CHAPTER 7

INTERPOLATION

Chapter Objectives

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7.1 Introduction

Suppose we are given the following values of $y = f(x)$ for a set of values of x:

x:	x_0	x_1	$x_2 \cdots x_n$
y:	Y_0	y_1	$y_2 \cdots y_n$.

Then the process of finding the value of y corresponding to any value of $x = x_i$ between x_0 and x_n is called *interpolation*. Thus *interpolation is the technique of estimating the value of a function for any intermediate value of the independent variable* while the process of computing the value of the function outside the given range is called *extrapolation*. The term interpolation however, is taken to include extrapolation.

If the function $f(x)$ is known explicitly, then the value of y corresponding to any value of x can easily be found. Conversely, if the form of $f(x)$ is not known (as is the case in most of the applications), it is very difficult to determine the exact form of $f(x)$ with the help of tabulated set of values (x_i, y_i). In such cases, $f(x)$ is replaced by a simpler function $\phi(x)$ which assumes the same values as those of $f(x)$ at the tabulated set of points. Any other value may be calculated from $\phi(x)$ which is known as the *interpolating function* or *smoothing function*. If $\phi(x)$ is a polynomial, then it called the *interpolating polynomial* and the process is called the *polynomial interpolation*. Similarly when $\phi(x)$ is a finite trigonometric series, we have trigonometric interpolation. But we shall confine ourselves to polynomial interpolation only.

The study of interpolation is based on the calculus of finite differences. We begin by deriving two important *interpolation formulae* by means of forward and backward differences of a function. These formulae are often employed in engineering and scientific investigations.

7.2 Newton's Forward Interpolation Formula

Let the function $y = f(x)$ take the values $y_0, y_1, \cdots, y_n$ corresponding to the values $x0, x1, \cdots, x_n$ of x. Let these values of x be equispaced such that $x_i = x_0 + ih$ $(i = 0, 1, \cdots)$. Assuming $y(x)$ to be a polynomial of the nth degree in x such that $y(x_0) = y_0, y(x_1) = y_1, \cdots, y(x_n) = y_n$. We can write

$$y(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + a_3(x - x_0)(x - x_1)(x - x_2) + \cdots + a_n(x - x_0)(x - x_1)\cdots(x - x_{n-1}) \quad (1)$$

Putting $x = x_0, x_1, \cdots, x_n$ successively in (1), we get

$$y_0 = a_0, y_1 = a_0 + a_1(x_1 - x_0), y_2 = a_0 + a_1(x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1)$$

and so on.

From these, we find that $a_0 = y_0, \Delta y_0 = y_1 - y_0 = a_1(x_1 - x_0) = a_1 h$

$$\therefore \quad a_1 = \frac{1}{h}\Delta y_0$$

Also $\quad \Delta y_1 = y_2 - y_1 = a_1(x_2 - x_1) + a_2(x_2 - x_0)(x_2 - x_1)$

$$= a_1 h + a_2 hh = \Delta y_0 + 2h^2 a_2$$

$$\therefore \quad a_2 = \frac{1}{2h^2}(\Delta y_1 - \Delta y_0) = \frac{1}{2!h^2}\Delta^2 y_0$$

Similarly $a_3 = \frac{1}{3!h^3}\Delta^3 y_0$ and so on.

Substituting these values in (1), we obtain

$$y(x) = y_0 + \frac{\Delta y_0}{h}(x - x_0) + \frac{\Delta^2 y_0}{2!h^2}(x - x_0)(x - x_1) + \frac{\Delta^3 y_0}{3!h^3}(x - x_0)(x - x_1)(x - x_2) + \cdots \quad (2)$$

Now if it is required to evaluate y for $x = x_0 + ph$, then

$$(x - x_0) = ph, x - x_1 = x - x_0 - (x - x_0) = ph - h = (p-1)h,$$

$$(x - x_0) = x - x_0 - (x - x_0) = (p-1)h - h = (p-2)h \quad \text{etc.}$$

Hence, writting $y(x) = y(x_0 + ph) = y_p$, (2) becomes

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \cdots + \frac{p(p-1)\cdots(p - \overline{n-1})}{3!}\Delta^n y_0 \quad (3)$$

It is called Newton's forward interpolation formula as (3) contains y_0 and the forward differences of y_0

Otherwise: Let the function $y = f(x)$ take the values $y_0, y_1, y_2, \cdots$ corresponding to the values $x_0, x_0 + h, x_0 + 2h, \cdots$ of x. Suppose it is required to evaluate $f(x)$ for $x = x_0 + ph$, where p is any real number.

For any real number p, we have defined E such that

$$E^p f(x) = f(x+ph)$$

$$y_p = f(x_0 + ph) = E^p f(x_0) = (1+\Delta)^p y_0 \qquad [\because E = 1+\Delta]$$

$$= \left\{1 + p\Delta + \frac{p(p-1)}{2!}\Delta^2 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \cdots\right\} y_0 \qquad (4)$$

[Using binomial theorem]

i.e., $$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \cdots$$

If $y = f(x)$ is a polynomial of the nth degree, then $\Delta^{n+1} y_0$ and higher differences will be zero.

Hence (4) will become

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \cdots + \frac{p(p-1)\cdots\left(p-\overline{n-1}\right)}{3!}\Delta^n y_0$$

Which is same as (3)

NOTE

Obs. 1. *This formula is used for interpolating the values of y near the beginning of a set of tabulated values and extrapolating values of y a little backward (i.e., to the left) of* y_0.

Obs. 2. *The first two terms of this formula give the linear interpolation while the first three terms give a parabolic interpolation and so on.*

7.3 Newton's Backward Interpolation Formula

Let the function $y = f(x)$ take the values y_0, y_1, y_2, … corresponding to the values x_0, $x_0 + h$, $x_0 + 2h$, $\cdots$ of x. Suppose it is required to evaluate $f(x)$ for $x = x_n + ph$, where p is any real number. Then we have

$$y_p = f(x_n + ph) = Ep\, f(x_n) = (1-\nabla)^{-p}\, y_n \qquad [\because E^{-1} = 1-\nabla]$$

$$= \left[1 + p\nabla + \frac{p(p+1)}{2!}\nabla^2 + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_0 + \cdots\right] y_n$$

[using binomial theorem]

$$i.e., \quad y_p = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \cdots \qquad (1)$$

It is called *Newton's backward interpolation formula* as (1) contains y_n and backward differences of y_n

NOTE

Obs. *This formula is used for interpolating the values of y near the end of a set of tabulated values and also for extrapolating values of y a little ahead (to the right) of* y_n

EXAMPLE 7.1

The table gives the distance in nautical miles of the visible horizon for the given heights in feet above the earth's surface:

x = height:	100	150	200	250	300	350	400
y = distance:	10.63	13.03	15.04	16.81	18.42	19.90	21.27

Find the values of y when

(*i*) $x = 160$ *ft*. (*ii*) $x = 410$.

Solution:

The difference table is as under:

x	y	Δ	Δ^2	Δ^3	Δ^4
100	10.63				
		2.40			
150	**13.03**		– 0.39		
		2.01		0.15	
200	15.04		**– 0.24**		– 0.07
		1.77		**0.08**	
250	16.81		– 0.16		**– 0.05**
		1.61		0.03	
300	18.42		– 0.13		– 0.01
		1.48		**0.02**	
350	19.90		– **0.11**		
		1.37			
400	**21.27**				

(*i*) If we take $x_0 = 160$, then $y_0 = 13.03$, $\Delta y_0 = 2.01$, $\Delta^2 y_0 = -0.24$, $\Delta^3 = 0.08$, $\Delta^4 y_0 = -0.05$

Since $x = 160$ and $h = 50$, $\quad \therefore \; p = \dfrac{x - x_0}{h} = \dfrac{10}{50} = 0.2$

$\therefore$ Using Newton's forward interpolation formula, we get

$$y_{218} = y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0 + \cdots$$

$y_{160} = 13.03 + 0.402 + 0.192 + 0.0384 + 0.00168 = 13.46$ nautical miles

(*ii*) Since $x = 410$ is near the end of the table, we use Newton's backward interpolation formula.

$\therefore$ Taking $x_n = 400$, $p = \dfrac{x - x_n}{h} = \dfrac{10}{50} = 0.2$

Using the line of backward difference

$y_n = 21.27$, $\nabla y_n = 1.37$, $\nabla^2 y_n = -0.11$, $\nabla^3 y_n = 0.02$ etc.

$\therefore$ Newton's backward formula gives

$$y_{410} = y_{400} + p\nabla y_{400} + \frac{p(p+1)}{2!}\nabla^2 y_{400} + \frac{p(p+1)(p+2)}{3!}\Delta^3 y_{400} + \frac{p(p+1)(p+2)(p+3)}{4!}\nabla^4 y_{400} + \cdots$$

$$= 21.27 + 0.2(1.37) + \frac{0.2(1.2)}{2!}(-0.11) + \frac{0.2(1.2)(2.2)}{3!}(0.02) + \frac{0.2(1.2)(2.2)(3.2)}{4!}(-0.01)$$

$$= 21.27 + 0.274 - 0.0132 + 0.0018 - 0.0007$$

$= 21.53$ nautical miles

EXAMPLE 7.2

From the following table, estimate the number of students who obtained marks between 40 and 45:

Marks:	30—40	40—50	50—60	60—70	70—80
No. of students:	31	42	51	35	31

Solution:

First we prepare the cumulative frequency table, as follows:

Marks less than (x):	40	50	60	70	80
No. of students (y_x):	31	73	124	159	190

Now the difference table is

x	y_x	Δy_x	$\Delta^2 y_x$	$\Delta^3 y_x$	$\Delta 4yx$
40	**31**				
		42			
50	73		**9**		
		51		– **25**	
60	124		**– 16**		**37**
		35		**12**	
70	159		**– 4**		
		31			
80	190				

We shall find y_{45}, *i.e., the* number of students with marks less than 45.

Taking $x_0 = 40$, $x = 45$, we have

$$p = \frac{x - x_0}{h} = \frac{5}{10} = 0.5 \qquad [\because h = 10]$$

$\therefore$ Using Newton's forward interpolation formula, we get

$$y_{45} = y_{40} + p\Delta y_{40} + \frac{p(p-1)}{2!}\Delta^2 y_{40} + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_{40} + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_{40}$$

$$= 31 + 0.5 \times 42 + \frac{(0.5)(-0.5)}{2} \times 9 + \frac{(0.5)(-0.5)(-15)}{6} \times (-25) + \frac{(0.5)(-0.5)(-15)(-2.5)}{24} \times 37$$

$= 31 + 21 - 1.125 - 1.5625 - 1.4453$

$= 47.87$, on simplification.

The number of students with marks less than 45 is 47.87, i.e., 48. But the number of students with marks less than 40 is 31.

Hence the number of students getting marks between 40 and 45 = 48 – 31 = 17.

EXAMPLE 7.3.

Find the cubic polynomial which takes the following values:

x:	0	1	2	3
$f(x)$:	1	2	1	10

Hence or otherwise evaluate $f(4)$.

Solution:

The difference table is

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
0	**1**			
		1		
1	2		**– 2**	
		– 1		**12**
2	1		**10**	
		9		
3	**10**			

We take $x_0 = 0$ and $p = \dfrac{x-0}{h} = x$ $[\because h = 1]$

$\therefore$ Using Newton's forward interpolation formula, we get

$$f(x) = f(0) + \frac{x}{1}\Delta f(0) + \frac{x(x-1)}{1.2}\Delta^2 f(0) + \frac{x(x-1)(x-2)}{1.2.3}\Delta^3 f(0)$$

$$= 1 + x(1) + \frac{x(x-1)}{2}(-2) + \frac{x(x-1)(x-2)}{6}(12)$$

$$= 2x^3 - 7x^2 + 6x + 1$$

which is the required polynomial.

To compute $f(4)$, we take $x_n = 3$, $x = 4$ so that $p = \dfrac{x - x_n}{h} = 1$ $[\because h = 1]$

NOTE **Obs.** *Using Newton's backward interpolation formula, we get*

$$f(4) = f(3) + p\nabla f(3) + \frac{p(p+1)}{1.2}\nabla^2 f(3) + \frac{p(p+1)(p+2)}{1.2.3}\nabla^3 f(3)$$

$$= 10 + 9 + 10 + 12 = 41$$

which is the same value as that obtained by substituting $x = 4$ in the cubic polynomial above.

The above example shows that if a tabulated function is a polynomial, then interpolation and extrapolation give the same values.

EXAMPLE 7.4

Using Newton's backward difference formula, construct an interpolating polynomial of degree 3 for the data: $f(-0.75) = -0.0718125$, $f(-0.5) = -0.02475$, $f(-0.25) = 0.3349375$, $f(0) = 1.10100$. Hence find $f(-1/3)$.

Solution:

The difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
– 0.75	– 0.0718125			
		0.0470625		
– 0.50	– 0.02475		0.312625	
		0.3596875		**0.09375**
– 0.25	0.3349375		**0.400375**	
		0.7660625		
0	**1.10100**			

We use Newton's backward difference formula

$$y(x) = y_3 + \frac{p}{1!}\nabla y_3 + \frac{p(p+1)}{2!}\nabla^2 y_3 + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_3$$

taking $$x_3 = 0, p = \frac{x-0}{h} = \frac{x}{0.25} = 4x \qquad [\because h = 0.25]$$

$$y(x) = 1.10100 + 4x(0.7660625) + \frac{4x(4x+1)}{2}(0.400375) + \frac{4x(4x+1)(4x+2)}{6}(0.09375)$$

$$= 1.101 + 3.06425x + 3.251x^2 + 0.81275x + x^3 + 0.75x^2 + 0.125x$$

$$= x^3 + 4.001x^2 + 4.002x + 1.101$$

Put $x = -\frac{1}{3}$, so that

$$y\left(-\frac{1}{3}\right)=\left(-\frac{1}{3}\right)^3+4.001\left(-\frac{1}{3}\right)^2+4.002\left(-\frac{1}{3}\right)+1.101$$
$$= 0.1745$$

EXAMPLE 7.5

In the table below, the values of y are consecutive terms of a series of which 23.6 is the 6th term. Find the first and tenth terms of the series:

x:	3	4	5	6	7	8	9
y:	4.8	8.4	14.5	23.6	36.2	52.8	73.9

Solution:

The difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
3	4.8				
		3.6			
4	8.4		2.5		
		6.1		0.5	
5	14.5		3.0		0
		9.1		0.5	
6	23.6		3.5		0
		12.6		0.5	
7	36.2		4.0		0
		16.6		0.5	
8	52.8		4.5		
		21.1			
9	73.9				

To find the first term, use Newton's forward interpolation formula with $x_0 = 3$, $x = 1$, $h = 1$, and $p = -2$. We have

$$y(1)=4.8+\frac{(-2)}{1}\times 3.6+\frac{(-2)(-3)}{1.2}\times 2.5+\frac{(-2)(-3)(-4)}{1.2.3}\times 0.5=3.1$$

To obtain the tenth term, u se Newton's backward interpolation formula with $x_n = 9$, $x = 10$, $h = 1$, and $p = 1$.This gives

$$y(10)=73.9+\frac{1}{1}\times 21.1+\frac{1(2)}{1.2}\times 4.5+\frac{1(2)(3)}{1.2.3}\times 0.5=100$$

EXAMPLE 7.6

Using Newton's forward interpolation formula show

$$\sum n^3 = \left\{\frac{n(n+1)}{2}\right\}^2$$

Solution:

If $s_n = sn^3$, then $s_{n+1} = \Sigma(n+1)^3$

$$\therefore \qquad \Delta s_n = s_{n+1} - s_n = \sum(n+1)^3 - \sum n^3 = (n+1)^3$$

Then $\quad \Delta^2 s_n = \Delta s_{n+1} - \Delta s_n = (n+2)^3 - (n+1)^3 = 3n^2 + 9n + 7$

$$\Delta^3 s_n = \Delta^2 s_{n+1} - \Delta^2 s_n$$

$$= \left[3(n+1)^2 + 9(n+1) + 7\right] - \left(3n^2 + 9n + 7\right) = 6n + 12$$

$$\Delta^4 s_n = \Delta^3 s_{n+1} - \Delta^3 s_n = [6(n+1) + 12] - [6n + 12] = 6$$

and $\quad \Delta^5 s_n = \Delta^5 s_n = \ldots\ldots = 0$

Since the first term of the given series is 1, therefore taking $n = 1$, $s_1 = 1$, $\Delta s_1 = 8$, $\Delta^2 s_1 = 19$, $\Delta^3 s_1 = 18$, $\Delta^4 s_1 = 6$.

Substituting these in the Newton's for war d interpolation formula, *i.e.,*

$$s = s + (n-1)\Delta s_1 + \frac{(n-1)(n-2)}{2!}\Delta^2 s_1 + \frac{(n-1)(n-2)(n-3)}{3!}\Delta^3 s_1$$

$$+ \frac{(n-1)(n-2)(n-3)(n-4)}{4!}\Delta^4 s_1$$

$$sn = 1 + 8(n-1) + \frac{19}{2}(n-1)(n-2) + 3(n-1)(n-2)(n-3)$$

$$+ \frac{1}{4}(n-1)(n-2)(n-3)(n-4) = \frac{1}{4}(n^4 + 2n^3 + n^2) = \left\{\frac{n(n+1)}{2}\right\}^2$$

Exercises 7.1

1. Using Newton 's forward formula, fin d the value of $f(1.6)$, if

x:	1	1.4	1.8	2.2
$f(x)$:	3.49	4.82	5.96	6.5

2. From the following table find y when $x = 1.85$ an d 2.4 by Newton's interpolation formula:

x:	1.7	1.8	1.9	2.0	2.1	2.2	2.3
$y = e^x$:	5.474	6.050	6.686	7.389	8.166	9.025	9.974

3. Express the value of θ in terms of x using the following data:

x:	40	50	60	70	80	90
θ:	184	204	226	250	276	304

Also find θ at $x = 43$.

4. Given $\sin 45° = 0.7071$, $\sin 50° = 0.7660$, $\sin 55° = 0.8192$, $\sin 60° = 0.8660$, find $\sin 52°$ using Newton's forward formula.

5. From the following table:

x:	0.1	0.2	0.3	0.4	0.5	0.6
$f(x)$:	2.68	3.04	3.38	3.68	3.96	4.21

find $f(0.7)$ approximately.

6. The area A of a circle of diameter d is given for the following values:

d:	80	85	90	95	100
A:	5026	5674	6362	7088	7854

Calculate the area of a circle of diameter 105

7. From the following table:

$x°$:	10	20	30	40	50	60	70	80
$\cos x$:	0.9848	0.9397	0.8660	0.7660	0.6428	0.5000	0.3420	0.1737

Calculate $\cos 25°$ and $\cos 73°$ using the Gregory-1 Newton formula.

8. A test performed on a *NPN* transistor gives the following result:

Base current f (mA)	0	0.01	0.02	0.03	0.04	0.05
Collector current I_C (mA)	0	1.2	2.5	3.6	4.3	5.34

Calculate (*i*) the value of the collector current for the base current of 0.005 mA.
(*ii*) the value of base current required for a collector correct of 4.0 mA.

9. Find $f(22)$ from the following data using Newton's backward formulae.

x:	20	25	30	35	40	45
$f(x)$:	354	332	291	260	231	204

10. Find the number of men getting wages between Rs. 10 and 15 from the following data:

Wages in Rs:	0—10	10—20	20—30	30—40
Frequency:	9	30	35	42

11. From the following data, estimate the number of persons having incomes between 2000 and 2500:

Income	Below 500	500–1000	1000–2000	2000–3000	3000–4000
No. of persons	6000	4250	3600	1500	650

12. Construct Newton's forward interpolation polynomial for the following data:

x:	4	6	8	10
y:	1	3	8	16

Hence evaluate y for $x = 5$.

13. Find the cubic polynomial which takes the following values:
$y(0) = 1$, $y(1) = 0$, $y(2) = 1$ and $y(3) = 10$.

Hence or otherwise, obtain $y(4)$.

14. Construct the difference table for the following data:

x:	0.1	0.3	0.5	0.7	0.9	1.1	1.3
$f(x)$:	0.003	0.067	0.148	0.248	0.370	0.518	0.697

Evaluate $f(0.6)$

15. Apply Newton's backward difference formula to the data below, to obtain a polynomial of degree 4 in x:

x:	1	2	3	4	5
y:	1	– 1	1	– 1	1

16. The following table gives the population of a town during the last six censuses. Estimate the increase in the population during the period from 1976 to 1978:

Year:	1941	1951	1961	1971	1981	1991
Population: (*in thousands*)	12	15	20	27	39	52

17. In the following table, the values of y are consecutive terms of a series of which 12.5 is the fifth term. Find the first and tenth terms of the series.

x:	3	4	5	6	7	8	9
y:	2.7	6.4	12.5	21.6	34.3	51.2	72.9

18. Using a polynomial of the third degree, complete the record given below of the export of a certain commodity during five years:

Year:	1989	1990	1991	1992	1993
Export: (*in tons*)	443	384	—	397	467

19. Given $u_1 = 40$, $u_3 = 45$, $u_5 = 54$, find u_2 and u_4.

20. If $u_{-1} = 10$, $u_1 = 8$, $u_2 = 10$, $u_4 = 50$, find u_0 and u_3.

21. Given $y_0 = 3$, $y_1 = 12$, $y_2 = 81$, $y_3 = 200$, $y_4 = 100$, $y_5 = 8$, without forming the difference table, find $\Delta^5 y0$.

7.4 Central Difference Interpolation Formulae

In the preceding sections, we derived Newton's forward and backward interpolation formulae which are applicable for interpolation near the beginning and end of tabulated values. Now we shall develop central difference formulae which are best suited for interpolation near the middle of the table.

If x takes the values $x_0 - 2h$, $x_0 - h$, x_0, $x_0 + h$, $x_0 + 2h$ and the corresponding values of $y = f(x)$ are y_{-2}, y_{-1}, y_0, y_1, y_2, then we can write the difference table in the two notations as follows:

x	y	*1st diff.*	*2nd diff.*	*3rd diff.*	*4th diff.*
$x_0 - 2h$	y_{-2}				
		$\Delta y-_2 \ (= \Delta y-_{3/2})$			
$x_0 - h$	y_{-1}		$\Delta^2 y-_2 (= \Delta^2 y_{-1})$		
		$\Delta y_{-1} \ (= \Delta y_{-1/2})$		$\Delta^3 y_{-2} \ (= \Delta^3 y_{-1/2})$	
x_0	y_0		$\Delta^2 y_{-1} \ (= \Delta^2 y_0)$		$\Delta^3 y_{-2} (= \Delta^4 y_0)$
		$\Delta y_0 \ (= \Delta y_{1/2})$		$\Delta^3 y_{-1} \ (= \Delta^3 y_{1/2})$	
$x_0 + h$	y_1		$\Delta^2 y_0 (= \Delta^2 y_1)$		
		$\Delta y_1 (= \Delta y_{3/2})$			
$x_0 + 2h$	y_2				

7.5 Gauss's Forward Interpolation Formula

The Newton's forward interpolation formula is

$$y_0 = y_0 + p\Delta y_0 + \frac{p(p-1)}{1.2}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{1.2.3}\Delta^3 y_0 + \cdots \tag{1}$$

We have $\Delta^2 y_0 - \Delta^2 y_{-1} = \Delta^3 y_{-1}$

i.e., $$\Delta^2 y_0 = \Delta^2 y_{-1} + \Delta^3 y_{-1} \tag{2}$$

Similarly $$\Delta^3 y_0 = \Delta^3 y_{-1} + \Delta^4 y_{-1} \tag{3}$$

$$\Delta^4 y_0 = \Delta^4 y_{-1} + \Delta^5 y_{-1} \text{ etc.} \tag{4}$$

Also $\Delta^3 y_{-1} - \Delta^3 y_{-2} = \Delta^4 y_{-2}$

i.e., $$\Delta^3 y_{-1} = \Delta^3 y_{-2} + \Delta^4 y_{-2}$$

Similarly $$\Delta^4 y_{-1} = \Delta^4 y_{-2} + \Delta^5 y_{-2} \text{etc.} \tag{5}$$

Substituting for $\Delta^2 y_0$, $\Delta^3 y_0$, $\Delta^4 y_0$ from (2), (3), (4)..in (1), we get

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{1.2}\left(\Delta^2 y_{-1} + \Delta^3 y_{-1}\right) + \frac{p(p-1)(p-2)}{1.2.3}\left(\Delta^3 y_{-1} + \Delta^4 y_{-1}\right)$$
$$+ \frac{p(p-1)(p-2)(p-3)}{1.2.3.4}\left(\Delta^4 y_{-1} + \Delta^5 y_{-1}\right)$$

Hence $$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_{-1}$$
$$+ \frac{(p+1)(p-2)(p-3)}{4!}\Delta^4 y_{-2} + \cdots \quad \text{[using (5)]}$$

which is called Gauss's forward interpolation formula.

Cor. In the central differences notation, this formula will be

$$y_p = y_0 + p\delta y_{1/2} + \frac{p(p-1)}{2!}\delta^2 y_{1/2} + \frac{p(p-1)(p-2)}{3!}\delta^3 y_{1/2}$$
$$+ \frac{p(p-1)(p-2)(p-3)}{4!}\delta^4 y_{1/2}$$

NOTE **Obs. 1.** *It employs odd differences just below the central line and even difference on the central line as shown below:*

$$\begin{array}{ccccccccc} y_0 & & \Delta^2 y_{-1} & & \Delta^4 y_{-2} & & \Delta^6 y_{-3} & \textit{Central line} \\ & \searrow \quad \nearrow & & \searrow \quad \nearrow & & \searrow \quad \nearrow & & \searrow \\ & \Delta y_0 & & \Delta^3 y_{-1} & & \Delta^5 y_{-2} & & \Delta^7 y_{-3} \end{array}$$

Obs. 2. *This formula is used to interpolate the values of y for p (0 < p < 1) measured forwardly from the origin.*

7.6 Gauss's Backward Interpolation Formula

The Newton's forward interpolation formula is

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{1.2}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{1.2.3}\Delta^3 y_0 + \cdots \tag{1}$$

We have $\Delta y_0 - \Delta y_{-1} = \Delta^2 y_{-1}$

i.e., $\Delta y_0 = \Delta y_{-1} + \Delta^2 y_{-1}$ (2)

Similarly $\Delta^2 y_0 = \Delta^2 y_{-1} + \Delta^3 y_{-1}$ (3)

$\Delta^3 y_0 = \Delta^3 y_{-1} + \Delta^4 y_{-1}$ etc. (4)

Also $\Delta^3 y_{-1} - \Delta^3 y_{-2} = \Delta^4 y_{-2}$

i.e., $\Delta^3 y_{-1} = \Delta^3 y_{-2} + \Delta^4 y_{-2}$ (5)

Similarly $\Delta^4 y_{-1} = \Delta^4 y_{-2} + \Delta^5 y_{-2}$ etc. (6)

Substituting for $\Delta y_0, \Delta^2 y_0, \Delta^3 y_0, \cdots$ from (2), (3), (4) in (1), we get

$$y_p = y_0 + p(\Delta y_{-1} + \Delta^2 y_{-1}) + \frac{p(p-1)}{1.2}\left(\Delta^2 y_{-1} + \Delta^3 y_{-1}\right)$$

$$+\frac{p(p-1)(p-2)}{1.2.3}\left(\Delta^3 y_{-1} + \Delta^4 y_{-1}\right) + \frac{p(p-1)(p-2)(p-3)}{1.2.3.4} \times \left(\Delta^4 y_{-1} + \Delta^5 y_{-1}\right) + \cdots$$

$$= y_0 + p\Delta y_{-1} + \frac{p(p+1)}{1.2}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{1.2.3}\Delta^3 y_{-1}$$

$$+\frac{(p+1)p(p-1)(p-2)}{1.2.3.4}\Delta^4 y_{-1} + \frac{p(p-1)(p-2)(p-3)}{1.2.3.4}\Delta^5 y_{-1} + \cdots$$

$$= y_0 + p\Delta y_{-1} + \frac{(p+1)p}{1.2}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{1.2.3}\left(\Delta^3 y_{-2} + \Delta^4 y_{-2}\right)$$

$$+\frac{(p+1)p(p-1)(p-2)}{1.2.3.4}\left(\Delta^4 y_{-2} + \Delta^5 y_{-2}\right) + \cdots$$

[using (5) and (6)

Hence
$$yp = y_0 + p\Delta_{y-1} + \frac{p(p+1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-2}$$

$$+\frac{(p+1)p(p+1)(p-1)}{4!}\Delta^4 y_{-2} + \cdots$$

which is called Gauss's backward interpolation formula.

Cor. In the central differences notation, this formula will be

$$y_p = y_0 + p\delta y_{-1/2} + \frac{(p+1)p}{2!}\delta^2 y_0 + \frac{(p+1)p(p-1)}{3!}\delta^3 y_{-1/2} + \frac{(p+2)(p+1)p(p-1)}{4!}\delta^4 y_0 + \cdots$$

NOTE

Obs. 1. *This formula contains odd differences above the central line and even differences on the central line as shown below:*

$$\begin{array}{ccccccccccc}
 & & \Delta y_{-2} & & & \Delta^3 y_{-2} & & & \Delta^5 y_{-3} & & \\
 & \nearrow & & \searrow & \nearrow & & \searrow & \nearrow & & \searrow & \\
y_0 & & & & \Delta^2 y_{-1} & & & \Delta^4 y_{-2} & & & \Delta^6 y_{-3} \textit{Central line}
\end{array}$$

Obs. 2. *It is used to interpolate the values of y for a negative value of p lying between – 1 and 0.*

Obs. 3. *Gauss's forward and backward formulae are not of much practical use. However, these serve as intermediate steps for obtaining the important formulae of the following sections.*

7.7 Stirling's Formula

Gauss's forward interpolation formula is

$$y_p = y_0 + p\Delta y_0 + \frac{(p+1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-1} + \frac{(p+1)p(p-1)(p-2)}{4!}\Delta^4 y_{-2} + \cdots \tag{1}$$

Gauss's backward interpolation formula is

$$y_p = y_0 + p\Delta y_{-1} + \frac{(p+1)p}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-2} + \frac{(p+2)(p+1)p(p-1)}{4!}\Delta^4 y_{-2} + \cdots \tag{2}$$

Taking the mean of (1) and (2), we obtained

$$y_p = y_0 + p\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + \frac{p^2}{2!}\Delta^2 y_{-1} + \frac{p(p^2-1)}{3!} \times \left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \frac{p^2(p^2-1)}{4!}\Delta^4 y_{-2} + \cdots \tag{3}$$

Which is called Stirling's formula.

Cor. *In the central difference notation, (3) takes the form*

$$y_p = y_0 + p\mu\delta y_0 + \frac{p^2}{2!}\delta^2 y_0 + \frac{p(p^2-1^2)}{3!}\mu\delta^3 y_0 + \frac{p^2(p^2-1^2)}{4!}\delta^4 y_0 + \cdots$$

For $$\frac{1}{2}\left(\Delta y_0 + \Delta y_{-1}\right) = \frac{1}{2}\left(\delta y_{1/2} + \delta y_{-1/2}\right) = \mu\delta y_0$$

$$\frac{1}{2}\left(\Delta^3 y_{-1} + \Delta^3 y_{-2}\right) = \frac{1}{2}\left(\delta^3 y_{1/2} + \delta^3 y_{-1/2}\right) = \mu\delta^3 y_0 \text{ etc.}$$

NOTE **Obs.** *This formula involves means of the odd differences just above and below the central line and even differences on this line as shown below:*

$$\cdots y_0 \cdots \begin{pmatrix} \Delta y_{-1} \\ \Delta y_0 \end{pmatrix} \cdots \Delta^2 y_{-1} \cdots \begin{pmatrix} \Delta^3 y_{-2} \\ \Delta^3 y_{-1} \end{pmatrix} \cdots \Delta^4 y_{-2} \cdots \begin{pmatrix} \Delta^5 y_{-1} \\ \Delta^5 y_0 \end{pmatrix} \cdots \Delta^6 y_{-3} \cdots$$

Central line.

7.8 Bessel's Formula

Gauss's forward interpolation formula is

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-1} + \frac{(p+1)p(p-1)(p+1)}{4!}\Delta^4 y_{-2} + \cdots$$

We have $\Delta^2 y_0 - \Delta^2 y_{-1} = \Delta^3 y_{-1}$ (1)

i.e., $\Delta^2 y_{-1} = \Delta^2 y_0 - \Delta^3 y_{-1}$ (2)

Similarly $\Delta^4 y_{-2} = \Delta^4 y_{-1} - \Delta^4 y_{-2}$ etc.

Now (1) can be written as

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\left(\frac{1}{2}\Delta^2 y_{-1} + \frac{1}{2}\Delta^2 y_{-1}\right) + \frac{p(p^2-1)}{3!}\Delta^3 y_{-1} + \frac{p(p^2-1)(p-2)}{4!}\left(\frac{1}{2}\Delta^4 y_{-2} + \frac{1}{2}\Delta^4 y_{-2}\right) + \cdots$$

$$= y_0 + p\Delta y_0 + \frac{1}{2}\frac{p(p+1)}{2!}\Delta^2 y_{-1} + \frac{1}{2}\frac{p(p-1)}{2!}\left(\Delta^2 y_0 + \Delta^3 y_{-1}\right) + \frac{p(p^2-1)}{3!}\Delta^3 y_{-1} + \frac{1}{2}\frac{p(p^2-1)(p-2)}{4!}\Delta^4 y_{-2} + \frac{1}{2}\frac{p(p^2-1)(p-2)}{4!} \times \left(\Delta^4 y_{-1} - \Delta^5 y_{-1}\right) + \cdots$$

[Using (2), (3) etc.]

Hence $$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2} + \frac{\left(p-\frac{1}{2}\right)p(p-1)}{3!}\Delta^3 y_{-1} \tag{4}$$

Which is known as Bessel's formula.

Cor. *In the central difference notation, (4) becomes*

$$y_p = y_0 + p\delta y_{1/2} + \frac{p(p-1)}{2!}\mu\delta^2 y_{1/2} + \frac{\left(p-\frac{1}{2}\right)p(p-1)}{2!}\delta^3 y_{1/2} + \frac{(p+1)p(p-1)(p-2)}{4!}\mu\delta^4 y_{1/2} + \cdots$$

for $\frac{1}{2}\left(\Delta^2 y_{-1} + \Delta^2 y_0\right) = \mu\delta^2 y_{1/2}, \frac{1}{2}\left(\Delta^4 y_{-2} + \Delta^4 y_{-1}\right) = \mu\delta^4 y_{1/2}$ etc.,

NOTE **Obs.** *This is a very useful formula for practical purposes. It involves odd differences below the central line and means of even differences of and below this line as shown below*

$$\begin{matrix} y_0 & & \begin{Bmatrix} \Delta^2 y_{-1} \\ \\ \Delta^2 y_0 \end{Bmatrix} & & \begin{Bmatrix} \Delta^4 y_{-2} \\ \\ \Delta^4 y_{-1} \end{Bmatrix} & & \begin{Bmatrix} \Delta^6 y_{-1} \\ \\ \Delta^6 y_0 \end{Bmatrix} & & \textit{Central line} \\ & \Delta y_0 & & \Delta^3 y_{-1} & & \Delta^5 y_{-2} & & \Delta^7 y_{-3} & . \end{matrix}$$

7.9 Laplace-Everett's Formula

Gauss's forward interpolation formula is

$$y_p = y_0 + p\Delta y_0 + \frac{(p-1)p}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-1} + \frac{(p+1)p(p-1)(p-2)}{4!}\Delta^4 y_{-2} + \frac{(p+2)(p+1)p(p-1)(p-2)}{5!}\times\Delta^5 y_{-2} + \cdots \tag{1}$$

We eliminate the odd differences in (1) by using the relations

$\Delta y_0 = y_1 - y_0, \Delta^3 y_{-1} = \Delta^2 y_0 - \Delta^2 y_{-1}, \Delta^5 y_{-2} = \Delta^4 y_{-1} - \Delta^4 y_{-2}$ etc.

Then (1) becomes

$$y_p = y_0 + p(y_1 - y_0) + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\left(\Delta^2 y_0 - \Delta^2 y_{-1}\right) + \frac{(p+1)p(p-1)(p-2)}{4!}\Delta^4 y_{-2} + \frac{(p+2)(p+1)p(p-1)(p-2)}{5!} \times\left(\Delta^4 y_{-1} - \Delta^4 y_{-2}\right) + \cdots$$

$$= (1-p)y_0 + py_1 - \frac{p(p-1)(p-2)}{3!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^2 y_0$$
$$-\frac{(p+1)p(p-1)(p-2)(p-3)}{5!}\Delta^4 y_{-2}$$
$$+\frac{(p+2)(p+1)p(p-1)(p-2)}{5!}\times\Delta^4 y_{-1} - \cdots$$

To change the terms with negative sign, putting $p = 1 - q$, we obtain

$$y_p = qy_0 + \frac{q(q^2-1^2)}{3!}\Delta^2 y_{-1} + \frac{q(q^2-1^2)(q^2-2^2)}{5!}\Delta^4 y_{-2} + \cdots$$
$$+py_1 + \frac{p(p^2-1^2)}{3!}\Delta^2 y_0 + \frac{p(p^2-1^2)(p^2-2^2)}{5!}\Delta^4 y_{-2} + \cdots$$

This is known as *Laplace-Everett's formula*.

NOTE **Obs. 1.** *This formula is extensively used and involves only even differences on and below the central line as shown below:*

y_0	$\Delta^2 y_{-1}$	$\Delta^4 y_{-2}$	$\Delta^6 y_{-3}$	*Central line*
y_1	$\Delta^2 y_0$	$\Delta^4 y_{-1}$	$\Delta^6 y_{-2}$	

Obs. 2. *There is a close relationship between Bessel's formula and Everett's formula and one can be deduced from the other by suitable rearrangements. It is also interesting to observe that Bessel's formula truncated after third differences is Everett's formula truncated after second differences.*

7.10 Choice of an Interpolation Formula

So far we have derived several interpolation formulae such as Newton's forward, Newton's backward, Gauss's forward, Gauss's backward, Stirling's, Bessel's and Everett's formulae for calculating y_p from equispaced values which are called **classical formulae**. Now, we have to see which formula yields most accurate results in a particular problem.

The coefficients in the central difference formulae are smaller and converge faster than those in Newton's formulae. After a few terms, the coefficients in the Stirling's formula decrease more rapidly than those of the Bessel's formula and the coefficients of Bessel's formula decrease more rapidly than those of Newton's formula. As such, whenever possible, central difference formulae should be used in preference to Newton's formulae.

The right choice of an interpolation formula however, depends on the position of the interpolated value in the given data.

The following rules will be found useful:

1. To find a tabulated value near the beginning of the table, use Newton's forward formula.
2. To find a value near the end of the table, use Newton's backward formula.
3. To find an interpolated value near the center of the table, use either Stirling's or Bessel's or Everett's formula.

If interpolation is required for p lying between $-\frac{1}{4}$ and $\frac{1}{4}$, prefer Stirling's formula

If interpolation is desired for p lying between $\frac{1}{4}$ and $\frac{3}{4}$, use Bessel's or Everett's formula.

EXAMPLE 7.7

Find $f(22)$ from the Gauss forward formula:

x:	20	25	30	35	40	45
$f(x)$:	354	332	291	260	231	204

Solution:

Taking $x_0 = 25$, $h = 5$, we have to find the value of $f(x)$ for $x = 22$.

i.e., for $p = \frac{x - x_0}{h} = \frac{22 - 25}{5} = -\ 0.6$

The difference table is as follows:

x	p	y_p	Δy_p	$\Delta^2 y_p$	$\Delta^3 y_p$	$\Delta^4 y_p$	$\Delta^5 y_p$
20	– 1	354 ($= y_{-1}$)	– 22				
25	0	332 ($= y_0$)	– 41	– 19	29		
30	1	291 ($= y_1$)	– 31	10	– 8	– 37	45
35	2	260 ($= y_2$)	– 29	2	0	8	
40	3	231 ($= y_3$)	– 27	2			
45	4	204 ($= y_4$)					

Gauss forward formula is

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-1}$$
$$+\frac{(p+1)p(p-1)(p-2)}{4!}\Delta^4 y_{-2}$$
$$+\frac{(p+1)(p-1)p(p-2)(p+2)}{5!}\Delta^5 y_{-2} + \ldots..$$

$$\therefore f(22) = 332 + (0.6)(-41) + \frac{(-0.6)(-0.6-1)}{2!}(-19)$$
$$+\frac{(-0.6+1)(-0.6)(-0.6-1)}{3!}(-8)$$
$$+\frac{(-0.6+1)(-0.6)(-0.6-1)(-0.6-2)}{4!}(-37)$$
$$+\frac{(-0.6+1)(-0.6)(-0.6-1)(-0.6-2)(-0.6+2)}{5!}(45)$$

$= 332 + 24.6 - 9.12 - 0.512 + 1.5392 - 0.5241$

Hence $f(22) = 347.983$.

EXAMPLE 7.8

Use Gauss's forward formula to evaluate y_{30}, given that $y_{21} = 18.4708$, $y_{25} = 17.8144$, $y_{29} = 17.1070$, $y_{33} = 16.3432$ and $y_{37} = 15.5154$.

Solution

Taking $x_0 = 29$, $h = 4$, we require the value of y for $x = 30$

i.e., for $p = \dfrac{x - x_0}{h} = \dfrac{30-29}{4} = 0.25$

The difference table is given below:

x	p	y_p	Δy_p	$\Delta^2 y_p$	$\Delta^3 y_p$	$\Delta^4 y_p$
21	– 2	18.4708				
			– 0.6564			
25	– 1	17.8144		– 0.0510		
			– 0.7074		– 0.7074	
29	0	17.1070		– 0.0564		– 0.0022
			– 0.7638		– 0.0076	
33	1	16.3432		– 0.0640		
			– 0.8278			
37	2	15.5154				

Gauss's forward formula is

$$y_p = y_0 + p\Delta y_0 + \frac{p(p+1)}{1.2}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{1.2.3}\Delta^3 y_{-1} + \frac{(p+1)p(p-1)(p-2)}{1.2.3.4}\Delta^4 y_{-2} + \cdots$$

$$y_{30} = 17.1070 + (0.25)(-0.7638) + \frac{(0.25)(-0.75)}{2}(-0.0564) + \frac{(1.25)(0.25)(-0.75)}{6}(-0.0076) + \frac{(1.25)(0.25)(-0.75)(-1.75)}{24} \times (-0.0022)$$

$$= 17.1070 - 0.19095 + 0.00529 + 0.0003 - 0.00004 = 16.9216 \text{ approx.}$$

EXAMPLE 7.9

Using Gauss backward difference formula, find y (8) from the following table.

x	0	5	10	15	20	25
y	7	11	14	18	24	32

Solution:

Taking $x_0 = 10$, $h = 5$, we have to find y for $x = 8$, *i.e.,* for

$$p = \frac{x - x_0}{h} = \frac{8-10}{5} = -0.4.$$

The difference table is as follows:

x	p	y_p	Δy_p	$\Delta^2 y_p$	$\Delta^3 y_p$	$\Delta^4 y_p$	$\Delta^5 y_p$
0	2	7					
			4				
5	1	11		– 1			
			3		2		
10	0	14		1		– 1	
			4		1		0
15	1	18		2		– 1	
			6		0		
20	2	24		2			
			8				
25	3	32					

Gauss backward formula is

$$y_p = y_0 + p\Delta y_{-1} + \frac{(p+1)p}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-2} + \frac{(p+2)p(p+1)p(p-1)}{4!}\Delta^4 y_{-2} + \cdots$$

$$y(8) = 14 + (-0.4)(3) + \frac{(-0.4+1)(-0.4)}{2!}(1) + \frac{(-0.4+1)(-0.4)(-0.4-1)}{3!}(2) + \frac{(-0.4+2)(-0.4+1)(-0.4)(-0.4-1)}{4!}(-1)$$

$$= 14 - 1.2 - 0.12 + 0.112 + 0.034$$

Hence $y_{(8)} = 12.826$

EXAMPLE 7.10

Interpolate by means of Gauss's backward formula, the population of a town for the year 1974, given that:

Year:	1939	1949	1959	1969	1979	1989
Population: (*in thousands*)	12	15	20	27	39	52

Solution:

Taking $x_0 = 1969$, $h = 10$, the population of the town is to be found for

$$p = \frac{1974 - 1969}{10} = 0.5$$

The Central difference table is

x	p	y_p	Δy_p	$\Delta^2 y_p$	$\Delta^3 y_p$	$\Delta^4 y_p$	$\Delta^5 y_p$
1939	– 3	12	3	2	0	3	–10
1949	– 2	15					
			5				
1959	– 1	20		2			
			7		3		
1969	0	27		5		–7	
			12		– 4		
1979	1	39		1			
			13				
1989	2	52					

Gauss's backward formula is

$$y_p = y_0 + p\Delta y_{-1} + \frac{(p+1)p}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-2}$$
$$+\frac{(p+2)p(p+1)p(p-1)}{4!}\Delta^4 y_{-2}$$
$$+\frac{(p+2)(p+1)p(p-1)(p-2)}{5!}\Delta^5 y_3 + \cdots$$

$$y_{0.5} = 27 + (0.5)(7) + \frac{(1.5)(0.5)}{2}(5) + \frac{(1.5)(0.5)(-0.5)}{6}$$
$$+\frac{(2.5)(1.5)(-0.5)}{24}(-7) + \frac{(2.5)(1.5)(0.5)(-0.5)(1.5)}{120}(-10)$$
$$= 27 + 3.5 + 1.875 - 0.1875 + 0.2743 - 0.1172$$
$$= 32.532 \text{ thousands approx.}$$

EXAMPLE 7.11

Employ Stirling's formula to compute $y_{12.2}$ from the following table ($y_x = 1 + \log_{10}\sin x$):

$x°$:	10	11	12	13	14
$10^5\, y_x$:	23,967	28,060	31,788	35,209	38,368

Solution:

Taking the origin at $x_0 = 12°$, $h = 1$ and $p = x - 12$, we have the following central difference table:

p	y_x	Δy_x	$\Delta^2 y_x$	$\Delta^3 y_x$	$\Delta^4 y_x$
$-2 = x_{-2}$	$0.23967 = y_{-2}$				
		$0.04093 = \Delta y_{-2}$			
$-1 = x_{-1}$	$0.28060 = y_{-1}$		$-0.00365 = \Delta 2 y_{-2}$		
		$0.03728 = \Delta y_{-1}$		$0.00058 = \Delta^3 y_{-2}$	
$0 = x_0$	$0.31788 = y_0$		$-0.00307 = \Delta^2 y_{-1}$		$-0.00013 = \Delta^4 y_{-2}$
		$0.03421 = \Delta y_0$		$-0.00045 = \Delta^2 y_{-1}$	
$1 = x_1$	$0.35209 = y_1$		$-0.00062 = \Delta^2 y_0$		
		$0.03159 = \Delta y_1$			
$2 = x_2$	$0.38368 = y_2$				

At $x = 12.2$, $p = 0.2$. (As p lies between $-\frac{1}{4}$ and $\frac{1}{4}$, the use of String's formula will be Quite suitable.)

Stirling's formula is

$$y_p = y_0 + \frac{p}{1}\frac{\Delta y_{-1} + \Delta y_{-0}}{2} + \frac{p^2}{2!}\Delta^2 y_{-1} + \frac{p(p^2-1)}{3!}.\frac{\Delta^3 y_{-2} + \Delta^3 y_{-1}}{2}$$
$$+\frac{p^2(p^2-1)}{4!}\Delta^4 y_{-2} + \cdots$$

When $p = 0.2$, we have

$$\therefore \quad y_{0.2} = 0.3178 + 0.2\left(\frac{0.03728 + 0.03421}{2}\right) + \frac{(0.2)^2}{2}(-0.00307)$$
$$+\frac{(0.2)^2\left[(0.2)^2 - 1\right]}{6}\left(\frac{0.00058 + 0.00054}{2}\right) + \frac{(0.2)^2\left[(0.2)^2 - 1\right]}{24}(-0.00013)$$

$= 0.31788 + 0.00715 - 0.00006 - 0.000002 + 0.0000002$

$= 0.32497.$

EXAMPLE 7.12

Given

$\theta°$:	0	5	10	15	20	25	30
$\tan\theta$:	0	0.0875	0.1763	0.2679	0.3640	0.4663	0.5774

Using Stirling's formula, estimate the value of tan16°.

Solution:

Taking the origin at $\theta° = 15°$, $h = 5°$ and $p = \dfrac{\theta - 15}{5}$, we have the following central difference table:

p	$y = \tan\theta$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
– 3	0.0000					
		0.0875				
– 2	0.0875		0.0013			
		0.0888		0.0015		
– 1	0.1763		0.0028		0.0002	
		0.0916		0.0017		– 0.0002
0	0.2679		0.0045		0.0000	
		0.0961		0.0017		0.0009
1	0.3640		0.0062		0.0009	
		0.1023		0.0026		
2	0.4663		0.0088			
		0.1111				
3	0.5774					

At $\theta = 16°$, $p = \dfrac{16 - 15}{5} = 0.2$

Stirling's formula is

$$y_p = y_o + \frac{p}{1}\cdot\frac{\Delta y_{-1} + \Delta y_0}{2} + \frac{p^2}{2!}\Delta^2 y_{-1} + \frac{p^2(p^2-1)}{3!}\cdot\frac{\Delta^2 y_{-2} + \Delta^3 y_{-1}}{2}$$
$$+ \frac{p^2(p^2-1)}{4!}\Delta^4 y_{-2} + \cdots$$

$$\therefore\ y_{0.2} = 0.2679 + 0.2\left(\frac{0.0916 + 0.0916}{2}\right) + \frac{(0.2)^2}{2}(0.0045) + \cdots$$
$$= 0.2679 + 0.01877 + 0.00009 + \cdots = 0.28676$$

Hence, $\tan 16^{\circ} = 0.28676$.

EXAMPLE 7.13

Apply Bessel's formula to obtain y_{25}, given $y_{20} = 2854$, $y_{24} = 3162$, $y_{28} = 3544$, $y_{32} = 3992$.

Solution:

Taking the origin at $x_0 = 24$, $h = 4$, we have $p = (x - 24)$.

$\therefore$ The central difference table is

p	y	Δy	$\Delta^2 y$	$\Delta^3 y$
– 1	2854			
		308		
0	$\underline{3162}$		$\underline{74}$	
		$\underline{382}$		$\underline{-8}$
1	3544		$\underline{66}$	
		448		
2	3992			

At $x = 25, p = \dfrac{(25-24)}{4} = \dfrac{1}{4}$.. (As p lies between $\dfrac{1}{4}$ and $\dfrac{3}{4}$, the use of Bessel's formula will yield accurate results)

Bessel's formula is

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2} + \frac{\left(p-\frac{1}{2}\right)p(p-1)}{2!}\Delta^3 y_{-1} + \cdots \quad (1)$$

When $p = 0.25$, we have

$$y_p = 3162 + 0.25 \times 382 + \frac{0.25(-0.75)}{2!}\left(\frac{74+66}{2}\right) + \frac{(0.25)0.25(-0.75)}{2!} - 8$$

$= 3162 + 95.5 - 6.5625 - 0.0625$

$= 3250.875$ approx.

EXAMPLE 7.14

Apply Bessel's formula to find the value of f (27.5) from the table:

x:	25	26	27	28	29	30
$f(x)$:	4.000	3.846	3.704	3.571	3.448	3.333

Solution:

Taking the origin at $x_0 = 27$, $h = 1$, we have $p = x - 27$

The central difference table is

x	p	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
25	–2	4.000				
			–0.154			
26	–1	3.846		0.012		
			– 0.142		– 0.003	
27	0	3.704		0.009		0.004
			– 0.133		– 0.001	
28	1	3.571		0.010		– 0.001
			– 0.123		– 0.002	
29	2	3.448		0.008		
			– 0.115			
30	3	3.333				

At $x = 27.5$, $p = 0.5$ (As p lies between 1/4 and 3/4, the use of Bessel's formula will yield an accurate result),

Bessel's formula is

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2} + \frac{\left(p-\frac{1}{2}\right)p(p-1)}{3!}\Delta^3 y_{-1}$$
$$+\frac{(p+1)p(p-1)(p-2)}{4!}\left(\frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2}\right) + \cdots$$

When $p = 0.5$, we have

$$y_p = 3.704 - \frac{(0.5)(0.5-1)}{2}\left(\frac{0.009+0.010}{2}\right) + 0$$
$$+\frac{(0.5+1)(0.5)(0.5-1)(0.5-2)}{2}\frac{(-0.001-0.004)}{2}$$
$$= 3.704 - 0.11875 - 0.00006 = 3.585$$

Hence $f(27.5) = 3.585$.

EXAMPLE 7.15

Using Everett's formula, evaluate $f(30)$ if $f(20) = 2854$, $f(28) = 3162$, $f(36) = 7088$, $f(44) = 7984$

Solution:

Taking the origin at $x_0 = 28$, $h = 8$, we have $p = \frac{x-28}{8}$. The central table is

x	p	y	Δy	$\Delta^2 y$	$\Delta^3 y$
20	– 1	2854			
			308		
28	0	3162		3618	
			3926		– 6648
36	1	7088		– 3030	
			896		
44	2	7984			

At $\quad x = 30, p = \dfrac{30-28}{8} = 0.25$ and $q = 1 - p = 0.75$

Everett's formula is

$$y_p = qy_0 + \frac{q(q^2-1^2)}{3!}\Delta^2 y_{-1} + \frac{q(q^2-1^2)(q^2-2^2)}{5!}\Delta^4 y_{-2} + \cdots$$

$$+py_1 + \frac{p(p^2-1^2)}{3!}\Delta^2 y_0 + \frac{p(p^2-1^2)(p^2-2^2)}{5!}\Delta^4 y_{-2} + \cdots$$

$$= (0.75) + (3162) + \frac{0.75(0.75^2-1)}{6}(3618) + \cdots$$

$$+0.25 + (7080) + \frac{0.25\left(0.25^2-1\right)}{6}(-3030) + \cdots$$

$$= 2371.5 - 351.75 + 1770 + 94.69 = 3884.4$$

Hence $\ f(30) = 3884.4$

EXAMPLE 7.16

Given the table

x:	310	320	330	340	350	360
$\log x$:	2.49136	2.50515	2.51851	2.53148	2.54407	2.55630

find the value of log 337.5 by Everett's formula.

Solution:

Taking the origin at $x_0 = 330$ and $h = 10$, we have $p = \dfrac{x - 330}{10}$

∴ The central difference table is

p	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
–2	2.49136					
		0.01379				
–1	2.50515		– 0.00043			
		0.01336		0.00004		
0	2.51881		– 0.00039		– 0.00003	
		0.01297		0.00001		0.00004
1	2.53148		– 0.00038		0.00001	
		0.01259		0.00002		
2	2.54407		– 0.00036			
		0.01223				
3	2.55630					

To evaluate log 337.5, *i.e.*, for $x = 337.5$, $p = \dfrac{337.5 - 330}{10} = 0.75$

(As $p > 0.5$ and =0.75, Everett's formula will be quite suitable)

Everett's formula is

$$y_p = qy_0 + \frac{q(q^2-1^2)}{3!}\Delta^2 y_{-1} + \frac{q(q^2-1^2)(q^2-2^2)}{5!}\Delta^4 y_{-2} + \cdots$$

$$+py_1 + \frac{p(p^2-1^2)}{3!}\Delta^2 y_0 + \frac{p(p^2-1^2)(p^2-2^2)}{5!}\Delta^4 y_{-1} + \cdots$$

$$= 0.25 \times 2.51851 + \frac{0.25(0.0625\text{-}1)}{6} \times (-0.00039)$$

$$+\frac{0.25(0.0625\text{-}1)(0.0625\text{-}4)}{120} \times (-0.00003)$$

$$+0.75 \times 2.53148 + \frac{0.75(0.5625-1)}{6} \times (-0.00038)$$

$$+\frac{0.75(0.5625-1)(0.5625-4)}{6} \times (-0.00001)$$

= 0.62963 + 0.00002 – 0.0000002 + 1.89861 + 0.00002 + 0.0000001

= 2.52828 nearly.

Exercises 7.2

1. Find they y (25), given that $y_{20} = 24$, $y_{24} = 32$, $y_{28} = 35$, $y_{32} = 40$, using Gauss for ward difference formula.

2. Using Gauss's forward formula, fin d a polynomial of degree four which takes the following values of the function $f(x)$:

x:	1	2	3	4	5
$f(x)$:	1	– 1	1	– 1	1

3. Using Gauss's forward formula, evaluate $f(3.75)$ from the table:

x:	2.5	3.0	3.5	4.0	4.5	5.0
Y:	24.145	22.043	20.225	18.644	17.262	16.047

4. From the following table:

x:	1.00	1.05	1.10	1.15	1.20	1.25	1.30
e^x:	2.7183	2.8577	3.0042	3.1582	3.3201	3.4903	3.6693

Find $e^{1.17}$, using Gauss forward formula.

5. Using Gauss's backward formula, estimate the number of persons earning wages between Rs. 60 and Rs. 70 from the following data:

Wages (Rs.):	Below 40	40—60	60—80	80—100	100—120
No. of persons: (in thousands)	250	120	100	70	50

6. Apply Gauss's backward formula to find sin 45° from the following table:

$\theta°$:	20	30	40	50	60	70	80
sin θ:	0.34202	0.502	0.64279	0.76604	0.86603	0.93969	0.98481

7. Using Stirling's formula find y_{35}, given $y_{20} = 512$, $y_{30} = 439$, $y_{40} = 346$, $y_{50} = 243$, where y_x represents the number of persons at age x years in a life table.

8. The pressure p of wind corresponding to velocity v is given by the following data. Estimate p when $v = 25$.

v:	v:	10	20	30	40
p:	1.1	2	4.4	7.9	

9. Use Stirling's formula to evaluate $f(1.22)$, given

x:	1.0	1.1	1.2	1.3	1.4
$f(x)$:	0.841	0.891	0.932	0.963	0.985

10. Calculate the value of f (1.5) using Bessels' interpolation formula, from the table

x:	0	1	2	3
$f(x)$:	3	6	12	15

11. Use Bessel's formula to obtain y_{25}, given $y_{20} = 24$, $y_{24} = 32$, $y_{28} = 35$, $y_{32} = 40$.

12. Employ Bessel's formula to find the value of F at $x = 1.95$, given that

x:	1.7	1.8	1.9	2.0	2.1	2.2	2.3
F:	2.979	3.144	3.283	3.391	3.463	3.997	4.491

Which other interpolation formula can be used here? Which is more appropriate? Give reasons.

13. From the following table:

x:	20	25	30	35	40
$f(x)$:	11.4699	12.7834	13.7648	14.4982	15.0463

Find $f(34)$ using Everett's formula.

14. Apply Everett's formula to obtain u_{25}, given $u_{20} = 2854$, $u_{24} = 3162$, $u_{28} = 3544$, $u_{32} = 3992$.

15. Given the table:

x:	310	320	330	340	350	360
$\log x$:	2.4914	2.5052	2.5185	2.5315	2.5441	2.5563

16. Find the value of log 337.5 by Gauss, Stirling, Bessel, and Everett's formulae.

If $y_0, y_1, y_2, y_3, y_4, y_5$ (y_5 being constant) are given, prove that

$$y_{5/2} = \frac{3(a-c)+2.5(c-b)}{256} + \frac{c}{2} \text{ where } a = y_0 + y_5, b = y_1 + y_4, c = y_2 + y_3.$$

[***HINT*:** Use Bessel's formula taking $p = 1/2$.]

7.11 Interpolation with Unequal Intervals

The various interpolation formulae derived so far possess the disadvantage of being applicable only to equally spaced values of the argument. It is, therefore, desirable to develop interpolation formulae for unequally spaced values of x. Now we shall study two such formulae:

(*i*) Lagrange's interpolation formula

(*ii*) Newton's general interpolation formula with divided differences.

7.12 Lagrange's Interpolation Formula

If $y = f(x)$ takes the value $y_0, y_1, \ldots\ldots, y_n$ corresponding to $x = x_0, x_1, \cdots, x_n$, then

$$f(x) = \frac{(x-x_1)(x-x_2)\cdots(x-x_n)}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)}y_0 + \frac{(x-x_0)(x-x_2)\cdots(x-x_n)}{(x_1-x_0)(x_1-x_2)\cdots(x_1-x_n)}y_1 + \cdots + \frac{(x-x_0)(x-x_1)\cdots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\cdots(x_n-x_{n-1})}y_n \quad (1)$$

This is known as *Lagrange's interpolation formula for unequal intervals*.

Proof: Let $y = f(x)$ be a function which takes the values $(x_0, y_0), (x_1, y_1), \cdots, (x_n, y_n)$. Since there are $n + 1$ pairs of values of x and y, we can represent $f(x)$ by a polynomial in x of degree n. Let this polynomial be of the form

$$y = f(x) = a_0(x-x_1)(x-x_2)\ldots(x-x_n) + a_1(x-x_0)(x-x_2)\cdots(x-x_n)$$
$$+a_2(x-x_0)(x-x_1)(x-x_3)\cdots(x-x_n) + \cdots + a_n(x-x_0)(x-x_1)\cdots(x-x_{n-1}) \quad (2)$$

Putting $x = x_0$, $y = y_0$, in (2), we get

$$y_0 = a_0(x_0-x_1)(x-x_2)\cdots(x-x_n)$$
$$a_0 = y_0 / [(x-x_1)(x-x_2)\cdots(x-x_n)]$$

Similarly putting $x = x_1$, $y = y_1$ in (2), we have

$$a_1 = y_1 / [(x_1-x_0)(x_1-x_2)\cdots(x_1-x_n)]$$

Proceeding the same way, we find $a_2, a_3 \ldots\ldots a_n$.

Substituting the values of $a_0, a_1, \cdots, a_n$ in (2), we get (1)

NOTE

Obs. *Lagrange's interpolation formula (1) for n points is a polynomial of degree (n – 1) which is known as the Lagrangian polynomial and is very simple to implement on a computer.*

This formula can also be used to split the given function into partial fractions.

For on dividing both sides of (1) by $(x-x_0)(x-x_1)\cdots(x-x_n)$, *we get*

$$\frac{f(x)}{(x-x_0)(x-x_1)\cdots(x-x_n)}=\frac{y_0}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)}\cdot\frac{1}{(x-x_0)}$$
$$+\frac{y_1}{(x_1-x_0)(x_1-x_2)\cdots(x_1-x_n)}\cdot\frac{1}{(x-x_1)}+\cdots$$
$$+\frac{y_n}{(x_n-x_0)(x_n-x_1)\cdots(x_n-x_{n-1})}\cdot\frac{1}{(x-x_n)}$$

EXAMPLE 7.17

Given the values

x:	5	7	11	13	17
f(*x*):	150	392	1452	2366	5202

evaluate *f*(9), using *Lagrange's formula*

Solution:

(*i*) Here $x_0 = 5, x_1 = 7, x_2 = 11, x_3 = 13, x_4 = 17$

and $y_0 = 150, y_1 = 392, y_2 = 1452, y_3 = 2366, y_4 = 5202.$

Putting $x = 9$ and substituting the above values in Lagrange's formula, we get

$$f(9)=\frac{(9-7)(9-11)(9-13)(9-17)}{(5-7)(5-11)(5-13)(5-17)}\times 150+\frac{(9-5)(9-11)(9-13)(9-17)}{(7-5)(7-11)(7-13)(7-17)}\times 392$$
$$+\frac{(9-5)(9-7)(9-13)(9-17)}{(11-5)(11-7)(11-13)(11-17)}\times 1452$$
$$+\frac{(9-5)(9-7)(9-11)(9-17)}{(13-5)(13-7)(13-11)(13-17)}\times 2366$$
$$+\frac{(9-5)(9-7)(9-11)(9-13)}{(17-5)(17-7)(17-11)(17-13)}\times 5202$$

$$=-\frac{50}{3}+\frac{3136}{15}+\frac{3872}{3}+\frac{2366}{3}+\frac{578}{5}=810$$

EXAMPLE 7.18

Find the polynomial $f(x)$ by using Lagrange's formula and hence find $f(3)$ for

x:	0	1	2	5
$f(x)$:	2	3	12	147

Solution:

Here $x_0 = 0, x_1 = 1, x_2 = 2, x_3 = 5$

and $y_0 = 2, y_1 = 3, y_2 = 12, y_3 = 147.$

Lagrange's formula is

$$y = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}y_1 + \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}y_3$$

$$= \frac{(x-1)(x-2)(x-5)}{(0-1)(0-2)(0-5)}(2) + \frac{(x-0)(x-2)(x-5)}{(1-0)(1-2)(1-5)}(3) + \frac{(x-0)(x-1)(x-5)}{(2-0)(2-1)(2-5)}(12) + \frac{(x-0)(x-1)(x-2)}{(5-0)(5-1)(5-2)}(147)$$

Hence $f(x) = x^3 + x^2 - x + 2$

$\therefore \quad f(3) = 27 + 9 - 3 + 2 = 35$

EXAMPLE 7.19

A curve passes through the points (0, 18), (1, 10), (3, –18) and (6, 90). Find the slope of the curve at $x = 2$.

Solution:

Here $x_0 = 0, x_1 = 1, x_2 = 3, x_3 = 6$ and $y_0 = 18, y_1 = 10, y_2 = -18, y_3 = 90$.

Since the values of x are unequally spaced, we use the Lagrange's formula:

$$y = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}y_1 + \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}y_3$$

$$= \frac{(x-1)(x-3)(x-6)}{(0-1)(0-3)(0-6)}(18) + \frac{(x-0)(x-3)(x-6)}{(1-0)(1-3)(1-6)}(10)$$

$$+ \frac{(x-0)(x-1)(x-6)}{(3-0)(3-1)(3-6)}(-18) + \frac{(x-0)(x-1)(x-3)}{(6-0)(6-1)(6-3)}(90)$$

$$= (-x^3 + 10x^2 - 27x + 18) + (x^3 - 9x^2 + 18x)$$

$$+ (x^3 - 7x^2 + 6x) + (x^3 - 4x^2 + 3x)$$

i.e., $y = 2x^3 - 10x^2 + 18$

Thus the slope of the curve at $x = 2 = \left(\frac{dy}{dx}\right)_{x-2}$

$$= (6x^2 - 20x)_{x=2} = -16$$

EXAMPLE 7.20

Using Lagrange's formula, express the function $\frac{3x^2 + x + 1}{(x-1)(x-2)(x-3)}$ as a sum of partial fractions.

Solution:

Let us evaluate $y = 3x^2 + x + 1$ for $x = 1$, $x = 2$ and $x = 3$

These values are

x:	$x_0 = 1$	$x_1 = 2$	$x_2 = 3$
y:	$y_0 = 5$	$y_1 = 15$	$y_2 = 31$

Lagrange's formula is

$$y = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}y_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}y_1 + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}y_2$$

Substituting the above values, we get

$$y = \frac{(x-2)(x-3)}{(1-2)(1-3)}(5) + \frac{(x-1)(x-3)}{(2-1)(2-3)}(15) + \frac{(x-1)(x-2)}{(3-1)(3-2)}(31)$$

$$= 2.5\,(x-2)\,(x-3) - 15\,(x-1)\,(x-3) + 15.5\,(x-1)\,(x-2)$$

Thus $$\frac{3x^2 + x + 1}{(x-1)(x-2)(x-3)} = \frac{2.5\,(x-2)\,(x-3) - 15(x-1)\,(x-3) + 15.5\,(x-1)\,(x-2)}{(x-1)(x-2)(x-3)}$$

$$= \frac{25}{x-1} - \frac{15}{x-2} + \frac{15.5}{x-3}$$

EXAMPLE 7.21

Find the missing term in the following table using interpolation:

x:	0	1	2	3	4
y:	1	3	9	...	81

Solution:

Since the given data is unevenly spaced, therefore we use Lagrange's interpolation formula:

$$y=\frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}y_0+\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}y_1$$
$$+\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}y_2+\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}y_3$$

Here we have $x_0=0 \quad x_1=1 \quad x_2=2 \quad x_3=4$

$y_0=1 \quad y_1=3 \quad y_2=9 \quad y_3=81$

$$\therefore \quad y=\frac{(x-1)(x-2)(x-4)}{(0-1)(0-2)(0-4)}(1)+\frac{(x-0)(x-2)(x-4)}{(1-0)(1-2)(1-4)}(3)$$
$$+\frac{(x-0)(x-1)(x-4)}{(2-0)(2-1)(2-4)}(9)+\frac{(x-0)(x-1)(x-2)}{(4-0)(4-1)(4-2)}(81)$$

When $x=3$, then

$$\therefore \quad y=\frac{(3-1)(3-2)(3-4)}{-8}+3(3-2)(3-4)+\frac{3(3-1)(3-4)(9)}{-4}+$$
$$+\frac{3(3-1)(3-2)}{24}(81)=\frac{1}{4}-3+\frac{27}{2}+\frac{81}{24}=31$$

Hence the missing term for $x=3$ is $y=31$.

EXAMPLE 7.22

Find the distance moved by a particle and its acceleration at the end of 4 seconds, if the time verses velocity data is as follows:

t:	0	1	3	4
v:	21	15	12	10

Solution:

Since the values of t are not equispaced, we use Lagrange's formula:

$$v = \frac{(t-t_1)(t-t_2)(t-t_3)}{(t_0-t_1)(t_0-t_2)(t_0-t_3)}v_0 + \frac{(t-t_0)(t-t_2)(t-t_3)}{(t_1-t_0)(t_1-t_2)(t_1-t_3)}v_1$$
$$+\frac{(t-t_0)(t-t_1)(t-t_3)}{(t_1-t_0)(t_1-t_2)(t_1-t_3)}v_2 + \frac{(t-t_0)(t-t_1)(t-t_3)}{(t_1-t_0)(t_1-t_2)(t_1-t_3)}v_3$$

$$i.e.\ ,\ v = \frac{(t-1)(t-3)(t-4)}{(-1)(-2)(-4)}(21) + \frac{t(t-3)(t-4)}{(1)(-2)(-3)}(15)$$
$$+\frac{t(t-1)(t-4)}{(3)(2)(-1)}(12) + \frac{t(t-1)(t-3)}{(4)(3)(1)}(10)$$

$$i.e.,\ v = \frac{1}{12}(-5t^3 + 38t^2 - 105^t + 252)$$

$$\therefore \text{ Distance moved } s = \int_0^4 v dt = \int_0^4 (-5t^3 + 38t^2 - 105^t + 252) \qquad \left[\because v = \frac{ds}{dt}\right]$$

$$= \frac{1}{12}\left(-\frac{5t^4}{4} + \frac{38t^3}{3} - \frac{105t^2}{2} + 252t\right)_0^4$$

$$= \frac{1}{12}\left(-320 + \frac{2432}{3} - 840 + 1008\right) = 54.9$$

$$\text{Also acceleration} \qquad = \frac{dv}{dt} = \frac{1}{2}(-15t2 + 76t - 105 + 0)$$

$$\text{Hence acceleration at } (t = 4) = \frac{1}{2}(-15 \pm +76(4) - 105) = -3.4$$

Exercises 7.3

1. Use Lagrange's interpolation formula to find the value of y when $x = 10$, if the following values of x and y are given:

x:	5	6	9	11
y:	12	13	14	16

2. The following table gives the viscosity of oil as a function of temperature. Use Lagrange's formula to find the viscosity of oil at a temperature of 140°.

Temp°:	110	130	160	190
Viscosity:	10.8	8.1	5.5	4.8

3. Given $\log_{10}654 = 2.8156$, $\log_{10} 658 = 2.8182$, $\log_{10} 659 = 2.8189$, $\log_{10}661 = 2.8202$, find by using Lagrange's formula, the value of $\log_{10} 656$.

4. The following are the measurements T made on a curve recorded by oscilograph representing a change of current I due to a change in the conditions of an electric current.

T:	1.2	2.0	2.5	3.0
I:	1.36	0.58	0.34	0.20

Using Lagrange's formula, find I and $T = 1.6$.

5. Using Lagrange's interpolation, calculate the profit in the year 2000 from the following data:

Year:	1997	1999	2001	2002
Profit in Lakhs of Rs:	43	65	159	248

6. Use Lagrange's formula to find thee form of $f(x)$, given

x:	0	2	3	6
$f(x)$:	648	704	729	792

7. If $y(1) = -3$, $y(3) = 9$, $y(4) = 30$, $y(6) = 132$, fin d the Lagrange's interpolation polynomial that takes the same values as y at the given point s.

8. Given $f(0) = -18$, $f(1) = 0$, $f(3) = 0$, $f(5) = -248$, $f(6) = 0$, $f(9) = 13104$, find $f(x)$.

9. Find the missing term in the following table using interpolation

x:	1	2	4	5	6
y:	14	15	5	...	9

10. Using Lagrange's formula, express the function $\dfrac{x^2+x-3}{x^3-2x^2-x+2}$ as a sum of partial fractions.

11. Using Lagrange's formula, express the function $\dfrac{x^2+6x-1}{(x^2-1)(x-4)(x-6)}$ as a sum of partial fractions.

[***Hint***. Tabulate the values of $f(x) = x^2 + 6x - 1$ for $\underline{x} = -1, 1, 4, 6$ and apply Lagrange's formula.]

12. Using ***Lagrange's formula***, prove that

$$y_o = \frac{1}{2}(y_1 + y_{-1}) = \frac{1}{8}\left\{\frac{1}{2}(y_3 + y_1) - \frac{1}{2}(y_{-1} + y_{-3})\right\}.$$

[***Hint***: Here $x_0 = -3$, $x_1 = -1$, $x_2 = 1$, $x_3 = 3$.]

7.13 Divided Differences

The Lagrange's formula has the drawback that if another interpolation value were inserted, then the interpolation coefficients are required to be recalculated. This labor of recomputing the interpolation coefficients is saved by using Newton's general interpolation formula which employs what are called "**divided differences**." Before deriving this formula, we shall first define these differences.

If $(x_0,y_0),(x_1,y_1),(x_2,y_2),\cdots$ be given points, then the *first divided difference* for the arguments x_0, x_1 is defined by the relation $[x_0, x_1]$ or

$$\underset{x_1}{\Delta} y_0 = \frac{y_1 - y_0}{x_1 - x_0}$$

Similarly $[x_1, x_2]$ or $\underset{x_2}{\Delta} y_0 = \frac{y_2 - y_1}{x_2 - x_1}$ and $[x_2, x_3]$ or $\underset{x_3}{\Delta} y_0 = \frac{y_3 - y_2}{x_3 - x_2}$

The *second divided difference* for x_0, x_1, x_2 is defined as

$$[x_0, x_1, x_2] \text{ or } \underset{x_1,x_2}{\Delta^2} = \frac{[x_1,x_2]-[x_0,x_1]}{x_2 - x_0}$$

The *third divided difference* for x_0, x_1, x_2, x_3 is defined as

$$[x_0, x_1, x_2, x_3] \text{ or } \underset{x_1,x_2,x_3}{\Delta^3} y_0 = \frac{[x_1,x_{2,}x_3]-[x_0,x_1,x_2]}{x_2 - x_0}$$

Properties of Divided Differences

I. The divided differences are symmetrical in their arguments, i.e,. independent of the order of the arguments. For it is easy to write

$$[x_0,x_1] = \frac{y_0}{x_0 - x_1} + \frac{y_1}{x_1 - x_0} = [x_1,x_0],[x_0,x_1,x_2]$$

$$= \frac{y_0}{(x_0 - x_1)(x_0 - x_2)} + \frac{y_1}{(x_1 - x_0)(x_1 - x_2)} + \frac{y_2}{(x_2 - x_0)(x_2 - x_1)}$$

$= [x_{1,}\ x_{2,}\ x_0]$ or $[x_{2,}\ x_0,\ x_1]$ and so on

II. The nth divided differences of a polynomial of the nth degree are constant.

Let the arguments be equally spaced so that

$$x_1 - x_0 = x_2 - x_1 = \cdots = x_n - x_{n-1} = h. \text{ Then}$$

$$[x_0, x_1] = \frac{y_1 - y_0}{x_1 - x_0} = \frac{\Delta y_0}{h}$$

$$[x_0, x_1, x_2] = \frac{[x_1, x_2] - [x_0 - x_1]}{x_2 - x_0} = \frac{1}{2h}\left\{\frac{\Delta y_1}{h} - \frac{\Delta y_0}{h}\right\}$$

$$= \frac{1}{2!h^2}\Delta^2 y_0 \text{ and in general, } [x_0, x_1, x_2, \ldots\ldots, x_n] = \frac{1}{n!h^n}\Delta^n y_0$$

If the tabulated function is a nth degree polynomial, then $\Delta^n y_0$ will be constant. Hence the nth divided differences will also be constant

III. The divided difference operator $\blacktriangle$ is linear

i.e., $$\blacktriangle\{au_x + bv_x\} = a\blacktriangle u_x + b\blacktriangle v_x$$

We have $$\underset{x_1}{\blacktriangle}\left(au_{x_0} + bv_{x_0}\right) = \frac{\left(au_{x_1} + bv_{x_1}\right) - \left(au_{x_0} + bv_{x_0}\right)}{x_1 - x_0}$$

$$= a\left\{\frac{u_{x_1} - u_{x_0}}{x_1 - x_0}\right\} + b\left\{\frac{v_{x_1} - v_{x_0}}{x_1 - x_0}\right\}$$

$$= a\underset{x_1}{\blacktriangle}u_{x_0} + b\underset{x_0}{\blacktriangle}v_{x_0}$$

In general $\blacktriangle\left(au_x + bv_x\right) = a\blacktriangle u_x + b\blacktriangle v_x$ This property is also true for higher order differences.

7.14 Newton's Divided Difference Formula

Let $y_0, y_1, \cdots, y_n$ be the values of $y = f(x)$ corresponding to the arguments $x_0, x_1, \cdots, x_n$. Then from the definition of divided differences, we have

$$[x, x_0] = \frac{y - y_0}{x - x_0}$$

So that $$y = y_0 + (x - x_0)[x, x_0]$$

Again $$[x, x_0, x_1] = \frac{[x, x_0] - [x_0, x_1]}{x - x_1}$$

which gives $[x, x_0] = [x_0, x_1] + (x - x_1)[x, x_0, x_1]$

Substituting this value of $[x, x_0]$ in (1), we get

$$y = y_0 + (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x, x_0, x_1] \tag{2}$$

Also $\quad [x, x_0, x_1, x_2] = \dfrac{[x.x_{0,} x_1] - [x.x_{0,} x_2]}{x - x_2}$

which gives $[x, x_0, x_1] = [x_0, x_1, x_2] + (x - x_2)[x, x_0, x_1, x_2]$

Substituting this value of $[x, x_0, x_1]$ in (2), we obtain

$$y = y_0 + (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x_0, x_1, x_2]$$
$$+ (x - x_0)(x - x_1)(x - x_2)[x, x_0, x_1, x_2]$$

Proceeding in this manner, we get

$$y = y_0 + (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x_0, x_1, x_2]$$
$$+ (x - x_0)(x - x_1)\cdots(x - x_n)[x, x_0, x_1, \cdots x_n]$$
$$+ (x - x_0)(x - x_1)(x - x_2)[x, x_0, x_1, x_2] + \cdots \tag{3}$$

which is called *Newton's general interpolation formula with divided differences*.

7.15 Relation Between Divided and Forward Differences

If $(x_0, y_0), (x_1, y_1), (x_2, y_2), \cdots$ be the given points, then

$$[x_0, x_1] = \frac{y_1 - y_0}{x_1 - x_0}$$

Also $\quad \Delta y_0 = y_1 - y_0$

If $x_0, x_1, x_2, \cdots$ are equispaced, then $x_1 - x_0 = h$, so that

$$[x_0, x_1] = \frac{\Delta y_0}{h}$$

Similarly $\quad [x_1, x_2] = \dfrac{\Delta y_1}{h}$

Now $\quad [x_0, x_1, x_2] = \dfrac{[x_1, x_2] - [x_0, x_1]}{x_2 - x_0}$

$$= \frac{\Delta y_1 / h - \Delta y_0 / h}{2h} \qquad [\because x_2 - x_0 = 2h]$$

$$= \frac{\Delta y_1 - \Delta y_0}{2h^2}$$

Thus $\quad [x_0, x_1, x_2] = \dfrac{\Delta^2 y_0}{2! h^2}$

Similarly $[x_0, x_1, x_2] = \dfrac{\Delta^2 y_1}{2!h^2}$

$$\therefore [x_0, x_1, x_2, x_3] = \frac{\Delta^2 y_1/2h^2 - \Delta^2 y_0/2h^2}{x_3 - x_0} = \frac{\Delta^2 y_1 - \Delta^2 y_0}{2h^2(3)} \qquad [\because x_3 - x_0 = 3h]$$

Thus $[x_0, x_1, x_2, x_3] = \dfrac{\Delta^3 y_0}{3!h^3}$

In general, $[x_0, x_1, \cdots x_n] = \dfrac{\Delta^n y_0}{n!h^n}$

This is the relation between divided and forward differences.

EXAMPLE 7.23

Given the values

x:	5	7	11	13	17
$f(x)$:	150	392	1452	2366	5202

evaluate $f(9)$, using Newton's divided difference formula

Solution:

The divided differences table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
5	150	$\dfrac{392-150}{7-5} = 121$		
7	392		$\dfrac{265-121}{11-5} = 24$	
		$\dfrac{1452-392}{11-7} = 265$		$\dfrac{32-24}{13-5} = 1$
11	1452		$\dfrac{457-265}{13-7} = 32$	
		$\dfrac{2366-1452}{13-11} = 457$		$\dfrac{42-32}{17-7} = 1$
13	2366		$\dfrac{709-457}{17-11} = 42$	
		$\dfrac{5202-2366}{17-13} = 709$		
17	5202			

Taking $x = 9$ in the Newton's divided difference formula, we obtain

$$f(9) = 150 + (9-5) \times 121 + (9-5)(9-7) \times 24 + (9-5)(9-7)(9-11) \times 1$$

$$= 150 + 484 + 192 - 16 = 810.$$

EXAMPLE 7.24

Using Newton's divided differences formula, evaluate $f(8)$ and $f(15)$ given:

x:	4	5	7	10	11	13
$y = f(x)$:	48	100	294	900	1210	2028

Solution:

The divided differences table is

x	$f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
4	48				0
		52			
5	100		15		
		97		1	
7	294		21		0
		202		1	
10	900		27		0
		310		1	
11	1210		33		
		409			
13	2028				

Taking $x = 8$ in the Newton's divided difference formula, we obtain

$$f(8) = 48 + (8-4)\,52 + (8-4)\,(8-5)\,15 + (8-4)\,(8-5)\,(8-7)\,1$$

$$= \mathbf{448.}$$

Similarly $f(15) = 3150$.

EXAMPLE 7.25

Determine $f(x)$ as a polynomial in x for the following data:

x:	– 4	– 1	0	2	5
$y = f(x)$:	1245	33	5	9	1335

Solution:

The divided differences table is

x	$f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
– 4	1245				
		– 404			
– 1	33		94		
		– 28		– 14	
0	5		10		3
		2		13	
2	9		88		
		442			
5	1335				

Applying Newton's divided difference formula

$$f(x) = f(x_0) + (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x_0, x_1, x_2] + \cdots$$

$$= 1245 + (x + 4)(-404) + (x + 4)(x + 1)(94)$$

$$+ (x + 4)(x + 1)(x - 0)(-14) + (x + 4)(x + 1)x(x - 2)(3)$$

$$= 3x^4 - 5x^2 + 6x^2 - 14x + 5$$

EXAMPLE 7.26

Using Newton's divided difference formula, find the missing value from the table:

x:	1	2	4	5	6
y:	14	15	5	...	9

Solution:

The divided difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
1	14			
		$\frac{15-14}{2-1} = 1$		
2	15		$\frac{-5-1}{4-1} = -2$	
		$\frac{5-15}{4-2} = -5$		$\frac{7/4+2}{6-1} = \frac{3}{4}$

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
4	5		$\frac{2+5}{6-2}=\frac{7}{4}$	
		$\frac{9-6}{6-4}=2$		
6	9			

Newton's divided difference formula is

$$y = y_0 + (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x_0, x_1, x_2]$$
$$+ (x - x_0)(x - x_1)(x - x_2)[x_0, x_1, x_2, x_3] + \cdots$$
$$= 14 + (x - 1)\,(1) + (x - 1)\,(x - 2)\,(-2) + (x - 1)\,(x - 2)\,(x - 4)\,\frac{3}{4}$$

Putting $x = 5$, we get

$$y(5) = 14 + 4 + (4)\,(3)\,(-2) + (4)\,(3)\,(1)\,\frac{3}{4} = 3.$$

Hence missing value is 3

Exercises 7.4

1. Find the third divided difference with arguments 2, 4, 9, 10 of the function $f(x) = x^3 - 2x$.

2. Obtain the Newton 's divided difference interpolating polynomial and hence find $f(6)$:

x:	3	7	9	10
$f(x)$:	160	120	72	63

3. Using Newton's divided differences interpolation, find $u(3)$, given that $u(1) = -26$, $u(2) = 12$, $u(4) = 256$, $u(6) = 844$.

4. A thermocouple gives the following output for rise in temperature

Tem p (°C)	0	10	20	30	40	50
Output (m V)	0.0	0.4	0.8	1.2	1.6	2.0

Find the output of thermocouple for 37°C temperature using Newton's divided difference formula.

5. Using Newton 's divided difference interpolation, find the polynomial of the given data:

x:	-1	0	1	3
$f(x)$:	2	1	0	-1

6. For the following table, find $f(x)$ a s a polynomial in x using Newton's divided difference formula:

x:	5	6	9	11
$f(x)$:	12	13	14	16

7. Using the following data, find $f(x)$ a s a polynomial in x:

x:	– 1	0	3	6	7
$f(x)$:	3	– 6	39	822	1611

8. The observed values of a function are respectively 168, 120, 72, and 63 at the four positions 3, 7, 9, and 10 of the independent variable. What is the best estimate value of the function at the position 6?

9. Find the equation of the cubic curve which passes through the point s (4, – 43), (7, 83), (9, 327), and (12, 1053).

10. Find the missing term in the following table using Newton's divided difference formula.

x:	0	1	2	3	4
y:	1	3	9	...	81

7.16 Hermite's Interpolation Formula

This formula is similar to the Lagrange's interpolation formula. In Lagrange's method, the interpolating polynomial $P(x)$ agrees with $y(x)$ at the points $x_0, x_1, \ldots\ldots, x_n$, whereas in

Hermite's method $P(x)$ and $y(x)$ as well as $P'(x)$ and $y'(x)$ coincide at the $(n+1)$ points, *i.e.*,

$$P(x_i) = y(x_i) \text{ and } P'(x_i) = y'(x_i);\ i = 0, 1, \ldots\ldots, n \tag{1}$$

As there are $2(n + 1)$ conditions in (1), $(2n + 2)$ coefficients are to be determined.

Therefore $P(x)$ is a polynomial of degree $(2n + 1)$.

We assume that $P(x)$ is expressible in the form

$$p(x) = \sum_{i=0}^{n} U_i(x)\, y(x_i) + \sum_{i=0}^{n} V_i(x)\, y'(x_i) \tag{2}$$

where $U_i(x)$ and $V_i(x)$ are polynomials in x of degree $(2n + 1)$. These are to be determined. Using the conditions (1), we get

$$\left.\begin{aligned} U_i(x_j) &= \begin{cases} 0 \text{ when } i \neq j;\ V_i(x_j) = 0 \text{ for all } i \\ 1 \text{ when } i = j \end{cases} \\ U_i'(x_j) = 0 \text{ for all } i; V_i(x_j) &= \begin{cases} 0 \text{ when } i \neq j; \\ 1 \text{ when } i = j \end{cases} \end{aligned}\right\} \quad (3)$$

We now write

$$U_i(x) = A_i(x)\left[L_i(x)\right]^2; V_i(x) = B_i(x)\left[L_i(x)\right]^2$$

where
$$L_i(x) = \frac{(x-x_0)(x-x_1)\cdots(x-x_{i-1})(x-x_{i+1})\cdots(x-x_n)}{(x_i-x_0)(x_i-x_1)\cdots(x_i-x_{i-1})(x_i-x_{i+1})\cdots(x_i-x_n)}$$

Since $[L_i(x)]^2$ is of degree $2n$ and $U_i(x)$, $V_i(x)$ are of degree $(2n + 1)$, therefore $A_i(x)$ and $Bi(x)$ are both linear functions

$\therefore$ We can write
$$\left.\begin{aligned} U_i(x) &= (a_i + b_i x)\,[L_i(x)]^2 \\ \text{and} \quad V_i(x) &= (c_i + d_i x)[L_i(x)]^2 \end{aligned}\right\} \quad (4)$$

Using conditions (3) in (4), we get
$$\left.\begin{aligned} a_i + b_i x = 1,\ c_i + d_i x &= 0 \\ \text{and} \quad b_i + 2L_i'(x_i) = 0,\ d_i &= 1 \end{aligned}\right\} \quad (5)$$

Solving these equations, we obtain

$$\left.\begin{aligned} &b_i = -\,2L_i'(x_i), a_i = 1 + 2x_i L_i'(x_i) \\ &d_i = 1 \text{ and } c_i = -x_i \end{aligned}\right\} \quad (6)$$

Now putting the above values in (4), we get

$$\begin{aligned} U_i(x) &= [1 + 2x_i L_i'(x_i) - 2xL_i'(x_i)][L_i(x)]^2 \\ &= [1 - 2(x - x_i)L_i'(x_i)][L_i(x)]^2 \end{aligned}$$

and $V_i(x) = (x - x_i)\,[L_i(x)]^2$

Finally substituting $U_i(x)$ and $V_i(x)$ in (2), we obtain

$$p(x) = \sum_{i=0}^{n}\left[1 - 2\left(x - x_i\right)Li(x_i)\right][Li(x)]^2 y(x_i) + \sum_{i=0}^{n}\left(x - x_i\right)[Li(x)]^2 y'(x_i) \quad (7)$$

This is the required *Hermite's interpolation formula* which is sometimes known as *osculating interpolation formula*.

NOTE **Obs.** *In comparison to Lagrange's interpolation formula, the Hermite interpolation formula is computationally uneconomical*

EXAMPLE 7.27

For the following data:

x:	$f(x)$	$f'(x)$
0.5	4	– 16
1	1	– 2

Find the Hermite interpolating polynomial.

Solution:

We have $x_0 = 0.5$, $x_1 = 1$, $y(x_0) = 4$, $y(x_1) = 1$; $y'(x_0) = -16$, $y'(x_1) = -2$

Also $$L_i(x_0) = \frac{(x - x_0)}{(x_i - x_0)} = \frac{x-1}{-0.5} = -2(x-1); L_i'(x_0) = -2$$

$$L_i(x_1) = \frac{(x - x_0)}{(x_i - x_0)} = \frac{x-0.5}{1-0.5} = 2x-1; L_i'(x_1) = 2$$

Hermite's interpolation formula in this case, is

$$P(x) = [1-2(x-x_0)L'(x_0)][L(x_0)]^2 y(x_0) + (x-x_0)[L(x_0)]^2 y'(x_0)$$

$$+[1-2(x-x_1)L'(x_1)][L(x_1)]^2 y(x_1) + (x-x_1)[L(x_1)]^2 y'(x_1)$$

$$= [1-2(x-0.5)(-2)][-2(x-1)]^2(4) + (x-0.5)[-2(x-1)]^2(-16)$$

$$+[1-2(x-1)(2)](2x-1)^2(1) + (x-1)(2x-1)^2(-2)$$

$$= 16[1+4(x-0.5](x^2-2x+1) \; -164(x-0.5)(x^2-2x+1)$$

$$+[1 \; -4(x-1)](4x^2-4x+1) \; -2(x-1)(4x^2 - \; 4x+1)$$

Hence $P(x) = -24x^3 + 324x^2 - 130x + 23$

EXAMPLE 7.28

Apply Hermite's formula to interpolate for sin (1.05) from the following data:

x	$\sin x$	$\cos x$
1.00	0.84147	0.54030
1.10	0.89121	0.45360

Solution:

Here $y(x) = \sin x$ and $y'(x) = \cos x$

so that $y(x_0) = 0.84147$, $y'(x_0) = 0.54030$, $y(x_1) = 0.89121$,

$y'(x_1) = 0.45360$

Also $\quad L_i(x_0) = \dfrac{(x - x_0)}{(x_i - x_0)} = 11 - 10x, L_i'(x_0) = -10$

$$L_i(x_1) = \frac{(x - x_0)}{(x_i - x_0)} = -10 + 10x, L_i'(x_1) = 10$$

Hence the Hermite's interpolation formula in this case is

$$P(x) = [1 - 2(x - x_0)L'(x_0)][L(x_0)]^2 y(x_0) + (x - x_0)[L(x_0)]^2 y'(x_0)$$
$$+[1 - 2(x - x_1)L'(x_1)][L(x_1)]^2 y(x_1) + (x - x_1)[L(x_1)]^2 y'(x_1)$$

$$= [1 - 2(x - 1)(-10)](11 - 10x)^2 (0.84147) + (x - 1)(-11 + 10x)^2 (0.54030)$$
$$+[1 - 2(x - 1.1)(10)](-10 \ +10x)^2 (0.89121)$$
$$+(x - 1.1)(-10 \ +10x)^2 (0.4536)$$

Putting $x = 1.05$ in $P(x)$, we get

$$\sin(1.05) = 1 - 2(0.05)(-10)[-10(1.05) \ +11]^2 (0.84147)$$
$$+(0.05)(-0.5)^2 (0.54030) + [1 - 2(0.05)(10)](0.5)^2 \times (0.89121)$$
$$+(-0.05)(0.5)^2 (0.4536) = 0.86742$$

EXAMPLE 7.29

Determine the Hermite polynomial of degree 4 which fits the following data:

x:	0	1	2
$y(x)$:	0	1	0
$y'(x)$:	0	0	0

Solution:

Here $x_0 = 0$, $x_1 = 1$, $x_2 = 2$, $y(x_0) = 0$, $y(x_1) = 1$, $y(x_2) = 0$ and $y'(x_0) = 0$, $y'(x_1) = 0$, $y'(x2) = 0$.

Hermite's formula in this case is

$$P(x) = [1 - 2L_0'(x_0)(x - x_0)][L_0(x)]^2 y(x_0) + (x - x_0)[L_0(x)]^2 y'(x_0)$$
$$+[1 - 2L_1'(x_1)(x - x_1)][L1(x)]^2 \times y(x_1) + (x - x_1)[L1(x)]^2 y'(x_1)$$
$$+[1 - 2L_2'(x_2)(x - x_2)] \times [L_2(x)]^2 y(x_2) + (x - x_2)[L_2(x)]^2 y'(x_2)$$

Substituting the above values in $P(x)$, we get

$$P(x) = [1 - 2L_1'(x_1)(x - 1)][L_1(x)]^2$$

Where $L_1(x) = \dfrac{(x - x_0)(x - x_2)}{(x_1 - x_2)(x_1 - x_2)} = 2x - x^2$ and $L_1'(x_1) = (2 - 2x)_{x-1} = 0$

Hence $p(x) = [L_1(x)]^2 = (2x - x^2)^2$.

EXAMPLE 7.30

Using Hermite's intropolation, find the value of $f(-0.5)$ from the following

x:	– 1	0	1
$f(x)$:	1	1	3
$f'(x)$:	– 5	1	7

Solution:

Here $x_0 = -1$, $x_1 = 0$, $x_2 = 1$; $f(x_0) = 1$, $f(x_1) = 1$, $f(x_2) = 3$ and $f'(x_0) = -5$, $f'(x_1) = 1$, $f'(x_2) = 7$.

Hermite's formula in this case is

$$P(x) = U_0 f(x_0) + V_0 f'(x_0) + U_1 f(x_1) + V_1 f'(x_1) + U_2 f(x_2) + V_2 f'(x_2) \qquad (i)$$

where $U_0 = [1 - 2L_0'(x_0)(x - x_0)][L_0(x)]^2, V_0 = (x - x_0)[L_0(x)]^2$

$$U_1 = [1 - 2L_1'(x_1)(x - x_1)][L_1(x)]^2, V_1 = (x - x_1)[L_1(x)]^2$$
$$U_2 = [1 - 2L_2'(x_2)(x - x_2)][L_2(x_2)]^2, V_2 = (x - x_2)[L_2(x)]^2$$

and $L_0(x) = \dfrac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} = \dfrac{x(x-1)}{2}, L_0'(x) = x - \dfrac{1}{2}$

$$L_1(x) = \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} = 1 - x^2 = L_1'(x) = -2x$$

$$L_2(x) = \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} = \frac{x(x+1)}{2} = L_2'(x) = x + \frac{1}{2}$$

Substituting the values of $L_0, L_0'; L_1, L_1'$ and L_2, L_2', we get

$$U_0 = [1+3(x+1)]\ \frac{x^2(x-1)^2}{4} = \frac{1}{4}\left(3x^5 - 2x^4 - 5x^3 + 4x^2\right)$$

$$V_0 = (x+1)\ \frac{x^2(x-1)^2}{4} = \frac{1}{4}\left(x^5 - x^4 - x^3 + x^2\right)$$

$$U_1 = x^4 - 2x^2 + 1, V\ 1 = x^5 - 2x^3 + x$$

$$U_2 = \frac{1}{4}\left(3x^5 - 2x^4 - 5x^3 + 4x^2\right), V_2 = \frac{1}{4}\left(x^5 - x^4 - x^3 + x^2\right)$$

Substituting the values of $U_{0,,}$ V_0, U_1, V_1; U_2, V_2 in (i), we get

$$P(x) = \frac{1}{4}\left(3x^5 - 2x^4 - 5x^3 + 4x^2\right)(1) + \frac{1}{4}\left(x^5 - x^4 - x^3 + x^2\right)$$

$$+(x^4 - 2x^2 + 1)(1) + (x^5 - 2x^3 + x)(1)$$

$$-\frac{1}{4}(3x^5 - 2x^4 - 5x^3 + 4x^2)(3) + \frac{1}{4}(x^5 - x^4 - x^3 + x^2)(7)$$

$$= 2x^4 - x^2 + x + 1$$

Hence $f(-0.5) = 2(-0.5)^4 - (-0.5)^2 + (-0.5) + 1 = 0.375$

Exercises 7.5

1. Find the Hermite's polynomial which fits the following data:

x:	0	1	2
$f(x)$:	1	3	21
$f'(x)$:	0	3	36

2. A switching path between parallel rail road tracks is to be a cubic polynomial joining positions (0, 0) and (4, 2) and tangents to the lines $y = 0$ and $y = 2$. Using Hermite's method, find the polynomial, given:

x	y	y'
0	0	0
4	2	0

3. Apply Herm it e's formula estimate the values of log 3.2 from the following data:

x	$y = \log x$	$y' = 1/x$
3.0	1.0986	0.3333
3.5	1.2528	0.2857
4.0	1.3863	0.2500

7.17 Spline Interpolation

In the interpolation methods so far explained, a single polynomial has been fitted to the tabulated points. If the given set of points belong to the polynomial, then this method works well, otherwise the results are rough approximations only. If we draw lines through every two closest points, the resulting graph will not be smooth. Similarly we may draw a quadratic curve through points A_i, A_{i+1} and another quadratic curve through A_{i+1}, A_{i+2}, such that the slopes of the two quadratic curves match at A_{i+1} (Fig. 7.1). The resulting curve looks better but is not quite smooth. We can ensure this by drawing a cubic curve through A_i, A_{i+1} and another cubic through A_{i+1}, A_{i+2}su ch that the slopes and curvatures of the two curves match at A_{i+1}. Such a curve is called a **cubic spline.** We may use polynomials of higher order but the resulting graph is not better. As such, cubic splines are commonly used. This technique of "spline-fitting" is of recent origin and has important applications.

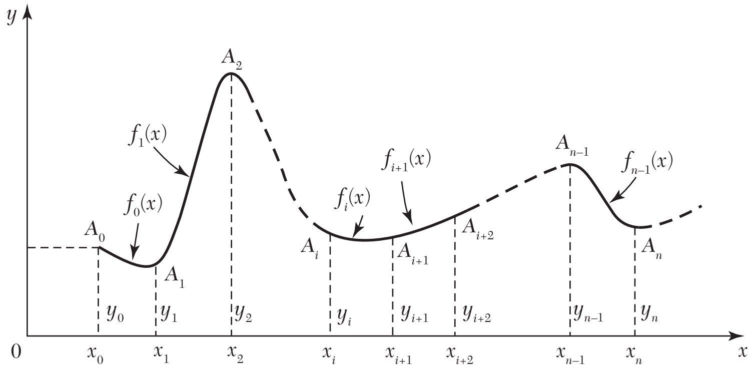

FIGURE 7.1

Cubic spline

Consider the problem of interpolating between the data points (x_0, y_0), $(x_1, y_1), \cdots (x_n, y_n)$ by means of spline fitting.

Then the cubic spline $f(x)$ is such that

(*i*) $f(x)$ is a linear polynomial outside the interval (x_0, x_n),

(*ii*) $f(x)$ is a cubic polynomial in each of the subintervals,

(*iii*) $f'(x)$ and $f''(x)$ are continuous at each point.

Since $f(x)$ is cubic in each of the subintervals $f''(x)$ shall be linear.

$\therefore$ Taking equally-spaced values of x so that $x_{i+1} - x_i = h$, we can write

$$f''(x) = \frac{1}{h}\left[\left(x_{i+1} - x\right)f''(x_i) + (x - x_i)f''(x_{i+1})\right]$$

Integrating twice, we have

$$f(x) = \frac{1}{h}\left[\frac{\left(x_{i+1} - x\right)}{3!}f''(x_i) + \frac{(x - x_i)}{3!}f''(x_{i+1})\right]a_i\left(x_{i+1} - x\right) + b_i(x - x_i) \qquad (1)$$

The constants of integration a_i, b_i are determined by substituting the values of $y = f(x)$ at x_i and x_{i+1}. Thus,

$$a_i = \frac{1}{h}\left[y_i - \frac{h^2}{3!}f''(x_i)\right] \text{ and } b_i = \frac{1}{h}\left[y_{i+1} - \frac{h^2}{3!}f''(x_{i+1})\right]$$

Substituting the values of *ai*, *bi* and writing $f''(x_i) = M_i$, (1) takes the form

$$f(x) = \frac{\left(x_{i+1} - x\right)^3}{6h}M_i + \frac{\left(x - x_i\right)^3}{6h}M_{i+1}$$
$$+ \frac{x_{i+1} - x}{h}\left(y_i - \frac{h^2}{6}M_i\right) + \frac{x - x_i}{h}\left(y_{i+1} - \frac{h^2}{6}M_{i+1}\right) \qquad (2)$$

$$\therefore f'(x) = -\frac{\left(x_{i+1} - x\right)^2}{2h}M_i + \frac{\left(x - x_i\right)^2}{6h}M_{i+1} - \frac{h}{6}\left(M_{i+1} - M_i\right) + \frac{1}{h}\left(y_{i+1} - y_i\right)$$

To impose the condition of continuity of $f'(x)$, we get
$f'(x - \varepsilon) = f'(x + \varepsilon)$ as $\varepsilon \to 0$

$$\therefore \frac{h}{6}(2M_i + M_{i-1}) + \frac{1}{h}(y_i - y_{i-1}) = -\frac{h}{6}(2M_i + M_{i+1}) + \frac{1}{h}(y_{i+1} - y_i)$$

$$M_{i-1} + 4M_i + M_{i+1} = \frac{6}{h^2}(y_{i-1} - 2y_i + y_{i+1}), i = 1 \text{ to } n - 1 \qquad (3)$$

Now since the graph is linear for $x < x_0$ and $x > x_n$, we have

$$M_0 = 0, M_n = 0 \qquad (4)$$

(3) and (4) give $(n + 1)$ equations in $(n + 1)$ unknowns M_i $(i = 0, 1, \cdots n)$ which can be solved. Substituting the value of M_i in (2) gives the concerned cubic spline.

EXAMPLE 7.31

Obtain the cubic spline for the following data

x:	0	1	2	3
y:	2	– 6	– 8	2

Solution:

Since the points are equispaced with $h = 1$ and $n = 3$, the cubic spline can be determined from $M_{i-1} + 4M_i + M_{i+1} = 6\,(y_{i-1} - 2y_i + y_{i+1})$, $i = 1, 2$.

$$\therefore \quad M_0 + 4M_1 + M_2 = 6\,(y_0 - 2y_1 + y_2)$$

$$M_1 + 4M_2 + M_3 = 6\,(y_1 - 2y_2 + y_3)$$

i.e., $4M_1 + M_2 = 36;\ M_1 + 4M_2 = 72$ $\quad [\because M_0 = 0,\ M_3 = 0]$

Solving these, we get $M_1 = 4.8$, $M_2 = 16.8$.

Now the cubic spline in $(x_i \le x \le x_i + 1)$ is

$$f(x) = \frac{1}{6}(x_{i+1} - x)^3 M_i + \frac{1}{6}(x - x_i)^3 M_{i+1} + (x_{i+1} - x)\left(y_i - \frac{1}{6}M_i\right)$$

$$+ (x - x_i)\left(y_{i+1} - \frac{1}{6}M_{i+1}\right)$$

Taking $i = 0$ in (A) the cubic spline in $(0 \le x \le 1)$ is

$$f(x) = \frac{1}{6}(1-x)^3(0) + \frac{1}{6}(x-0)^3(4.8) + (1-x)(x-0) + x[-6 - \frac{1}{6}(4.8)$$

$$= 0.8x^3 - 8.8x + 2 \qquad (0 \le x \le 1)$$

Taking $i = 1$ in (A), the cubic spline in $(1 \le x \le 2)$ is

$$f(x) = \frac{1}{6}(2-x)^3(4.8) + \frac{1}{6}(x-1)^3(16.8) + (2-x)[-\,6 - \frac{1}{6}(4.8)]$$

$$+ (x-1)[-\,8 - 1(16.8)]$$

$$= 2x^3 - 5.84x^2 - 1.68x + 0.8$$

Taking $i = 2$ in (A), the cubic spline in $(2 \le x \le 3)$ is

$$f(x) = \frac{1}{6}(3\ - x)^3(4.8) + \frac{1}{6}(x-2)^3(0) + (3-x)[-\,8\ - 1(16.8)]$$

$$+ (x-2)[2 - 1(2)]$$

$$= -\,0.8x^3 + 2.64x^2 + 9.68x - 14.8$$

EXAMPLE 7.32

The following values of x and y are given:

x:	1	2	3	4
y:	1	2	5	11

Find the cubic splines and evaluate $y(1.5)$ and $y'(3)$.

Solution:

Since the points are equispaced with $h = 1$ and $n = 3$, the cubic splines can be obtained from

$$M_{i-1} + 4M_i + M_{i+1} = 6(y_{i-1} - 2y_i + y_{i+1}),\ i = 1, 2.$$

$$\therefore \quad M_0 + 4M_1 + M_2 = 6(y_0 - 2y_1 + y_2)$$

$$M_1 + 4M_2 + M_3 = 6(y_1 - 2y_2 + y_3)$$

i.e., $\quad 4M_1 + M_2 = 12,\ M_1 + 4M_2 = 18 \qquad [\because M_0 = 0, M_3 = 0]$

which give, $\quad M_1 = 2,\ M_2 = 4.$

Now the cubic spline in $(x_i \le x \le x_{i+1})$ is

$$f(x) = \frac{1}{6}(x_{i+1} - x)^3 M_i + \frac{1}{6}(x - x_i)^3 M_{i+1} + (x_{i+1} - x)\left(y_i - \frac{1}{6}M_i\right)$$

$$+(x - x_i)\left(y_{i+1} - \frac{1}{6}M_{i+1}\right) \qquad \text{(A)}$$

Thus, taking $i = 0$, $i = 1$, $i = 2$ in (A), the cubic splines are

$$f(x) = \begin{cases} \frac{1}{3}\left(x^3 - 3x^2 + 5x\right) 1 \le x \le 2 \\ \frac{1}{3}\left(x^3 - 3x^2 + 5x\right) 2 \le x \le 3 \\ \frac{1}{3}\left(-2x^3 - 24x^2 - 76x + 81\right) 3 \le x \le 4 \end{cases}$$

$$\therefore \quad y(1.5) = f(1.5) = 11/8$$

EXAMPLE 7.33

Find the cubic spline interpolation for the data:

x:	1	2	3	4	5
$f(x)$:	1	0	1	0	1

Solution:

Since the points are equispaced with $h = 1$, $n = 4$, the cubic spline can be found by means of

$$M_{i-1} + 4M_i + M_{i+1} = 6(y_{i-1} - 2y_i + y_{i+1}), i = 1, 2, 3$$

$$\therefore \quad M_0 + 4M_1 + M_2 = 6\left(y_0 - 2y_1 + y_2\right) = 12$$

$$M_1 + 4M_2 + M_3 = 6(y_1 - 2y_2 + y_3) = -12$$

$$M_2 + 4M_3 + M_4 = 6(y_2 - 2y_3 + y_4) = 12$$

Since $M_0 = y''_0 = 0$ and $M_4 = y''_4 = 0$

$$\therefore \quad 4M_1 + M_2 = 12; M_1 + 4M_2 + M_3 = -12; M_1 + 4M_3 = 12$$

Solving these equations, we get $M_1 = 30/7$, $M_2 = -36/7$, $M_3 = 30/7$

Now the cubic spline in $(xi \le x \le xi{+}1)$ is

$$f(x) = \frac{1}{6}\left(x_{i+1} - x\right)^3 M_i + \frac{1}{6}\left(x - x_i\right)^3 M_{i+1} + \left(x_{i+1} - x\right)\left(y_i - \frac{1}{6}M_i\right) + \left(x - x_i\right)\left(y_{i+1} - \frac{1}{6}M_{i+1}\right) \qquad (A)$$

Taking $i = 0$, in (A), the cubic spline in $(1 \le x \le 2)$ is

$$y = \frac{1}{6}\left[\left(x_1 - x\right)^3 M_0 + \left(x - x_0\right)^3 M_1\right] + \left(x_1 - x\right)\left(y_0 - \frac{1}{6}M_0\right) + \left(x - x_0\right)\left(y_1 - \frac{1}{6}M_1\right)$$

$$= \frac{1}{6}\left[(2 - x)^3(0) + \left(x - x_0\right)^3(30/7)\right] + (2 - x)\left(1 - \frac{1}{6}(0)\right) + (x - 1)\left(0 - \frac{1}{6}\left(\frac{30}{7}\right)\right)$$

i.e., $\quad y = 0.71x^3 - 2.14x^2 + 0.42x + 2 \quad (1 < x \le 2)$

Taking $i = 1$ in (A), the cubic spline in $(2 \le x \le 3)$ is

$$y = \frac{1}{6}\left[(3 - x)^3 \frac{30}{7} + (x - 2)^3\left(-\frac{36}{7}\right)\right] + (3 - x)\left(0 - \frac{1}{6}\left(\frac{30}{7}\right)\right) + (x - 2)\left(1 - \frac{1}{6}\left(-\frac{36}{7}\right)\right)$$

i.e., $y = -1.57\, x^3 + 11.57x^2 - 27x + 20.28.$ $(2 \le x \le 3)$

Taking $i = 2$ in (A), the cubic spline in $(3 \le x \le 4)$ is

$$y = \frac{1}{6}(4-x)^3\left(-\frac{36}{7}\right) + \frac{1}{6}(x-3)^3\,\frac{30}{7} + (4-x)\left(1 - \frac{1}{6}\left(-\frac{36}{7}\right)\right) + (x-3)\left(0 - \frac{5}{7}\right)$$

i.e., $y = 1.57\, x^3 - 16.71\, x^2 + 57.86x - 64.57$ $(3 \le x \le 4)$

Taking $i = 3$ in (A), the cubic spline in $(4 \le x \le 5)$ is

$$y = \frac{1}{6}(1-x)^3\left(\frac{30}{7}\right) + (5-x)^3\left(-\frac{5}{7}\right) + (x-4)(1)$$

i.e., $y = -0.71x^3 + 2.14x^2 - 0.43x - 6.86.$ $(4 \le x \le 5)$

Exercises 7.6

1. Find the cubic splines for the following table of values:

x:	1	2	3
y:	– 6	– 1	16

Hence evaluate $y(1.5)$ and $y'(2)$.

2. The following values of x and y are given:

x:	1	2	3	4
y:	1	5	11	8

Usin g cubic spline, show that

(*i*) $y(1.5) = 2.575$ (*ii*) $y'(3) = 2.067.$

3. Find the cubic spline corresponding to the interval [2,3] from the following table:

x:	1	2	3	4	5
y:	30	15	32	18	25

Hence compute (*i*) $y(2.5)$ and (*ii*) $y'(3)$.

7.18 Double Interpolation

So far, we have derived interpolation formulae to approximate a function of a single variable. In the case of functions, of two variables, we interpolate with respect to the first variable keeping the other variable constant. Then interpolate with respect to the second variable.

Similarly, we can extend the said procedure for functions of three variables.

7.19 Inverse Interpolation

So far, given a set of values of x and y, we have been finding the value of y corresponding to a certain value of x. On the other hand, the process of estimating the value of x for a value of y (which is not in the table) is called *inverse interpolation*. When the values of x are unequally spaced *Lagrange's method* is used and when the values of x are equally spaced, the *Iterative method* should be employed.

7.20 Lagrange's Method

This procedure is similar to Lagrange's interpolation formula (p. 207), the only difference being that x is assumed to be expressible as a polynomial in y.

Lagrange's formula is merely a relation between two variables either of which may be taken as the independent variable. Therefore, on interchanging x and y in the Lagrange's formula, we obtain

$$x=\frac{(y-y_1)(y-y_2)\cdots(y-y_n)}{(y-y_1)(y-y_2)\cdots(y-y_n)}x_0+\frac{(y-y_0)(y-y_2)\cdots(y-y_n)}{(y_1-y_0)(y_1-y_2)\cdots(y1-y_n)}x_1 + \frac{(y-y_0)(y-y_1)\cdots(y-y_{n-1})}{(y_n-y_0)(y_n-y_1)\cdots(y_n-y_{n-1})} \quad (1)$$

EXAMPLE 7.34

The following table gives the values of x and y:

x:	1.2	2.1	2.8	4.1	4.9	6.2
y:	4.2	6.8	9.8	13.4	15.5	19.6

Find the value of x corresponding to $y = 12$, using Lagrange's technique.

Solution:

Here $x_0 = 1.2$, $x_1 = 2.1$, $x_2 = 2.8$, $x_3 = 4.1$, $x_4 = 4.9$, $x_5 = 6.2$ and $y_0 = 4.2$, $y_1 = 6.8$, $y_2 = 9.8$, $y_3 = 13.4$, $y_4 = 15.5$, $y_5 = 19.6$.

Taking $y = 12$, the above formula (1) gives

$$x = \frac{(12-6.8)(12-9.8)(12-13.4)(12-15.5)(12-19.6)}{(4.2-6.8)(4.2-9.8)(4.2-13.4)(4.2-15.5)(4.2-19.6)} \times 1.2$$
$$+ \frac{(12-4.2)(12-9.8)(12-13.4)(12-15.5)(12-19.6)}{(6.8-4.2)(6.8-9.8)(6.8-13.4)(6.8-15.5)(6.8-19.6)} \times 2.1$$
$$+ \frac{(12-4.2)(12-6.8)(12-13.4)(12-15.5)(12-19.6)}{(9.8-4.2)(9.8-6.8)(9.8-13.4)(9.8-15.5)(9.8-19.6)} \times 2.8$$
$$+ \frac{(12-4.2)(12-6.8)(12-9.8)(12-15.5)(12-19.6)}{(13.4-4.2)(13.4-6.8)(13.4-9.8)(13.4-15.5)(13.4-19.6)} \times 4.1$$
$$+ \frac{(12-4.2)(12-6.8)(12-9.8)(12-13.4)(12-19.6)}{(15.5-4.2)(15.5-6.8)(15.5-9.8)(15.5-13.4)(15.5-19.6)} \times 4.9$$
$$+ \frac{(12-4.2)(12-6.8)(12-9.8)(12-13.4)(12-15.5)}{(19.6-4.2)(19.6-6.8)(19.6-9.8)(19.6-13.4)(19.6-15.5)} \times 6.2$$

$= 0.022 - 0.234 + 1.252 + 3.419 - 0.964 + 0.055 = 3.55$

EXAMPLE 7.35

Apply Lagrange's formula inversely to obtain a root of the equation $f(x) = 0$, given that $f(30) = -30$, $f(34) = -13$, $f(38) = 3$, and $f'(42) = 18$.

Solution:

Here $x_0 = 30$, $x_1 = 34$, $x_2 = 38$, $x_3 = 42$

and $y_0 = -30$, $y_1 = -13$, $y_2 = 3$, $y_3 = 18$

It is required to find x corresponding to $y = f(x) = 0$.

Taking $y = 0$, Lagrange's formula gives

$$x = \frac{(y-y_1)(y-y_2)(y-y_3)}{(y_0-y_1)(y_0-y_2)(y_0-y_3)} x_0 + \frac{(y-y_0)(y-y_2)(y-y_3)}{(y_1-y_0)(y_1-y_2).(y1-y_3)} x_1$$
$$+ \frac{(y-y_0)(y-y_1).(y-y_3)}{(y_2-y_0)(y_2-y_1)(y_2-y_3)} x_2 + \frac{(y-y_0)(y-y_1)(y-y_2)}{(y_3-y_0)(y_3-y_1)(y_3-y_2)} x_3$$
$$= \frac{13(-3)(-18)}{(-17)(-33)(-48)} \times 30 + \frac{30(-3)(-18)}{17(-16)(-31)} \times 34$$
$$+ \frac{30(13)(-18)}{33(16)(-15)} \times 38 + \frac{30(13)(-3)}{48(31)(15)} \times 42$$

$= -0.782 + 6.532 + 33.682 - 2.202 = 37.23.$

Hence the desired root of $f(x) = 0$ is 37.23.

7.21 Iterative Method

Newton's forward interpolation formula (p. 274) is

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \ldots$$

From this, we get

$$p = \frac{1}{\Delta y_0}\left[y_p - y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \cdots\right] \tag{1}$$

Neglecting the second and higher differences, we obtain the first approximation to p as

$$p_1 = (y_p - y_0)/\Delta y_0 \tag{2}$$

To find the second approximation, retaining the term with second differences in (1) and replacing p by p_1, we get

$$p_2 = \frac{1}{\Delta y_0}\left[y_p - y_0 + \frac{p_1(p_1 - 1)}{2!}\Delta^2 y_0\right] \tag{3}$$

To find the third approximation, retaining the term with third differences in (1) and replacing every p by p_2, we have

$$p_3 = \frac{1}{\Delta y_0}\left[y_p - y_0 + \frac{p_2(p_2 - 1)}{2!}\Delta^2 y_0 + \frac{p_2(p_2 - 1)(p_2 - 2)}{3!}\Delta^3 y_0\right]$$

and so on. This process is continued till two successive approximations of p agree with each other

NOTE **Obs.** *This technique can be equally well be applied by starting with any other interpolation formula.*

This method is a powerful iterative procedure for finding the roots of an equation to a good degree of accuracy.

EXAMPLE 7.36

The following values of $y = f(x)$ are given

x:	10	15	20
y:	1754	2648	3564

Find the value of x for $y = 3000$ by iterative method.

Solution:

Taking $x_0 = 10$ and $h = 5$, the difference table is

x	y	Δy	$\Delta^2 y$
10	1754		
		894	
15	2648		22
		916	
20	3564		

Here $y_p = 3000$, $y_0 = 1754$, $\Delta y_0 = 894$ and $\Delta^2 y0 = 22$.

$\therefore$ The successive approximations to p are

$$p_1 = \frac{1}{894}(3000 - 1754) = 1.39$$

$$p_2 = \frac{1}{894}\left[3000 - 1754 - \frac{1.39(1.39-1)}{2} \times 22\right] = 1.387$$

$$p_3 = \frac{1}{894}\left[3000 - 1754 - \frac{1.387(1.387-1)}{2} \times 22\right] = 1.3871$$

We, therefore, take $p = 1.387$ correct to three decimal places. Hence the value of x (corresponding to $y = 3000$) $= x_0 + ph = 10 + 1.387 \times 5 = 16.935$.

EXAMPLE 7.37

Using inverse interpolation, find the real root of the equation $x3 + x - 3= 0$, which is close to 1.2.

Solution:

The difference table is

x	y	$Y = x^3 + x - 3$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1	– 0.2	– 1	0.431			
				0.066		
1.1	– 0.1	– 0.569	0.497		0.006	
				0.072		0
1.2	0	– 0.072			0.006	

x	y	$Y = x^3 + x - 3$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
			0.569			
1.3	0.1	0.497		0.078		
			0.647			
1.4	0.2	1.144				

Clearly the root of the given equation lies between 1.2 and 1.3. Assuming the origin at $x = 1.2$ and using Stirling's formula

$$y = y_0 + x\frac{\Delta y_0 + \Delta y_{-1}}{2} + \frac{x^2}{2}\Delta^2 y_{-1} + \frac{x(x^2-1)}{6} \times \frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}, \text{ we get}$$

$$0 = -0.072 + x.\frac{0.569 + 0.467}{2} + \frac{x^2}{2}(0.072) + \frac{x(x^2-1)}{6}\left(\frac{0.006 + 0.006}{2}\right)$$

$$(\because y = 0)$$

or $$0 = -0.072 + 0.532x + 0.036x^2 + 0.001x^3 \qquad (i)$$

This equation can be written a s

$$x = \frac{0.072}{0.532} - \frac{0.036}{0.532}x^2 - \frac{0.001}{0.532}x^3$$

$\therefore$ First approximation $x^{(1)} = \dfrac{0.072}{0.532} = 0.1353$

Putting $x = x^{(1)}$ on R.H.S. of (i), we get

Second approximation

$$x^{(2)} = 0.1353 - 0.067(0.1353)^2 - 1.8797(0.1353)^3 = 0.134$$

Hence the desired root = 1.2 + 0.1 × 0.134 = 1.2134.

Exercises 7.7

1. Apply Lagrange's method to find the value of x when f(x)=5 from the given data:

x:	5	6	9	11
$f(x)$:	12	13	14	16

2. Obtain the value of t when A = 85 from the following table, using Lagrange's method:

t:	2	5	8	14
A:	94.8	87.9	81.3	68.7

3. Apply Lagrange's formula inversely to obtain the root of the equation $f(x) = 0$, given that $f(30) = -30, f(34) = -13, f(38) = 3$ and $f(42) = 18$.

4. From the following data:

x:	1.8	2.0	2.2	2.4	2.6
y:	2.9	3.6	4.4	5.5	6.7,

find x when $y = 5$ u sin g the iterative method.

5. The equation $x^3 - 15x + 4 = 0$ ha s a root close to 0.3. Obtain this root upto four decimal places using inverse interpolation.

6. Solve the equation $x = 10 \log x$, by iterative method given that

x:	1.35	1.36	1.37	1.38
$\log x$:	0.1303	0.1355	0.1367	0.1392

7.22 Objective Type of Questions

Exercises 7.8

1. Select the correct answer or fill up the blanks in the following question:
Newton's back war d interpolation formula is.........

2. Bessel's formula is most appropriate when p lies between
(a) – 0. 25 an d 0.25 (b) 0.25 an d 0.75 (c) 0.75 an d 1.00

3. Form the divided difference table for the following data:

x:	5	15	22
y:	7	36	160

4. Interpolation is the technique of estimating the value of a function for any......

5. Bessel's formula for interpolation is......

6. The four divided differences for x_0, x_1, x_2, x_3, x_4 =.......

7. Stirling's formula is best suited for p lying between......

8. Newton's divided differences formula is.......

9. Given (x_0, y_0), (x_1, y_1), (x_2, y_2), Lagrange's interpolation formula is.......

10. If $f(0) = 1, f(2) = 5, f(3) = 10$ and $f(x) = 14$, then $x =$......

11. The difference between Lagrange's interpolating polynomial and Hermite's interpolating polynomial is.......

12. If $y(1) = 4, y(3) = 12, y(4) = 19$ and $y(x) = 7$, find x using Lagrange's formula.

13. Extrapolation is defined as.......

14. The second divided difference of $f(x) = 1/x$, with arguments a, b, c is......

15. The Gauss-forward interpolation formula is used to interpolate values of y for

(a) $0 < p < 1$ (b) $-1 < 1 < 0$
(c) $0 < p < -\alpha$ (d) $-\alpha < p < 0$

16. Given

x:	0	1	3	4
y:	–12	0	6	12

Using Lagrange's formula, a polynomial that can be fitted to the data is......

17. The nth divided difference of a polynomial of degree n is

(a) zero (b) a constant
(c) a variable (d) none of these.

18. The Gauss forward interpolation formula involves

(a) differences above the central line and odd differences on the central line
(b) even differences below the central line and odd differences on the central line
(c) odd differences below the central line and even differences on the central line
(d) odd differences above the central line and even differences on the central line.

19. Differentiate between interpolation polynomial and least square polynomial obtained for a set of data.