E-Content for Lie Algebras (Remaining Part)

Vivek Sahai

Lucknow University
Lucknow.

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Definition

Let F be a field of characteristic 0, V a finite dimensional vector space over F and T a linear operator on V. Then the trace of T, denoted by $\mathrm{tr}\,(T)$, is given by $\mathrm{tr}\,(T) = \sum_{i=1}^{n} \alpha_{ii}$, where $(\alpha_{ij}) = [T]_B$, B being a basis of V.

If ρ_i are the characterstic roots of T, then

$$\mathrm{tr}\,(T) = \sum_{i=1}^{n} \rho_i.$$

For a nonnegative integer k, we have

$$\mathrm{tr}\,(T^k) = \sum_{i=1}^{n} \rho_i^k.$$

If T is nilpotent, then $\rho_i = 0$ for all i. This gives $\mathrm{tr}\,(T) = \mathrm{tr}\,(T^k) = 0$, for $k = 1, 2, \ldots$. Conversely, if $\mathrm{tr}\,(T^k) = 0$, $k = 1, 2, \ldots$, then T is nilpotent.

Lemma

Let V be a vector space over a field F of characteristic 0, and let $T \in L(V)$ such that $T = \sum_{i=1}^{r}[A_i, B_i]$, $A_i, B_i \in L(V)$ and $[T, A_i] = 0$, $i = 1, 2, \ldots, r$. Then T is nilpotent.

Proof.

Let $[T^{k-1}, A_i] = 0$, then

$$\begin{aligned}
[T^k, A_i] &= T^k A_i - A_i T^k \\
&= T[T^{k-1}, A_i] + TA_i T^{k-1} - [A_i, T]T^{k-1} - TA_i T^{k-1} \\
&= 0.
\end{aligned}$$

Therefore $[T^k, A_i] = 0$ for all $i = 1, 2, \ldots, r$ and $k = 1, 2, \ldots$. This gives □

Proof . . .

$$\begin{aligned} T^k &= T^{k-1}T = T^{k-1}\sum_{i=1}^{r}[A_i, B_i] = \sum_{i=1}^{r}(A_i T^{k-1}B_i - T^{k-1}B_i A_i) \\ &= \sum_{i=1}^{r}[A_i, T^{k-1}B_i]. \end{aligned}$$

As trace of a commutator is zero, we have $\operatorname{tr}(T^k) = 0$, for all $k = 1, 2, \ldots, r$. Hence T is nilpotent.

This completes the proof

Theorem

Let $\operatorname{char} F = 0$ and let L be a Lie algebra of linear transformations in $L(V)$ such that L^* is semi-simple. Then $L = L_1 \oplus Z$ where $Z = Z(L)$, the centre of L, and L_1 is an ideal of L (which is a semi-simple Lie algebra).

Proof of the Theorem . . .

Let C be the radical of L. If $C \neq Z$, then $C_1 = [L, C]$ is a non-zero solvable ideal.
Therefore there exists $n \in \mathbb{N}$ such that $C_1^{(n)} = \{0\}$ and $C_1^{(n-1)} \neq \{0\}$. Let $C_2 = C_1^{(n-1)}$ and $C_3 = [C_2, L]$.
If $T \in C_3$, then $T = \sum_{i=1}[A_i, B_i]$, for some $A_i \in C_2$ and $B_i \in L$. This gives $[T, A_i] \in [C_3, C_2] \subseteq [C_2, C_2] = \{0\}$. Therefore by above lemma, T is nilpotent. Hence, every element of ideal C_3 is nilpotent, and so by theorem of Unit 3, Lecture 6,

$$C_3 \subseteq C_3^* \subseteq R = \text{ the radical of } L^* = \{0\},$$

as L^* is semi simple. Hence, $C_2 \subseteq Z$.

Proof . . .

Since $C_2 \subseteq C_1 \subseteq L' = [L, L]$, every element T of C_2 is of the type $T = \sum_{i=1}^{r} [A_i, B_i]$, $A_i, B_i \in L$ and $[T, A_i] = 0$ because $C_2 \subseteq Z$. Therefore T is nilpotent by above lemma. So $C_2 \subseteq C_2^* \subseteq R =$ radical of $L^* = \{0\}$, a contradiction. Therefore, $C = Z$.

Let $L' \cap C \neq \{0\}$. If $T \in L' \cap C$ then $T = \sum_{i=1}^{r} [A_i, B_i]$, $A_i, B_i \in L$ and $[T, A_i] = 0$ as $T \in C = Z$. Therefore T is nilpotent by the lemma, and so $L' \cap C \subseteq R = \{0\}$, a contradiction.

Therefore there exists L_1, a subspace of L, $L_1 \supseteq L'$, such that $L = L_1 \oplus Z$. So L_1 is an ideal and $L_1 \cong \frac{L}{Z} = \frac{L}{C}$. Hence L_1 is semi-simple.

This completes the proof.

Corollary

Let L be as in the above theorem. Then L is solvable if and only if L is abelian. More generally, if L is solvable and R is the radical of L^*, then $\frac{L^*}{R}$ is commutative.

Proof.

Clearly, if L is abelian then L is solvable.
Conversely, let L be solvable and L^* semi-simple. Then $L = L_1 \oplus C$, where $C = Z =$ centre of L and L_1 is a semi-simple ideal of L. Therefore L_1 is semi-simple and solvable. But then $L_1 = \{0\}$. Hence, $L = C$ is abelian.
Consider the Lie algebra $\frac{L+R}{R}$. Clearly $\left(\frac{L+R}{R}\right)^* = \frac{L^*}{R}$, which is semisimple.
As $\frac{L+R}{R}$ is a homomorphic image of L, so L solvable implies $\frac{L+R}{R}$ is solvable, which gives $\frac{L+R}{R}$ is abelian, and hence $\frac{L^*}{R}$ is commutative. □

Go through the following definitions:

Let V be a vector space over F, $\dim_F(V) < \infty$, and let Σ be a set of linear operators on V. Let $L(\Sigma)$ denotes the collection of subspaces invariant under Σ, that is,

$$L(\Sigma) = \{W \mid W \text{ is a subspace of } V \text{ and } T(W) \subseteq W \text{ for all } T \in \Sigma\}.$$

We say that $L(\Sigma)$ is the collection of Σ-subspaces of V.

Definition

Σ is called an irreducible set of linear transformations and V is called Σ-irreducible if $L(\Sigma) = \{V, 0\}$ and $V \neq \{0\}$.

Definition

Σ is called indecomposable and V is called Σ-indecomposable if V can not be written as $V = V_1 \oplus V_2$, $V_i \neq 0$ in $L(\Sigma)$. Clearly, Σ-irreduciblity implies Σ-indecomposability.

Definition

Σ is called completely reducible and V is called Σ-completely reducible if $V = \oplus_\alpha V_\alpha$, $V_\alpha \in L(\Sigma)$, V_α irreducible.

Note that $W \in L(\Sigma)$ implies that W is invariant under Σ^* and $\Sigma^\dagger$. Therefore, $L(\Sigma) = L(\Sigma^*) = L(\Sigma^\dagger)$.

Theorem

Let V be a vector space over F, $\dim_F(V) < \infty$, and let Σ be a set of linear operators on V. Then Σ is completely reducible if and only if for every $W \in L(\Sigma)$ there exists $W' \in L(\Sigma)$ such that $V = W \oplus W'$ (that is, $L(\Sigma)$ is complemented).

Proof of the Theorem . . .

Let Σ be completely reducible. Therefore $V = \oplus_\alpha V_\alpha$, where each V_α is irreducible in $L(\Sigma)$. Let $W \in L(\Sigma)$. If $\dim W = \dim V$, then $V = W \oplus \{0\}$ and we are done.

Assume that $\dim W < \dim V$ and let the theorem hold for all subspaces $W_1 \in L(\Sigma)$ such that $\dim W_1 > \dim W$. Since $W \subsetneq V$ and $V = \oplus_\alpha V_\alpha$, there exists a V_α such that $V_\alpha \not\subseteq W$. Consider $V_\alpha \cap W \in L(\Sigma)$, As $V_\alpha \cap W$ is a subspace of irreducible Σ-subspace V_α, we have either $V_\alpha \cap W = V_\alpha$ or $V_\alpha \cap W = \{0\}$. Now $V_\alpha \cap W = V_\alpha$ is not possible, so $V_\alpha \cap W = \{0\}$.

Let $W_1 = W \oplus V_\alpha$, by induction hypothesis, $V = W_1 \oplus W_1'$, $W_1' \in L(\Sigma)$. This gives $V = W \oplus V_\alpha \oplus W_1' = W \oplus W'$, $W' = V_\alpha \oplus W_1' \in L(\Sigma)$.

Proof . . .

Conversely, let $L(\Sigma)$ be complemented and $V_1(\neq 0)$ be a minimal element of $L(\Sigma)$. As V is finite dimensional, so V_1 exists and it has to be irreducible. Therefore $V = V_1 \oplus W$, for some $W \in L(\Sigma)$.

If B is a Σ-subspace of W, then $V = B \oplus B'$ and $W = V \cap W = B \cap W + B' \cap W = B + B' \cap W = B + B''$, where $B'' = B' \cap W \in L(\Sigma)$ is a subspace of W.

Also $B'' \cap B = B' \cap W \cap B = \{0\}$ implies that $W = B \oplus B''$. Thus for W also $L(\Sigma)$ is complemented. Repeating this process for W we have $W = V_2 \oplus W_1$, $V_2, W_1 \in L(\Sigma)$, V_2 is irreducible.

Continuing in this way, in a finite number of steps, we get $V = V_1 \oplus V_2 \oplus \cdots \oplus V_r$, where each $V_i \in L(\Sigma)$ and are irreducible.

This completes the proof.

Lecture-3

Theorem

Let A be an associative algebra of linear transformations in $L(V)$, $\dim V < \infty$. *If A is completely reducible then A is semi-simple.*

Proof.

Let R be the radical of A and let $V = \oplus_\alpha V_\alpha$, V_α's irreducible in $L(A)$. Let $R(V_\alpha)$ be the subspace spanned by $\{T(y) | y \in V_\alpha, T \in R\}$. Then $R(V_\alpha) \in L(A)$ and $R(V_\alpha) \subseteq V_\alpha$.

Since there exists $k \in N$ such that $R^k = \{0\}$, $R(V_\alpha) \subsetneq V_\alpha$. Therefore $R(V_\alpha) = \{0\}$ for all α, (as V_α is irreducible).

This gives $R(V) = 0$, that is, $R = 0$, and so A is semi-simple. □

Lecture-3 . . .

Corollary

If Σ is completely reducible then Σ^* and $\Sigma^\dagger$ are semi-simple.

Proof.

Σ is completely reducible $\Rightarrow L(\Sigma)$ is complemented $\Leftrightarrow L(\Sigma^*)$ and $L(\Sigma^\dagger)$ are complemented $\Leftrightarrow \Sigma^*$ and $\Sigma^\dagger$ are complemented $\Rightarrow \Sigma^*$, $\Sigma^\dagger$ are completely reducible $\Rightarrow \Sigma^*$, $\Sigma^\dagger$ are semisimple. □

Definition

An operator $T \in L(V)$ is said to be semi-simple if $m_T(x) = p_1(x)p_2(x)\cdots p_r(x)$, where each $p_i(x)$ is an irreducible polynomial in $F[x]$, $p_i(x) \neq p_j(x)$.

Theorem

An operator $T \in L(V)$ is semi-simple if and only if $\{T\}^\dagger$ has no non-zero nilpotent elements.

Proof.

Let T be not semi-simple. Then $m_T(x) = p_1^{r_1}(x)\cdots p_k^{r_k}(x)$, where $r_i > 1$ for some i. Let $W = p_1(T)\cdots p_k(T)$. Clearly $W \in \{T\}^\dagger$. If m is the lcm of $\{r_i\}$'s, then

$$W^m = p_1(T)^m \cdots p_k(T)^m = 0.$$

Further $W \neq 0$ because $W | m_T(x)$ and $\deg m_T(x) > \deg W$. So W is non-zero nilpotent elements of $\{T\}^\dagger$.
Conversely, if $T \in L(V)$ is semi-simple, then $m_T(x) = p_1(x)\cdots p_k(x)$. Let $W = f(T)$ be a nilpotent element of $\{T\}^\dagger$. Then $W^r = 0$ implies $m_T(x)|f^r(x)$, and so $m_T(x)|f(x)$. Hence $0 = f(T) = W$. □

Theorem

Let L be a completely reducible Lie algebra of linear operators on a finite dimensional vector space V over a field of characteristic 0. Then $L = C \oplus L_1$, where $C = Z$ and L_1 is a semi-simple ideal. Moreover, elements of C are semi-simple.

Proof.

We know that if L is completely reducible then L^* and $L^\dagger$ are semi-simple. So by Theorem of Lecture 1, $L = C \oplus L_1$. Further, let $T \in C$ be such that T is not semi-simple. Then there exists a nonzero nilpotent element $W \in \{T\}^\dagger$. Let $k \in N$ be such that $W^k = 0$. Now $\{T\}^\dagger = \{a_0 + a_1 T + a_2 T^2 + \cdots + a_n T^n | n \in N, a_i \in F\}$. As $T \in C$, we have $W \in \{T\}^\dagger$ is also in the centre of L. Therefore $L^\dagger W = WL^\dagger$ is an ideal in $L^\dagger$ and $(WL^\dagger)^k \subseteq W^k L^\dagger = 0$. But $0 \neq W \in WL^\dagger$ implies $WL^\dagger$ is a non-zero nilpotent ideal in $L^\dagger$, a contradiction as $L^\dagger$ is semi-simple. $\square$

Definition

Let V be a finite dimensional vector space over F and let $\Sigma \subseteq L(V)$. A chain $V = V_1 \supset V_2 \supset \cdots \supset V_s \supset V_{s+1} = \{0\}$ of Σ-subspaces of V is a composition series for V relative to Σ if for every i, there exists no $V' \in L(\Sigma)$ such that $V_i \supset V' \supset V_{i+1}$, that is, $\frac{V_i}{V_{i+1}}$ is irreducible.

Note that as $\dim_F(V) < \infty$, composition series of V do exist. For, let V_2 be the maximal Σ-subspace of $V_1 = V$, $V_2 \neq V_1$. Similarly, let V_3 be the maximal Σ-subspace of V_2, $V_3 \neq V_2$, etc. Continuing like this. we get a composition series $V = V_1 \supset V_2 \supset \cdots \supset V_s \supset V_{s+1} = \{0\}$.

Lecture-4 . . .

Lemma

Let L be an abelian Lie algebra of linear operators on a finite dimensional vector space V over an algebraically closed field F. If V is L-irreducible, then $\dim_F(V) = 1$.

Proof.

If $T \in L$, then T has a non-zero characterstic vector x. So $T(x) = \alpha x$, for some $\alpha \in F$. Let $V_\alpha = \{v \in V | T(v) = \alpha v\}$, the characterstic subspace of V corresponding to characterstic value α. If $U \in L$ then $T(U(y)) = UT(y) = TU(y) = U(T(y)) = U(\alpha y) = \alpha U(y)$, for all $y \in V_\alpha$, and so $U(y) \in V_\alpha$. This gives V_α is an L-subspace of V. As V is L-irreducible, we have $V = V_\alpha$. Hence $T = \alpha I$.

Thus every $T \in L$ is such that $T = \alpha I$, for some $\alpha \in F$. So every subspace of V is an L-subspace. Now V is L-irreducible, so V has no subspaces other than V and $\{0\}$. Therefore $\dim_F V = 1$. □

Theorem

(Lie's Theorem) *Let V be a finite dimensional vector space over an algebraic closed field F of characteristic 0 and let $L \subseteq L(V)$ be such that L is a solvable Lie algebra. Then there exists a basis $\mathcal{B}$ of V such that $[T]_{\mathcal{B}} \in T_n(F)$ for all $T \in L$.*

Proof:
Let $V = V_1 \supset V_2 \supset \cdots \supset V_s \supset V_{s+1} = \{0\}$ be a composition series of V relative to L. If $T \in L$, then $T_i = T|_{V_i} \in L(V_i)$.

Define $\bar{T}_i : \frac{V_i}{V_{i+1}} \to \frac{V_i}{V_{i+1}}$ by $\bar{T}_i(x + V_{i+1}) = T_i(x) + V_{i+1}$. Let $\bar{L}_i = \{\bar{T}_i : T \in L, \bar{T}_i \in L(\frac{V_i}{V_{i+1}})\}$.

Define $\theta : L \to \bar{L}_i$ by $\theta(T) = \bar{T}_i$. Clearly, θ is an epimorphism and so $\bar{L}_i$ is solvable.

Proof . . .

Every $\bar{L}_i$-subspace of $\frac{V_i}{V_{i+1}}$ is of the type $\frac{W_i}{V_{i+1}}$ where W_i is an L-subspace of V_i containing V_{i+1}. As $\frac{V_i}{V_{i+1}}$ is irreducible, we have $\bar{L}_i$ is irreducible. So $\bar{L}_i$ is completely reducible in $\frac{V_i}{V_{i+1}}$. This gives $\bar{L}_i = C_i \oplus L_{i1}$, where L_{i1} is a semisimple ideal of $\bar{L}_i$. But then L_{i1} is also solvable. So $L_{i1} = \{0\}$ and $\bar{L}_i = C_i = Z_i$, the centre of $\bar{L}_i$. Hence $\bar{L}_i$ is abelian and $\dim \frac{V_i}{V_{i+1}} = 1$. This gives $\dim V_s = 1$, $\dim V_{s-1} = 2, \cdots, \dim V_2 = s-1$ and $\dim V_1 = s$.

Let $\mathcal{B} = \{e_1, e_2, e_3, \cdots, e_s\}$ be a basis for V such that $\{e_1\}$ is a basis for V_s, $\{e_1, e_2\}$ is a basis for V_{s-1}, $\{e_1, e_2, e_3\}$ is a basis for V_{s-2} etc. Then for $T \in L$,

$$\begin{aligned}
T(e_1) &= \alpha_{11}e_1, \\
T(e_2) &= \alpha_{12}e_1 + \alpha_{22}e_2, \\
&\vdots \\
T(e_s) &= \alpha_{1s}e_1 + \alpha_{2s}e_2 + \cdots + \alpha_{ss}e_s.
\end{aligned}$$

Proof . . .

Hence,

$$[T]_{\mathcal{B}} = \begin{pmatrix} \alpha_{11} & \alpha_{12} & . & . & . & \alpha_{1s} \\ & \alpha_{22} & . & . & . & \alpha_{2s} \\ & & . & & & . \\ & & & . & & . \\ & & & & . & . \\ & & & & & \alpha_{ss} \end{pmatrix}.$$

This completes the proof of the Lie's theorem

Theorem

Let L be a finite dimensional solvable Lie algebra over an algebraic closed field of characterstic zero. Then there exists a chain of ideals $L = I_s \supset I_{s-1} \supset \cdots \supset I_1 \supset \{0\}$ *such that* $\dim I_j = j$.

Proof.

As L is solvable, $ad(L)$ is a solvable Lie algebra of linear transformations on the finite dimensional vector space L. Let $L = L_1 \supset L_2 \supset \cdots \supset L_{s+1} = \{0\}$ be a composition series of L relative to $\Sigma = ad(L)$. Then $ad(L)(L_i) \subseteq L_i$ gives L_i is an ideal of L. By Lie's theorem $\dim \frac{L_i}{L_{i+1}} = 1$. Hence, $\dim L_s = 1$, $\dim L_{s-1} = 2, ..., \dim L_1 = s$. Now put $I_j = L_{s-(j-1)}$ to get $\dim I_j = \dim L_{s-(j-1)} = j$, as required. □

Universal Enveloping Algebras

Definition

Let L be a Lie algebra. A pair $(\mathcal{A}, i)$ where $\mathcal{A}$ is an associative algebra and i a homomorphism of L into $\mathcal{A}_L$ is called a universal enveloping algebra of L if for an algebra A and homomorphism θ of L into A_L, there exists a unique homomorphism θ' of $\mathcal{A}$ into A such that $\theta = i\theta'$, that is, the diagram

$$\begin{array}{l} \mathcal{A} = \mathcal{A}_L \\ i \uparrow \quad \searrow \theta^{'} (unique) \\ \;\; L \underset{\theta}{\longrightarrow} A = A_L \end{array}$$

commutes.

Uniqueness of universal enveloping algebras

Let $(\mathcal{A}, i), (\mathcal{B}, j)$ be two universal enveloping algebras for L. Then there exists a unique isomorphism j' of $\mathcal{A}$ onto $\mathcal{B}$ such that $j = ij'$. We have the following commutative diagrams:

$$\begin{array}{l} \mathcal{A} = \mathcal{A}_L \\ i \uparrow \quad \searrow j'(unique) \\ \quad L \xrightarrow[j]{} \mathcal{B} = \mathcal{B}_L \end{array} \qquad ,$$

$$\begin{array}{l} \mathcal{B} = \mathcal{B}_L \\ \quad j \uparrow \quad \searrow i'(unique) \\ \qquad L \xrightarrow[i]{} \mathcal{A} = \mathcal{A}_L \end{array} \qquad .$$

So $j = ij'$and $i = ji'$. Clearly $i'j' : \mathcal{B} \to \mathcal{B}$ and $j'i' : \mathcal{A} \to \mathcal{A}$.

Uniqueness . . .

Again the following commutative diagrams

$$\begin{array}{ll} \mathcal{B} & \\ j \uparrow & \searrow i_{\mathcal{B}}(unique) \\ L \xrightarrow[j]{} \mathcal{B} & \end{array} \quad ,$$

and

$$\begin{array}{ll} \mathcal{A} & \\ i \uparrow & \searrow i_{\mathcal{A}}(unique) \\ L \xrightarrow[i]{} \mathcal{A} & \end{array}$$

imply $j = ji_{\mathcal{B}}$ and $i = ii_{\mathcal{A}}$. Thus $i'j' = i_{\mathcal{B}}$ and $j'i' = i_{\mathcal{A}}$. So, j' is an isomorphism. This shows uniqueness of universal enveloping algebras up to isomorphism.

Lecture-7

We now give the construction of universal enveloping algebras:
Let L be a Lie algebra over F. Denote by $T(L)$, the tensor algebra based on the vector space L. We have $T(L) = \oplus_{i=1}^{\infty} L_i$, where $L_0 = F$, $L_1 = L, \dots, L_i = \underbrace{L \otimes L \otimes \cdots \otimes L}_{i-times}$.

Note that $T(L)$ is a vector space over F with usual addition and scalar multiplication. Define multiplication $\otimes$ in $T(L)$ by

$$(x_1 \otimes \cdots \otimes x_i) \otimes (y_i \otimes \cdots \otimes y_j) = x_1 \otimes x_2 \otimes \cdots \otimes x_i \otimes y_1 \otimes \cdots \otimes y_j.$$

With this $T(L)$ becomes an associative algebra.

Let R be an ideal of $T(L)$ generated by $\{[a, b] - a \otimes b + b \otimes a | a, b \in L\}$ and take $\mathcal{A} = \frac{T(L)}{R}$ with $\lambda : T(L) \to \mathcal{A}$ the canonical epimorphism. If $i = \lambda|_L$. Then

$$\begin{aligned} & i([a,b]) - i(a)\otimes i(b) + i(b)\otimes i(a) \\ &= [a,b] + R - (a + R \otimes b + R) + (b + R \otimes a + R) \\ &= ([a,b] - a\otimes b + b \otimes a) + R = R = 0 \in \mathcal{A}. \end{aligned}$$

Therefore, i is a Lie algebra homomorphism of L into $\mathcal{A}_L$.
We now show that $(\mathcal{A}, i)$ is an universal enveloping algebra for L.

Step 1: Let A be an algebra. Any linear map $\theta : L \to A$ can be extended to a homomorphism θ'' of $T(L)$ into A.
If $\{u_j | j \in J\}$ is a basis for L, then $\{u_{j_1} \otimes u_{j_2} \otimes \cdots \otimes u_{j_n} | j_i \in J\}$ form a basis for L_n. Here

$$u_{j_1} \otimes u_{j_2} \otimes \cdots \otimes u_{j_n} = u_{k_1} \otimes u_{k_2} \otimes \cdots \otimes u_{k_n} \Leftrightarrow j_r = k_r, r = 1, 2, \ldots, n.$$

where $u_{j_1} \otimes u_{j_2} \otimes \cdots \otimes u_{j_n}$ are monomials of degree n.

Hence,

$$\{1, u_{j_1} \otimes u_{j_2} \otimes \cdots \otimes u_{j_n} | n \in \mathbb{N}, j_i \in J\}$$

form a basis for $T(L)$.

Define $\theta'' : T(L) \to A$ by

$$\theta''(1) = 1, \quad \theta''(u_{j_1} \otimes u_{j_2} \otimes \cdots \otimes u_{j_n}) = \theta(u_{j_1})\theta(u_{j_2}) \cdots \theta(u_{j_n}),$$

then θ'' is an algebra homomorphism and $\theta''(a) = \theta(a)$ for all $a \in L$.

Lecture-8

Step 2: Let $\theta : L \to A_L$ be a homomorphism and θ'' be the extension of θ to a homomorphism of $T(L)$ into A. If $a, b \in L$, then

$$\begin{aligned}
&\theta''([a,b] - a \otimes b + b \otimes a) \\
&= \theta''([a,b]) - \theta''(a)\theta''(b) + \theta''(b)\theta''(a) \\
&= \theta([a,b]) - \theta(a)\theta(b) + \theta(b)\theta(a) \\
&= [\theta(a), \theta(b)] - [\theta(a), \theta(b)] = 0.
\end{aligned}$$

So $R \subseteq \ker \theta''$.
Define $\theta' : \mathcal{A} \to A$ by $\theta'(a + R) = \theta''(a)$, for all $a \in T(L)$. Verify that θ' is a well defined homomorphism. Further, for all $a \in L$:

$$i\theta'(a) = \theta'(i(a)) = \theta'(a + R) = \theta''(a) = \theta(a).$$

So $i\theta' = \theta$.

Step 3: (Uniqueness of θ') As $T(L)$ is generated by L, we have $\mathcal{A}$ is generated by $i(L)$. Any two homomorphism which coincide on generators are identical. Therefore θ' is unique such that $i\theta' = \theta$.

(In other words, if there exists $\theta^* : \mathcal{A} \to A$ such that $i\theta^* = \theta$, then $\theta^*(i(a)) = \theta(a) = \theta'(i(a))$, for all $a \in L$. This gives $\theta^* = \theta'$.)

This completes the construction and uniqueness of universal eneloping algebras.

Poincare-Birkhoff-Witt Theorem

Let L be a Lie algebra over a field F and let $\{u_j | j \in J\}$ be a basis for L, then $\{u_{j_1} \otimes u_{j_2} \otimes \cdots \otimes u_{j_n} | j_i \in J\}$ form a basis for L_n, $n \geq 1$. Here L_n's are as defined in Lecture 7.
Let J be an ordered set. For i, k, $i < k$, put

$$\eta_{ik} = \begin{cases} 1 \text{ if } j_i > j_k \\ 0 \text{ if } j_i \leq j_k. \end{cases}$$

Define the index of a monomial by

$$\text{ind}\,(u_{j_1} \otimes u_{j_2} \otimes \cdots \otimes u_{j_n}) = \sum_{i<k} \eta_{ik}.$$

Clearly ind $= 0$ if and only if $j_1 \leq j_2 \leq j_3 \leq \cdots \leq j_n$. Such monomials are called standard monomials.

PBW Theorem . . .

Let $j_k > j_{k+1}$. If $\operatorname{ind}\left(u_{j_1} \otimes u_{j_2} \otimes \cdots \otimes u_{j_n}\right) = \sum_{i<k} \eta_{ik} = l$, then $\operatorname{ind}\left(u_{j_1} \otimes u_{j_2} \otimes \cdots \otimes u_{j_{k-1}} \otimes u_{j_{k+1}} \otimes u_{j_k} \otimes u_{j_{k+2}} \otimes \cdots \otimes u_{j_n}\right) = l - 1$. (Verify.)

Theorem

(Poincare-Birkhoff-Witt Theorem) *Let L be a Lie algebra over a field F with basis $\{u_j | j \in J\}$. Let J be an orderderd set, then cosets of 1 and the standard monomials form a basis for the universal enveloping algebra $\mathcal{A} = \frac{T(L)}{R}$ of L.*

The proof is not in the course.

This completes the Syllabus.

ALL THE BEST.