UNIT - IV

$LECTURE - I$

We have seen that a free R-module F has a property that every exact sequence of R-modules,

$$0 \longrightarrow M \longrightarrow N \longrightarrow F \longrightarrow 0$$

splits. Now we shall study a class of modules which satisfy this property.

DEFINITION: A module P over a ring R is **projective** if to each R-module epimorphism $\psi \colon M \to N$ and $f \in \mathrm{Hom}_R(P, N)$, there exists $g \in \mathrm{Hom}_R(P, M)$ such that $\psi \circ g = f$, i.e., the diagram

$$\begin{array}{ccc} P & & \\ g \downarrow & \searrow f & \\ M & \underset{\psi(epi)}{\longrightarrow} & N \end{array}$$

commutes.

THEOREM 0.1. *The following conditions on a module P over a ring R are equivalent:*

(i) P is projective;

(ii) if $\psi \colon M \to N$ is an R-module epimorphism, then

$$\psi_* \colon Hom_R(P, M) \to Hom_R(P, N)$$

is an epimorphism of abelian groups;
(iii) if $0 \longrightarrow M_1 \xrightarrow{\phi} M \xrightarrow{\psi} M_2 \longrightarrow 0$ *is a short exact sequence of R-modules, then*

$$0 \longrightarrow Hom_R(P, M_1) \xrightarrow{\phi_*} Hom_R(P, M) \xrightarrow{\psi_*} Hom_R(P, M_2) \longrightarrow 0$$

is a short exact sequence of $\mathbb{Z}$-modules.

Proof. $(i) \Leftrightarrow (ii)$ ψ_* is surjective if and only if to each $f \in \mathrm{Hom}_R(P, N)$, there exists $g \in \mathrm{Hom}_R(P, M)$ such that $\psi_*(g) = f$, that is, $\psi \circ g = f$.

$(ii) \Rightarrow (iii)$ If $0 \longrightarrow M_1 \xrightarrow{\phi} M \xrightarrow{\psi} M_2 \longrightarrow 0$ is a short exact sequence of R-modules, then since ψ_* is epimorphism, by a theorem in Unit II,

$$0 \longrightarrow \mathrm{Hom}_R(P, M_1) \xrightarrow{\phi_*} \mathrm{Hom}_R(P, M) \xrightarrow{\psi_*} \mathrm{Hom}_R(P, M_2) \longrightarrow 0$$

is a short exact sequence of $\mathbb{Z}$-modules.

$(iii) \Rightarrow (ii)$ If $\psi \colon M \to N$ is an R-module epimorphism, then we have an exact sequence $0 \longrightarrow \ker\psi \longrightarrow M \xrightarrow{\psi} N \longrightarrow 0$. Thus,

$$0 \longrightarrow \mathrm{Hom}_R(P, \ker\psi) \longrightarrow \mathrm{Hom}_R(P, M) \xrightarrow{\psi_*} \mathrm{Hom}_R(P, N) \longrightarrow 0$$

is an exact sequence of $\mathbb{Z}$-modules. In particular, ψ_* is surjective.

The first examples of projective modules are free modules.

Proposition 0.2. *Every free module is projective.*

Proof. Let F be a free module over a ring R with basis $B = \{ x_i \mid i \in I \}$. If $\psi \colon M \to N$ is an R-module epimorphism and $f \in \mathrm{Hom}_R(F, N)$, then to each $f(x_i)$ $(i \in I)$, there exists $y_i \in M$ so that $\psi(y_i) = f(x_i)$ for all $i \in I$. Let $g \colon F \to M$ be an R-module homomorphism defined by $g(x_i) = y_i$ $(i \in I)$. Then $\psi g(x_i) = f(x_i)$ for all $i \in I$, and so $\psi \circ g = f$. Hence, F is projective.

LECTURE – II

We have seen that every free module is projective. Now we will show that every projective module is a submodule of a free module.

THEOREM 0.3. *The following conditions on a module P over a ring R are equivalent:*
(i) P is projective;
(ii) if $0 \longrightarrow M_1 \xrightarrow{\phi} M \xrightarrow{\psi} P \longrightarrow 0$ is a short exact sequence of R-modules, then it splits;
(iii) there exists an R-module K such that $P \oplus K$ is free.

Proof. $(i) \Rightarrow (ii)$ Since P is projective and $0 \longrightarrow M_1 \xrightarrow{\phi} M \xrightarrow{\psi} P \longrightarrow 0$ is a short exact sequence, $\psi_*\colon \operatorname{Hom}_R(P, M) \to \operatorname{Hom}_R(P, P)$ is surjective (Theorem 0.1). Therefore, there exists $f \in \operatorname{Hom}_R(P, M)$ such that $\psi_*(f) = 1_P$, or $\psi \circ f = 1_P$ and the sequence splits.

$(ii) \Rightarrow (iii)$ Let F be a free R-module such that $F/K \simeq P$. Then $0 \longrightarrow K \longrightarrow F \longrightarrow P \longrightarrow 0$ is a short exact sequence. Hence, it splits and $F \simeq P \oplus K$.

$(iii) \Rightarrow (i)$ Let $F = P \oplus K$ be a free R-module and let $\psi\colon M \to N$ be an R-module epimorphism. Since $F = P \oplus K$ is free, so by Proposition 0.2, F is projective. Therefore, $\psi_*\colon \operatorname{Hom}_R(P \oplus K, M) \to \operatorname{Hom}_R(P \oplus K, N)$ is an epimorphism. Let $\pi\colon P \oplus K \to P$ and $\imath\colon P \to P \oplus K$ be the canonical projection and injection respectively. Then for $f \in \operatorname{Hom}_R(P, N)$, $f \circ \pi \in \operatorname{Hom}_R(P \oplus K, N)$, and so there exists $\bar{g} \in \operatorname{Hom}_R(P \oplus K, M)$ such that $\psi \circ \bar{g} = f \circ \pi$. This gives $\psi \circ (\bar{g} \circ \imath) = f \circ (\pi \circ \imath) = f$ as $\pi \circ \imath = 1_P$. Now $\bar{g} \circ \imath \in \operatorname{Hom}_R(P, M)$, so P is projective.

COROLLARY 0.4. *A projective module P over an integral domain R is torsion free.*

Proof. Since P is projective, P is a submodule of a free R-module (Theorem 0.3). Now a free module over an integral domain is torsion free and a submodule of a torsion free module is also torsion free. Hence, P is torsion free.

Corollary 0.5. *An R-module P is finitely generated and projective if and only if P is a direct summand of a finitely generated free R-module.*

Proof. Suppose that P is a finitely generated projective R-module. There exists a finitely generated free module F such that $F/K \simeq P$. Thus we have an exact sequence $0 \longrightarrow K \longrightarrow F \longrightarrow P \longrightarrow 0$, and it splits. Therefore, P is a direct summand of F.

Conversely, if $P \oplus K \simeq F$, F a free module, then P is projective and as $F/K \simeq P$, so P is finitely generated.

Corollary 0.6. *If R is a PID and P is a projective R-module, then P is free.*

Proof. By the above theorem, there exists an R-module K such that $P \oplus K$ is a free R-module. Since submodule of a free module over a PID is also free, P is a free R-module.

Proposition 0.7. *Let $\{ P_i \mid i \in I \}$ be a family of modules over a ring R and let $P = \oplus_{i \in I} P_i$. Then P is projective if and only if each P_i is projective.*

Proof. Let P be projective. Then there exists an R-module K such that $P \oplus K = F$ is a free R-module. Then $F = P \oplus K = P_i \oplus (\oplus_{i \neq j} P_j \oplus K)$. So each P_i is projective.

Conversely, if each P_i is projective, then there exists an R-module K_i such that $F_i = P_i \oplus K_i$ is a free R-module. Now, $F = \oplus_{i\in I} F_i = \oplus_{i\in I}(P_i \oplus K_i) \simeq P \oplus (\oplus_{i\in I} K_i)$ is free. Hence, P is projective.

PROPOSITION 0.8. *If P and Q are finitely generated projective R-modules over a commutative ring R, then $Hom_R(P, Q)$ is also a finitely generated projective R-module.*

Proof. Since P and Q are finitely generated projective R-modules, there exist modules K and L such that $P \oplus K$ and $Q \oplus L$ are finitely generated free R-modules (Corollary 0.5). Therefore, $\text{Hom}_R(P \oplus K, Q \oplus L)$ is a finitely generated free R-module Since, $\text{Hom}_R(P \oplus K, Q \oplus L) = \text{Hom}_R(P, Q) \oplus \text{Hom}_R(P, L) \oplus \text{Hom}_R(K, Q) \oplus \text{Hom}_R(K, L)$, so $\text{Hom}_R(P, Q)$ is a direct summand of a free R-module. Hence, $\text{Hom}_R(P, Q)$ is a finitely generated projective R-module (Corollary 0.5).

LECTURE – III

Now we study the class of modules which satisfy the dual of the property of projective modules, that is, a module Q for which every exact sequence of R-modules $0 \longrightarrow Q \longrightarrow M \longrightarrow N \longrightarrow 0$ splits.

DEFINITION: A module E over a ring R is said to be **injective** if to each R-module monomorphism $\phi \colon M \to N$ and $f \in \text{Hom}_R(M, E)$, there exists $g \in \text{Hom}_R(N, E)$ such that $g \circ \phi = f$, i.e., the diagram

$$\begin{array}{lll} E & & \\ f\uparrow & \nwarrow g & \\ M & \underset{\phi(mono)}{\longrightarrow} & N \end{array}$$

commutes.

THEOREM 0.9. *The following conditions on a module E over a ring R are equivalent:*
(i) E is injective;
(ii) if $\phi\colon M \to N$ is an R-module monomorphism, then

$$\phi^*\colon \mathit{Hom}_R(N, E) \to \mathit{Hom}_R(M, E)$$

is an epimorphism of abelian groups;
(iii) if $0 \longrightarrow M_1 \xrightarrow{\phi} M \xrightarrow{\psi} M_2 \longrightarrow 0$ is a short exact sequence of R-modules, then

$$0 \longrightarrow \mathit{Hom}_R(M_2, E) \xrightarrow{\psi^*} \mathit{Hom}_R(M, E) \xrightarrow{\phi^*} \mathit{Hom}_R(M_1, E) \longrightarrow 0$$

is an exact sequence of abelian groups.

Proof. Exercise. Similar to that of Theorem 0.1.

PROPOSITION 0.10. *Let $\{ E_i \mid i \in I \}$ be a family of modules over a ring R and let $E = \prod_{i\in I} E_i$. Then E is injective if and only if each E_i is injective.*

Proof. Let $\pi_j\colon \prod_{i\in I} E_i \to E_j$ be the j-th canonical projection and let $\imath_j\colon E_j \to \prod_{i\in I} E_i$ be the j-th canonical injection.

Suppose that E is injective. Let $\phi\colon M \to N$ be an R-module monomorphism. If $f \in \mathrm{Hom}_R(M, E_j)$, then $\imath_j f \in \mathrm{Hom}_R(M, E)$ and hence, there exists $\bar{g} \in \mathrm{Hom}_R(N, E)$ such that $\bar{g}\phi = \imath_j f$. Now $\pi_j \bar{g}\phi = \pi_j \imath_j f = f$, as $\pi_j \imath_j = 1_{E_j}$. So by taking $g = \pi_j \bar{g} \in \mathrm{Hom}_R(N, E_j)$, we have $g\phi = f$. Hence, E_j is injective.

Conversely, assume that each E_i is injective. Let $\phi\colon M \to N$ be an R-module monomorphism and let $f \in \mathrm{Hom}_R(M, E)$. Then for each $i \in I$, $\pi_i f \in \mathrm{Hom}_R(M, E_i)$, so there exists $g_i \in \mathrm{Hom}_R(N, E_i)$ such that $g_i\phi = \pi_i f$. Let $g \in \mathrm{Hom}_R(N, E)$ so that $\pi_i g = g_i$ for all $i \in I$. Then $\pi_i g\phi = \pi_i f$ for all $i \in I$. Thus, $g\phi = f$ and E is injective.

LECTURE – IV

Our aim is to prove that every module is a submodule of an injective module. For this purpose, we now introduce divisible abelian groups.

DEFINITION: An abelian group A is said to be **divisible** if $mA = A$ for all $m \in \mathbb{Z} \setminus \{0\}$.

Thus, an abelian group A is divisible if and only if for every nonzero integer m and $a \in A$, there exists $b \in A$ such that $mb = a$.

EXAMPLE 0.11. $\mathbb{Q}$ is a divisible abelian group, as for every nonzero integer m and $a \in \mathbb{Q}$, $b = a/m \in \mathbb{Q}$ such that $mb = a$. It is easy to see by the same logic that $\mathbb{Z}$ is not divisible.

PROPOSITION 0.12. (i) *Every homomorphic image of a divisible group is divisible.*
(ii) *Direct product and direct sum of divisible groups are divisible.*

Proof. (i) Let $f \colon A \to B$ be an epimorphism of abelian groups and let A be divisible. If $m \in \mathbb{Z} \setminus \{0\}$ and $b \in B$, then there are $a, c \in A$ such that $f(a) = b$ and $mc = a$. Thus, $b = f(a) = f(mc) = mf(c)$. Hence, B is divisible.

(ii) Let $A = \prod_{i \in I} A_i$ be a direct product of divisible abelian groups A_i. If $m \in \mathbb{Z} \setminus \{0\}$ and $a \in A$, then $a_i \in A_i$, for all i. So for every $i \in I$, there exists $b_i \in A_i$ such that $mb_i = a_i$. Now $b \in A$, where $b(i) = b_i$ and $mb = a$. Same proof works for direct sums also, if we write only finitely many $a_i \neq 0$ after $a_i \in A_i$ and take $b_i = 0$, if $a_i = 0$.

Using the Proposition 0.12, we have another example of a divisible abelian groups, namely, $\mathbb{Q}/\mathbb{Z}$.

Theorem 0.13. *Every abelian group is isomorphic to a subgroup of a divisible abelian group. Thus, every abelian group can be embedded in a divisible abelian group.*

Proof. Let A be an abelian group, equivalently a $\mathbb{Z}$-module. Then there is a free $\mathbb{Z}$-module $F = \oplus_{i\in I} F_i$, where $F_i = \mathbb{Z}$ for all $i \in I$, such that $A \simeq F/K$. Let $D = \oplus_{i\in I} D_i$, where $D_i = \mathbb{Q}$ for all $i \in I$. Then D is divisible (Proposition 0.12), and F is a subgroup of D. Clearly, F/K is a subgroup of D/K, and by Proposition 0.12, D/K is divisible. Thus, $A \simeq F/K$ and F/K is a subgroup of a divisible group D/K.

LECTURE – *V*

Lemma 0.14. *A module E over a ring R is injective if and only if for every left ideal I of R, any R-module homomorphism $I \to E$ may be extended to an R-module homomorphism $R \to E$.*

Proof. Let E be an injective R-module. Then for the inclusion map $\imath\colon I \to R$ and for $f \in \operatorname{Hom}_R(I, E)$, there exists $g \in \operatorname{Hom}_R(R, E)$ such that $g\imath = f$, that is, $g|_I = f$.

Conversely, let $\phi\colon M \to N$ be an R-module monomorphism and $f \in \operatorname{Hom}_R(M, E)$. Since $M \cong \operatorname{Im}\phi$, without any loss of generality, we can replace M by $\operatorname{Im}\phi$ and assume that ϕ is an inclusion mapping, so that M is a submodule of N. Then to prove that E is injective, we just have to extend f to N.

Let S be the set of all ordered pairs (K, h), where K is a submodule of N containing M and $h\colon K \to E$ is an R-module homomorphism such that $h|_M = f$. Then S is partially ordered by the relation: $(K, h) \leq (K', h')$ if $K \subseteq K'$ and $h'|_K = h$. We now apply Zorn's lemma. If $\{ (K_i, h_i) \mid i \in I \}$ is a chain in S then $K' = \cup_{i\in I} K_i$ and $h'\colon K' \to E$, an

R-module homomorphism defined by $h'(x) = h_i(x)$, whenever $x \in K_i$. Thus, (K', h') is an upper bound for the chain. Therefore, by Zorn's lemma we have a maximal element $(\bar{K}, \bar{h}) \in S$. We now claim that $\bar{K} = N$.

If $\bar{K} \neq N$, then choose $x \in N \setminus \bar{K}$. Let $L = \bar{K} + \langle x \rangle$ and let $I = \{ r \in R \mid rx \in \bar{K} \}$. Then I is a left ideal of R and the mapping $f' \colon I \to E$ defined by $f'(r) = \bar{h}(rx)$ is an R-module homomorphism. By hypothesis there exists $g \colon R \to E$ an R-module homomorphism such that $g|_I = f'$. Let $g(1) = y$ and let $h_1 \colon L \to E$ given by $h_1(a + rx) = \bar{h}(a) + ry$. Now h_1 is well defined: if $a_1 + r_1 x = a_2 + r_2 x$, then $a_1 - a_2 = (r_2 - r_1)x \in \bar{K} \cap \langle x \rangle$, and so $r_1 - r_2 \in I$, and this implies that $\bar{h}(a_1) - \bar{h}(a_2) = \bar{h}(a_1 - a_2) = \bar{h}((r_2 - r_1)x) = (r_2 - r_1)g(1) = (r_2 - r_1)y = r_2 y - r_1 y$, that is, $h_1(a_1 + r_1 x) = h_1(a_2 + r_2 x)$. It is easy to see that $h_1 \in \mathrm{Hom}_R(L, E)$. But then $(\bar{K}, \bar{h}) \leq (L, h_1)$ with $\bar{K} \subset L$ and $h_1|_{\bar{K}} = \bar{h}$, contradicting the maximality of $(\bar{K}, \bar{h})$. Hence, $\bar{K} = N$ and E is injective.

Theorem 0.15. *An abelian group is divisible if and only if it is an injective $\mathbb{Z}$-module.*

Proof. First suppose that A is an injective $\mathbb{Z}$-module. Let $a \in A$ and $m \in \mathbb{Z}^+$. Since A is injective, for the inclusion mapping $\imath \colon m\mathbb{Z} \to \mathbb{Z}$ and $f \in \mathrm{Hom}_{\mathbb{Z}}(m\mathbb{Z}, A)$ defined by $f(m) = a$, there exists $g \in \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, A)$ such that $g\imath = f$. Thus, $a = f(m) = g\imath(m) = g(m) = mg(1)$. Hence, A is divisible.

Conversely, we know that the only left ideals of $\mathbb{Z}$ are the cyclic groups $n\mathbb{Z}$, $n \in \mathbb{Z}$. If A is divisible and $f \colon n\mathbb{Z} \to A$ is a homomorphism, then there exists $a \in A$ with $na = f(n)$. Define $g \colon \mathbb{Z} \to A$ by $g(1) = a$.

Then g is a homomorphism that extends f. Therefore, A is injective by Lemma 0.14.

$$LECTURE = VI$$

Let R be a ring and let A be an abelian group. Then it can be verified, without much difficulty, that $\mathrm{Hom}_{\mathbb{Z}}(R, A)$ is an R-module with scalar multiplication defined by:

$$(rf)(x) = f(xr), \quad (r, x \in R,\ f \in \mathrm{Hom}_{\mathbb{Z}}(R, A)).$$

Take it as an exercise and verify.

LEMMA 0.16. *If A is a divisible abelian group and R is a ring, then $Hom_{\mathbb{Z}}(R, A)$ is an injective R-module.*

Proof. By Lemma 0.14, it is sufficient to show that for each left ideal I of R, every R-module homomorphism $f\colon I \to \mathrm{Hom}_{\mathbb{Z}}(R, A)$ may be extended to an R-module homomorphism $g\colon R \to \mathrm{Hom}_{\mathbb{Z}}(R, A)$. The mapping $h\colon I \to A$ defined by $h(x) = f(x)(1)$ is a group homomorphism. Since A is divisible, so A is an injective $\mathbb{Z}$-module (Theorem 0.15). Thus there exists $\bar{h}\colon R \to A$ such that $\bar{h}|_I = h$. Now let $g\colon R \to \mathrm{Hom}_{\mathbb{Z}}(R, A)$, where $g(r) \in \mathrm{Hom}_{\mathbb{Z}}(R, A)$ is defined by $g(r)(x) = \bar{h}(xr)$, $x \in R$. Also for r_1, r_2, $x \in R$, $g(r_1 + r_2)(x) = \bar{h}(x(r_1+r_2)) = \bar{h}(xr_1)+\bar{h}(xr_2) = g(r_1)(x)+g(r_2)(x)$. Thus, $g(r_1+r_2) = g(r_1) + g(r_2)$ for r_1, $r_2 \in R$. Further, if $s, r, x \in R$, then $g(sr)(x) = \bar{h}(x(sr)) = \bar{h}((xs)r) = g(r)(xs) = (sg(r))(x)$, that is, $g(sr) = sg(r)$. Hence, g is an R-module homomorphism.

We now show that $g|_I = f$. Let $r \in I$ and $x \in R$. Then $xr \in I$ and $g(r)(x) = \bar{h}(xr) = h(xr) = f(xr)(1)$. Since f is an R-module homomorphism, so $f(xr) = xf(r)$, and so $f(xr)(1) = (xf(r))(1) =$

$f(r)(x)$. Therefore, $f(r)(x) = g(r)(x)$ for all $x \in R$, that is, $f(r) = g(r)$.

Theorem 0.17. *Every module is isomorphic to a submodule of an injective module.*

Proof. Let A be a module over a ring R. Since A is an abelian group, there is a divisible abelian group D such that $\phi\colon A \to D$ is a monomorphism (Theorem 0.13). The mapping $\phi_*\colon \mathrm{Hom}_{\mathbb{Z}}(R, A) \to \mathrm{Hom}_{\mathbb{Z}}(R, D)$ defined by $\phi_*(f) = \phi f$ is an R-module monomorphism. Since every R-module homomorphism is also a $\mathbb{Z}$-module homomorphism, and as $\mathrm{Hom}_{\mathbb{Z}}(R, A)$ is an R-module, so $\mathrm{Hom}_R(R, A)$ is an R-submodule of $\mathrm{Hom}_{\mathbb{Z}}(R, A)$. Thus, there is an R-module monomorphism from $\mathrm{Hom}_R(R, A)$ to $\mathrm{Hom}_{\mathbb{Z}}(R, D)$. Also e know that $A \simeq \mathrm{Hom}_R(R, A)$. Therefore, there exists an R-module monomorphism from A to $\mathrm{Hom}_{\mathbb{Z}}(R, D)$. Since $\mathrm{Hom}_{\mathbb{Z}}(R, D)$ is injective R-module, the theorem is proved.

$LECTURE - VII$

We now discuss some examples.

(1) Every free module is projective. This is an example of a projective module which is not free.

If m and n are relatively prime integers, then $\mathbb{Z}_{mn} \simeq \mathbb{Z}_m \oplus \mathbb{Z}_n$ as abelian groups. It is easy to check that it is also a $\mathbb{Z}_{mn}$-module isomorphism. Since, $\mathbb{Z}_{mn}$ is a free $\mathbb{Z}_{mn}$-module, so $\mathbb{Z}_m$ and $\mathbb{Z}_n$ are projective $\mathbb{Z}_{mn}$-modules (Theorem 0.3). But neither $\mathbb{Z}_m$ nor $\mathbb{Z}_n$ are free $\mathbb{Z}_{mn}$-modules as both have lesser than mn elements. (If either $\mathbb{Z}_m$ or $\mathbb{Z}_n$ is a free $\mathbb{Z}_{mn}$-module, then it will be a direct sum of copies of $\mathbb{Z}_{mn}$ and hence it will have more than mn elements.)

(2) Example of modules which are projective and injective both.

Let R be a ring such that every exact sequence of R-modules is split exact. Let $\psi\colon M \to N$ be an R-module epimorphism, then $0 \longrightarrow \ker\psi \longrightarrow M \xrightarrow{\psi} N \longrightarrow 0$ is an exact sequence, and so it splits. Hence, for every R-module A,

$$0 \longrightarrow \operatorname{Hom}_R(A, \ker\psi) \longrightarrow \operatorname{Hom}_R(A, M) \xrightarrow{\psi_*} \operatorname{Hom}_R(A, N) \longrightarrow 0$$

is a split exact sequence. So, ψ_* is surjective and A is a projective R-module.

Similarly, if $\phi\colon M \to N$ is an R-module monomorphism, then

$$0 \longrightarrow M \xrightarrow{\phi} N \longrightarrow N/\operatorname{Im}\phi \longrightarrow 0$$

is an exact sequence, and so it splits. Hence, for every R-module A,

$$0 \longrightarrow \operatorname{Hom}_R(N/\operatorname{Im}\phi, A) \longrightarrow \operatorname{Hom}_R(N, A) \xrightarrow{\phi^*} \operatorname{Hom}_R(M, A) \longrightarrow 0$$

splits. So, ϕ^* is surjective and A is injective.

Now let D be a division ring. If $0 \longrightarrow M_1 \longrightarrow M \longrightarrow M_2 \longrightarrow 0$ is an exact sequence of D-modules, then it can be verified that $M \simeq M_1 \oplus M_2$, so, this sequence splits. Hence, every module over a division ring is projective as well as injective. In particular, a vector space over a field K is projective as well as injective.

(3) Example of an injective module which is not projective.

$\mathbb{Q}$ is an injective $\mathbb{Z}$-module as it is a divisible abelian group but it is not a projective $\mathbb{Z}$-module. (If $\mathbb{Q}$ is projective $\mathbb{Z}$-module, then since $\mathbb{Z}$ is a PID, it should be a free $\mathbb{Z}$-module, see Corollary 0.6, which it is not.)

(4) A finitely generated $\mathbb{Z}$-module is not an injective $\mathbb{Z}$-module. Equivalently, a finitely generated abelian group is not divisible. Thus, $\mathbb{Q}$ is not finitely generated abelian group. To prove this, let A be a finitely generated $\mathbb{Z}$-module. Then $A = T(A) \oplus F$, where F is a free module of finite rank. Also $T(A)$ is finitely generated (as $A/F \simeq T(A)$). Let $T(A) = \langle x_1, \ldots, x_k \rangle$. Then $\text{Ann}\,(T(A)) = \cap_{i=1}^{k} \text{Ann}\,(x_i) \neq \{0\}$. Let $m \in \text{Ann}\,(T(A))$. Then $mT(A) = 0$, and so $mA = mF \neq A$. Therefore, if A is a divisible abelian group (equivalently, an injective $\mathbb{Z}$-module), then A should be a free $\mathbb{Z}$-module of finite rank. Thus, $A \simeq \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}$ (finite summands). Again, this abelian group is not divisible. Since for any $m \in \mathbb{Z}$, $m \neq 0, \pm 1$, there does not exist any $x \in \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}$ such that $mx = (1, 0, \ldots, 0)$.

That's all in UNIT-IV students. Wishing you all the best.

Take care and stay safe