)} = \gamma'(z) + \frac{1}{z} + \frac{\varphi'(z)}{\varphi(z)}$$

$$\Rightarrow \quad \gamma'(z) = \frac{\varphi'(z-1)}{\varphi(z-1)} - \frac{\varphi'(z)}{\varphi(z)} - \frac{1}{z} \quad \text{——} (2)$$

Since $$\frac{\varphi_k'(z-1)}{\varphi_k(z-1)} - \frac{\varphi_k'(z)}{\varphi_k(z)} = \frac{1}{z} - \frac{1}{z+k} \quad \text{——} (3)$$

$\because$ differentiation is a continuous operator on entire functions. Hence

$$\lim_{k\to\infty} \varphi_k' = \varphi' \quad \text{and} \quad \lim_{k\to\infty} \varphi_k = \varphi.$$

from (3),

$$\lim_{k\to\infty} \left\{ \frac{\varphi_k'(z-1)}{\varphi_k(z-1)} - \frac{\varphi_k'(z)}{\varphi_k(z)} \right\} = \lim_{k\to\infty} \left\{ \frac{1}{z} - \frac{1}{z+k} \right\}$$

$$\Rightarrow \quad \frac{\varphi'(z-1)}{\varphi(z-1)} - \frac{\varphi'(z)}{\varphi(z)} = \frac{1}{z} \quad \text{——} (4)$$

from (2) and (4), we get $\gamma'(z) = 0$

$\Rightarrow$ $\gamma$ is a Contant function.

proved

## Euler's constant

The complex number $\gamma$ in

$$\varphi(z-1) = \exp(\gamma)\, z\, \varphi(z)$$

is known as Euler's constant.

## To find the value of $\gamma$.

Since
$$\varphi(z) = \prod_{j=1}^{\infty} (1+z/j)\exp(-z/j)$$

$$\therefore \quad \varphi(0) = 1$$

$$\varphi(1) = \prod_{j=1}^{\infty} \left(1+\frac{1}{j}\right)\exp(-1/j)$$

$$= \lim_{n\to\infty} \prod_{j=1}^{n} \left(1+\frac{1}{j}\right)\exp(-1/j)$$

$$= \lim_{n\to\infty} c_n.$$

where
$$c_n = \prod_{j=1}^{n}\left(1+\frac{1}{j}\right)\exp\left(-\frac{1}{j}\right)$$

$$= \exp\left(-\sum_{j=1}^{n}\left(\frac{1}{j}\right)\right)\prod_{j=1}^{n}\left(\frac{j+1}{j}\right)$$

$$\therefore \quad \log c_n = -\sum_{j=1}^{n}\frac{1}{j} + \sum_{j=1}^{n}\{\log(j+1) - \log j\}$$

$$\Rightarrow \quad -\log c_n = \left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right) + \log 1 - \log(n+1)$$

$$= \left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right) - \log\left\{n\left(1+\frac{1}{n}\right)\right\}$$

$$= \left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right) - \log n - \log\left(1+\frac{1}{n}\right)$$

$\therefore$ $$-\lim_{n\to\infty} \log C_n = \lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\log n\right)$$

Since $\lim_{n\to\infty}\left(1+\frac{1}{n}\right) = 0.$

Let $z=1$ in the equation

$\phi(z-1) = \exp(\gamma) z\, \phi(z)$, we have

$$\phi(0) = \exp\gamma \cdot \phi(1)$$

or $$\log \phi(0) = \gamma + \log \phi(1)$$

$\Rightarrow$ $$\log 1. = \gamma + \log(\lim C_n)$$

or $$\gamma = -\lim_{n\to\infty} \log C_n$$

$$\boxed{\gamma = \lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\log n\right).}$$

Euler's constant $\gamma$ is like $e$ and $\pi$,

## Gamma function (Euler's Gamma function)

The gamma function is defined by

$$\Gamma(z) = \frac{e^{-\gamma z}}{z\,\phi(z)}, \quad z \neq 0, -1, -2, \ldots$$

where $\gamma$ is Euler's constant.

It is meromorphic function with simple zeros at $0, -1, -2, -3, \ldots$