1 Reflection Principle

In general, some elementary functions $f(z)$ possess the property that

$$f(\overline{z}) = \overline{f(z)}$$

for all points z in some domain, and others do not.

Example 1 *The functions*

$$z, \quad z^2 + 1, \quad e^z, \quad \sin z$$

have that property. On the other hand, the functions

$$iz, \quad z^2 + i, \quad e^{iz}, \quad (1+i)\sin z$$

do not have this property.

The following theorem provides the conditions under which $f(\overline{z}) = \overline{f(z)}$ and is known as the **reflection principle**.

Theorem 1 *Let $f(z)$ be analytic inside the domain D which contains a segment of the real axis and whose lower half is the reflection of the upper half with respect to that axis. Then*

$$f(\overline{z}) = \overline{f(z)} \quad \forall\ z \in D \tag{1}$$

if and only if $f(x)$ is real for each point x on the segment.

Proof. Necessary condition: Let the domain D contains the segment ABC of the real axis. Also, let D be symmetrical about ABC and a function $f(x)$ be real for each point x on the segment ABC. Let $F(z) = \overline{f(\overline{z})}$. To prove result (1), we first show that the function $F(z)$ is analytic in the domain D. Let us write

$$f(z) = f(x+iy) = u(x,y) + iv(x,y)$$

and

$$F(z) = U(x,y) + iV(x,y). \tag{2}$$

Then

$$f(\overline{z}) = u(x,-y) + iv(x,-y)$$

and hence,

$$F(z) = \overline{f(\overline{z})} = u(x,-y) - iv(x,-y)$$

which on using (2) gives

$$U(x,y) = u(x,-y), \quad V(x,y) = -v(x,-y)$$

or

$$U(x,y) = u(x,\lambda), \quad V(x,y) = -v(x,\lambda), \tag{3}$$

where $\lambda = -y$. By hypothesis $f(\overline{z}) = f(x+i\lambda)$ is an analytic function of $x+i\lambda$, the functions $u(x,\lambda)$ and $v(x,\lambda)$, together with their partial derivatives, are continuous in D and they satisfy there the Cauchy-Riemann equations

$$u_x(x,\lambda) = v_\lambda(x,\lambda) \quad \text{and} \quad u_\lambda(x,\lambda) = -v_x(x,\lambda). \tag{4}$$

Now, by (3), we get

$$U_x(x,y) = u_x(x,\lambda), U_y(x,y) = u_\lambda(x,\lambda)\frac{d\lambda}{dy} = -u_\lambda(x,\lambda),$$

$$V_x(x,y) = -v_x(x,\lambda), V_y(x,y) = -v_\lambda(x,\lambda)\frac{d\lambda}{dy} = v_\lambda(x,\lambda)$$

which in view of (4) gives

$$\begin{aligned} U_x(x,y) &= u_x(x,\lambda) = v_\lambda(x,\lambda) = V_y(x,y), \\ U_y(x,y) &= -u_\lambda(x,\lambda) = v_x(x,\lambda) = -V_x(x,y). \end{aligned}$$

Thus the partial derivatives U_x, U_y, V_x, V_y are continuous (as $u_x(x,\lambda), u_y(x,\lambda), v_x(x,\lambda), v_y(x,\lambda)$ are continuous), and satisfy Cauchy-Riemann equations, hence, $F(z)$ is analytic in D .

Since $f(x)$ is real, $v(x,0) = 0$. Hence

$$F(x) = U(x,0) + iV(x,0) = u(x,0).$$

Thus $F(z) = f(z)$ at each point on ABC in the domain D, where both the functions are analytic. Hence, by analytic continuation $F(z) = f(z)$ in D which proves the result (1).

Sufficient condition: Let the function $f(z)$ has the property (1) that is $\overline{f(\overline{z})} = f(z)$ in D. Hence, in particular, $u(x,0) - iv(x,0) = u(x,0) + iv(x,0)$ which at once proves that $v(x,0) = 0$. Thus $f(x)$ is real for each point x on the segment ABC in D. ■

2 Poisson's Integral Formula

Theorem 2 (Poisson's Integral Formula) *Let $f(z)$ be analytic in a region including the disc $|z| \leq R$, and let $u(r,\theta)$ be its real part. Then for $0 \leq r < R$*

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{R^2 - r^2}{R^2 - 2Rr\cos(\theta-\phi) + r^2} u(R,\phi)\mathrm{d}\phi.$$

Proof. We may suppose without loss of generality that $f(z) = \sum a_n z^n$, where all the coefficients a_n are real. For, in the general case, if $a_n = \alpha_n + i\beta_n$, then

$$f(z) = \sum \alpha_n z^n + i\sum \beta_n z^n = f_1(z) + if_2(z)$$

and hence, we find

$$\Re e f(z) = \Re e f_1(z) - \Im m f_2(z),$$

where $f_1(z)$ and $f_2(z)$ are also analytic for $|z| \leq R$, since $|\alpha_n| \leq |a_n|$ and $|\beta_n| \leq |a_n|$. So the general case follows from the special case. Thus we prove the formula for this special case. Let z_1 be a point on the circle $|z| = R$ and $z = re^{i\theta}$ be any interior point of the circle $|z| = R$ and let $f(re^{i\theta}) = u + iv$ and $f(z_1) = f(Re^{i\phi}) = u_1 + iv_1$. Then by the *reflection principle* $f(Re^{-i\phi}) = u_1 - iv_1$, and by *Cauchy's integral formula*

$$u + iv = \frac{1}{2\pi i} \int \frac{u_1 + iv_1}{z_1 - z} \mathrm{d}z_1 = \frac{1}{2\pi} \int_0^{2\pi} \frac{u_1 + iv_1}{Re^{i\phi} - re^{i\theta}} Re^{i\phi} \mathrm{d}\phi. \tag{5}$$

Further, since the point R^2/z is outside the circle $|z| = R$, we have

$$0 = \frac{1}{2\pi i} \int \frac{u_1 + iv_1}{z_1 - R^2/z} \mathrm{d}z_1 = \frac{1}{2\pi} \int_0^{2\pi} \frac{u_1 + iv_1}{Re^{i\phi} - R^2 r^{-1} e^{-i\theta}} Re^{i\phi} \mathrm{d}\phi.$$

Also, on replacing ϕ by $-\phi$ and v_1 by $-v_1$, we obtain

$$0 = \frac{1}{2\pi} \int_0^{2\pi} \frac{u_1 - iv_1}{Re^{-i\phi} - R^2 r^{-1} e^{-i\theta}} Re^{-i\phi} \mathrm{d}\phi$$

which on simplifying gives

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{u_1 - iv_1}{re^{i\theta} - Re^{i\phi}} re^{i\theta} \mathrm{d}\phi = 0$$

or

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{u_1 - iv_1}{Re^{i\phi} - re^{i\theta}} re^{i\theta} \mathrm{d}\phi = 0. \tag{6}$$

on adding (5) and (6), we get

$$u + iv = \frac{1}{2\pi} \int_0^{2\pi} \left\{ u_1 \frac{Re^{i\phi} + re^{i\theta}}{Re^{i\phi} - re^{i\theta}} + iv_1 \right\} \mathrm{d}\phi$$

which on taking real parts proves the result. ■

3 Jensen's Formula

Theorem 3 (Jensen's Theorem) *Let $f(z)$ be analytic for $|z| < R$. Suppose that $f(0)$ is not zero, and let $r_1, r_2, ..., r_n, ...$ be the moduli of the zeros of $f(z)$ in the disc $|z| < R$, arranged as a non-decreasing sequence. Then, if $r_n \leq r \leq r_{n+1}$,*

$$\log \frac{r^n |f(0)|}{r_1 r_2 ... r_n} = \frac{1}{2\pi} \int_0^{2\pi} \log \left| f(re^{i\theta}) \right| \mathrm{d}\theta. \tag{7}$$

Proof. Assume that the zeros are counted as often as its multiplicity. First we write the formula (7) in terms of the number of zeros inside the disc. Let $n(x)$

denote the number of zeros of $f(z)$ for $|z| \leq x$. Then, if $r_n \leq r \leq r_{n+1}$,

$$\begin{aligned}
\log \frac{r^n}{r_1 r_2 ... r_n} &= n \log r - \sum_{m=1}^{n} \log r_m \\
&= \sum_{m=1}^{n-1} m(\log r_{m+1} - \log r_m) + n(\log r - \log r_n) \\
&= \sum_{m=1}^{n-1} m \int_{r_m}^{r_{m+1}} \frac{\mathrm{d}x}{x} + n \int_{r_n}^{r} \frac{\mathrm{d}x}{x}.
\end{aligned}$$

We have $m = n(x)$ when $r_m \leq x < r_{m+1}$ and $n = n(x)$ when $r_n \leq x < r$. Hence,

$$\begin{aligned}
\log \frac{r^n}{r_1 r_2 ... r_n} &= \sum_{m=1}^{n-1} \int_{r_m}^{r_{m+1}} \frac{n(x)}{x} \mathrm{d}x + \int_{r_n}^{r} \frac{n(x)}{x} \mathrm{d}x \\
&= \int_{r_1}^{r} \frac{n(x)}{x} \mathrm{d}x = \int_{0}^{r} \frac{n(x)}{x} \mathrm{d}x
\end{aligned}$$

as $n(x) = 0$ when $0 \leq x < r_1$. Thus the formula (7) may also be given by

$$\int_0^r \frac{n(x)}{x} \mathrm{d}x = \frac{1}{2\pi} \int_0^{2\pi} \log \left| f(re^{i\theta}) \right| \mathrm{d}\theta - \log |f(0)| \,. \tag{8}$$

Now in order to prove the formula (7) or (8), we consider number of stages (cases).

(i) If $f(z)$ has no zeros for $|z| \leq r$, then $\log f(z)$ is analytic for $|z| \leq r$, and hence, on applying *Cauchy's integral formula f*or the function $\log f(z)$, we get

$$\log f(0) = \frac{1}{2\pi i} \int\limits_{|z|=r} \frac{\log f(z)}{z} \mathrm{d}z = \frac{1}{2\pi} \int_0^{2\pi} \log f(re^{i\theta}) \mathrm{d}\theta,$$

which on equating the real parts proves the result (8):

$$\log |f(0)| = \frac{1}{2\pi} \int_0^{2\pi} \log \left| f(re^{i\theta}) \right| \mathrm{d}\theta. \tag{9}$$

(ii) If $a_1 = r_1 e^{i\theta_1}, 0 < r_1 < r$, then again on applying *Cauchy's integral formula f*or the function $\log(1 - w\overline{a_1})$, we get

$$\int\limits_{|w|=1/r} \frac{\log(1 - w\overline{a_1})}{w} \mathrm{d}w = 0 \tag{10}$$

since the function $\log(1 - w\overline{a_1})$ is analytic on and inside the circle $|w| = 1/r$ (as the singularity $\frac{1}{\overline{a_1}}$ of $\log(1 - w\overline{a_1})$ lies out side of the circle $|w| = 1/r$). On writing $1 - w\overline{a_1} = -w\overline{a_1}\left(1 - \frac{1}{w\overline{a_1}}\right)$, result (10) gives

$$\begin{aligned}
\frac{1}{2\pi i}\int\limits_{|w|=1/r} \log\left(1 - \frac{1}{w\overline{a_1}}\right)\frac{\mathrm{d}w}{w} &= \frac{1}{2\pi i}\int\limits_{|w|=1/r} \log\left(-\frac{1}{w\overline{a_1}}\right)\frac{\mathrm{d}w}{w} \\
&= \frac{1}{2\pi i}\log\left(-\frac{1}{\overline{a_1}}\right)\int\limits_{|w|=1/r}\frac{\mathrm{d}w}{w} - \frac{1}{2\pi i}\int\limits_{|w|=1/r}\log w\frac{\mathrm{d}w}{w} \\
&= \log\left(-\frac{1}{\overline{a_1}}\right) - \frac{1}{4\pi i}\left[(\log w)^2\right]_{|w|=1/r} \\
&= \log\left(-\frac{1}{\overline{a_1}}\right) - \frac{1}{4\pi i}\left[(\log 1/r + i\theta)^2\right]_{\theta=0}^{2\pi} \\
&= \log\left(-\frac{1}{\overline{a_1}}\right) - \log 1/r - i\pi
\end{aligned}$$

which on equating the real parts proves that

$$\frac{1}{2\pi}\int_0^{2\pi}\log\left|1 - \frac{1}{w\overline{a_1}}\right|\mathrm{d}\theta = \frac{1}{2\pi}\int_0^{2\pi}\log\left|1 - \frac{r}{r_1}e^{i(\theta_1-\theta)}\right|\mathrm{d}\theta = \log\frac{r}{r_1} \qquad (11)$$

which is the Jensen's formula for the function

$$f(z) = 1 - \frac{z}{a_1}.$$

(iii) The result in the above case may be extended to the case $r = r_1$. In that case we make a small circular indentation so that the singularity $\frac{1}{\overline{a_1}}$ of $\log(1 - w\overline{a_1})$ is excluded. The integral around the indentation tends to 0 with the radius and the result we get the same as above.

(iv) In general case, if

$$f(z) = \left(1 - \frac{z}{a_1}\right)\left(1 - \frac{z}{a_2}\right)...\left(1 - \frac{z}{a_n}\right)\phi(z),$$

where $\phi(z)$ is not zero for $|z| < r_{n+1}$, and $\phi(0) = f(0)$, then

$$\log f(z) = \sum_{j=1}^{n}\log\left(1 - \frac{z}{a_j}\right) + \log\phi(z)$$

which yields on equating the real parts that

$$\log|f(z)| = \sum_{j=1}^{n}\log\left|1 - \frac{z}{a_j}\right| + \log|\phi(z)|$$

and hence, on considering $|a_j| = r_j$ we obtain

$$\frac{1}{2\pi}\int_0^{2\pi}\log\left|f(re^{i\theta})\right|\mathrm{d}\theta = \sum_{j=1}^{n}\frac{1}{2\pi}\int_0^{2\pi}\log\left|1 - \frac{r}{r_j}e^{i(\theta_j-\theta)}\right|\mathrm{d}\theta + \frac{1}{2\pi}\int_0^{2\pi}\log\left|\phi(re^{i\theta})\right|\mathrm{d}\theta$$

which on using results (9) and (11) proves the formula

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi}\log\left|f(re^{i\theta})\right|\mathrm{d}\theta &= \sum_{j=1}^{n}\log\frac{r}{r_j}+\log|\phi(0)| \\ &= \log\frac{r^n}{r_1 r_2 ... r_n}+\log|f(0)|.\end{aligned}$$

■

The Theorem 3 may be extended to a function having zeros as well as poles.

Theorem 4 (Generalized Jensen's formula) *Let $f(z)$ satisfy the same conditions as in Theorem 3, with zeros $a_1, a_2, ..., a_m$ and poles $b_1, b_2, ..., b_n$ with moduli not exceeding r. Then*

$$\log\left\{\left|\frac{b_1,...,b_n}{a_1,...,a_m}f(0)\right|r^{m-n}\right\}=\frac{1}{2\pi}\int_0^{2\pi}\log\left|f(re^{i\theta})\right|\mathrm{d}\theta.$$

Proof. The function $f(z)$ may be expressed as $f(z) = g(z)/h(z)$, where the functions $g(z)$ and $h(z)$ are analytic in $|z| \le r$ having zeros, respectively, at $a_1, a_2, ..., a_m$ and $b_1, b_2, ..., b_n$ in $|z| \le r$. Thus, on applying Theorem 3 for these functions, we obtain

$$\log\left\{\left|\frac{g(0)}{b_1,...,b_n}\right|r^{n}\right\}=\frac{1}{2\pi}\int_0^{2\pi}\log\left|g(re^{i\theta})\right|\mathrm{d}\theta$$

and

$$\log\left\{\left|\frac{h(0)}{a_1,...,a_m}\right|r^{m}\right\}=\frac{1}{2\pi}\int_0^{2\pi}\log\left|h(re^{i\theta})\right|\mathrm{d}\theta$$

which on subtracting proves the result. ■

4 The Poisson-Jensen Formula

Theorem 5 *Let $f(z)$ have zeros at the points $a_1, a_2, ..., a_m$ and poles $b_1, b_2, ..., b_n$, inside the disc $|z| \le R$, and be analytic elsewhere inside and on the circle. Then for any $r < R$*

$$\begin{aligned}\log\left|f(re^{i\theta})\right| &= \frac{1}{2\pi}\int_0^{2\pi}\frac{R^2-r^2}{R^2-2Rr\cos(\theta-\phi)+r^2}\log\left|f(Re^{i\phi})\right|\mathrm{d}\phi \\ &\quad -\sum_{\mu=1}^{m}\log\left|\frac{R^2-\overline{a_\mu}\; re^{i\theta}}{R\,(re^{i\theta}-a_\mu)}\right|+\sum_{\nu=1}^{m}\log\left|\frac{R^2-\overline{b_\nu}\; re^{i\theta}}{R\,(re^{i\theta}-b_\nu)}\right|.\end{aligned} \tag{12}$$

Proof. The Poisson-Jensen Formula (12) contains both Poisson and Jensen's formula in particular cases. If there are no zeros or poles, it reduces to the Poisson formula for the function $\log f(z)$. On the other hand, if $r = 0$, we get Generalized Jensen's formula

$$\log|f(0)|=\frac{1}{2\pi}\int_0^{2\pi}\log\left|f(Re^{i\phi})\right|\mathrm{d}\phi-\log\left\{\left|\frac{b_1,...,b_n}{a_1,...,a_m}f(0)\right|R^{m-n}\right\}.$$

Further, in particular,

(i) if $f(z) = z - a$, where $|a| < R$, then the formula (12) is equivalent to

$$\log\left|re^{i\theta} - a\right| = \frac{1}{2\pi}\int_0^{2\pi} \frac{R^2 - r^2}{R^2 - 2Rr\cos(\theta - \phi) + r^2}\log\left|Re^{i\phi} - a\right| \mathrm{d}\phi - \log\left|\frac{R^2 - \overline{a}\ re^{i\theta}}{R\left(re^{i\theta} - a\right)}\right| \tag{13}$$

or

$$\log\left|R - \frac{\overline{a}\ re^{i\theta}}{R}\right| = \frac{1}{2\pi}\int_0^{2\pi} \frac{R^2 - r^2}{R^2 - 2Rr\cos(\theta - \phi) + r^2}\log\left|Re^{i\phi} - a\right| \mathrm{d}\phi,$$

where $\left|R - \frac{\overline{a}\ Re^{i\phi}}{R}\right| = \left|R - \overline{a}\ e^{i\phi}\right| = \left|Re^{i\phi} - a\right|$, and this is the Poisson formula for the function $\log\left(R - \frac{\overline{a}\ z}{R}\right)$.

(ii) if $f(z) = 1/(z - b)$, where $|b| < R$, then the formula (12) is equivalent to

$$\log\left|\frac{1}{re^{i\theta} - b}\right| = \frac{1}{2\pi}\int_0^{2\pi} \frac{R^2 - r^2}{R^2 - 2Rr\cos(\theta - \phi) + r^2}\log\left|\frac{1}{Re^{i\phi} - b}\right| \mathrm{d}\phi + \log\left|\frac{R^2 - \overline{b}\ re^{i\theta}}{R\left(re^{i\theta} - b\right)}\right|$$

or

$$\log\left|R - \frac{\overline{b}\ re^{i\theta}}{R}\right| = \frac{1}{2\pi}\int_0^{2\pi} \frac{R^2 - r^2}{R^2 - 2Rr\cos(\theta - \phi) + r^2}\log\left|Re^{i\phi} - b\right| \mathrm{d}\phi$$

which is the Poisson formula for the function $\log\left(R - \frac{\overline{b}\ z}{R}\right)$.

(iii) If $f(z)$ is analytic and has no zeros or poles in $|z| \leq R$, the formula (12) is the Poisson formula for $\log f(z)$:

$$\log\left|f(re^{i\theta})\right| = \frac{1}{2\pi}\int_0^{2\pi} \frac{R^2 - r^2}{R^2 - 2Rr\cos(\theta - \phi) + r^2}\log\left|f(Re^{i\phi})\right| \mathrm{d}\phi. \tag{14}$$

In general, if

$$f(z) = \frac{(z - a_1)\ldots(z - a_m)}{(z - b_1)\ldots(z - b_n)}\phi(z),$$

where $\phi(z)$ is analytic with $\phi(z) \neq 0$ in $|z| \leq R$, then at $z = re^{i\theta}$,

$$\log f(re^{i\theta}) = \sum_{j=1}^{m}\log\left(re^{i\theta} - a_j\right) - \sum_{j=1}^{n}\log\left(re^{i\theta} - b_j\right) + \log\phi(re^{i\theta})$$

which on equating the real parts yields

$$\log\left|f(re^{i\theta})\right| = \sum_{j=1}^{m}\log\left|re^{i\theta} - a_j\right| - \sum_{j=1}^{n}\log\left|re^{i\theta} - b_j\right| + \log\left|\phi(re^{i\theta})\right|.$$

and by (13) and (14), it proves the formula (12). ■

5 Convex functions

Definition 6 *A function $\phi(x)$ of a real variable x is said to be convex , if the curve $y = \phi(x)$ between x_1 and x_2 always lies below the chord joining the points $(x_1, \phi(x_1))$ and $(x_2, \phi(x_2))$. Analytically the condition is given by*

$$\phi(x) \leq \frac{x_2 - x}{x_2 - x_1}\phi(x_1) + \frac{x - x_1}{x_2 - x_1}\phi(x_2) \quad (x_1 < x < x_2)\,. \tag{15}$$

Theorem 7 *A convex function is continuous.*

Proof. Let $\phi(x)$ be a convex function of x and let $x_1 < x < x_2$. Then ϕ satisfy the inequality (15). If $x_1 \to x$ and $x \to x_2$, then from inequality (15), we get, respectively,

$$\phi(x) \leq \phi(x - 0) \;\; \text{and} \;\; \phi(x_2 - 0) \leq \phi(x_2)$$

which proves that for any x,

$$\phi(x) = \phi(x - 0).$$

Similarly, If $x_2 \to x$ and $x \to x_1$, then from inequality (15), we get, respectively,

$$\phi(x) \leq \phi(x + 0) \;\; \text{and} \;\; \phi(x_1 + 0) \leq \phi(x_1)$$

which proves that for any x,

$$\phi(x) = \phi(x + 0).$$

Hence, ϕ is a continuous function of x. ∎

As an application of Maximum Modulus Principle we have proved following Hadamard's three circle Theorem:

Theorem 8 (Hadamard's three-circle theorem) *Let $f(z)$ be an analytic function, regular for $r_1 \leq |z| \leq r_3$. Let $r_1 < r_2 < r_3$, and let M_1, M_2, M_3 be the maxima of $|f(z)|$ on the three circles $|z| = r_1, r_2, r_3$ respectively. Then*

$$M_2^{\log(r_3/r_1)} \leq M_1^{\log(r_3/r_2)} M_3^{\log(r_2/r_1)}. \tag{16}$$

Theorem 9 (The three-circles theorem as a convexity theorem) *Under the same hypothesis of Theorem 8, let $M(r)$ be the maxima of $|f(z)|$ on the circle $|z| = r$. Then $\log M(r)$ is convex function of $\log r$.*

Proof. For $r_1 < r_2 < r_3$, we have

$$M\left(r_2\right)^{\log(r_3/r_1)} \leq M\left(r_1\right)^{\log(r_3/r_2)} M\left(r_3\right)^{\log(r_2/r_1)} \tag{17}$$

which on taking logarithms is given by

$$\log\left(r_3/r_1\right)\log M(r_2) \leq \log\left(r_3/r_2\right)\log M(r_1) + \log\left(r_2/r_1\right)\log M\left(r_3\right)$$

and it is equivalent to

$$\log M(r_2) \leq \frac{\log r_3 - \log r_2}{\log r_3 - \log r_1}\log M(r_1) + \frac{\log r_2 - \log r_1}{\log r_3 - \log r_1}\log M\left(r_3\right)$$

this proves the result. ∎

6 Harmonic Functions

A real valued function $u(x, y)$ is said to be harmonic in a domain of the xy plane if throughout that domain, it has continuous partial derivatives of the first and second order and satisfies the partial differential equation

$$u_{xx}(x, y) + u_{yy}(x, y) = 0 \tag{18}$$

Equation (18) is called Laplace's equation. If $u(x, y)$ is harmonic in a disc, then there exists an analytic function $f(z) = u(x, y) + iv(x, y)$, where $v(x, y)$ is the harmonic conjugate of $u(x, y)$.

Theorem 10 (Mean Value Property) *Let U be an open disc $B(a, R)$ and let $u : U \to \mathbb{R}$ be harmonic. Then*

$$u(a) = \frac{1}{2\pi}\int_0^{2\pi} u(a + re^{i\theta})\mathrm{d}\theta.$$

Proof. Let $u : U \to R$ be harmonic. Then there exists $f(z) = u(x, y) + iv(x, y)$ analytic in U. Hence, we have Gauss's Mean Value Formula for any $0 < r < R$,

$$f(a) = \frac{1}{2\pi}\int_0^{2\pi} f(a + re^{i\theta})\mathrm{d}\theta$$

which on equating the real parts proves the result. ■

Theorem 11 (Harnack's inequality) *Let $u : clB(a, R) \to \mathbb{R}$ be continuous, harmonic in $B(a, R)$ and $u \geq 0$. Then for $0 \leq r < R$ and for all θ,*

$$\frac{R - r}{R + r}u(a) \leq u(a + re^{i\theta}) \leq \frac{R + r}{R - r}u(a).$$

Proof. Let $u : clB(a, R) \to \mathbb{R}$ be continuous, harmonic in $B(a, R)$ and $u \geq 0$. Then there exists a function $f(a + re^{i\theta})$ analytic in $B(a, R)$, hence, by Poisson integral formula we have

$$u(a + re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{R^2 - r^2}{R^2 - 2Rr\cos(\theta - \phi) + r^2}u(a + Re^{i\phi})\mathrm{d}\phi,$$

where

$$\frac{R^2 - r^2}{R^2 - 2Rr\cos(\theta - \phi) + r^2} = \frac{R^2 - r^2}{|Re^{i\phi} - re^{i\theta}|^2}$$

and

$$\frac{R^2 - r^2}{(R + r)^2} \leq \frac{R^2 - r^2}{|Re^{i\phi} - re^{i\theta}|^2} \leq \frac{R^2 - r^2}{(R - r)^2}$$

or

$$\frac{R - r}{R + r} \leq \frac{R^2 - r^2}{|Re^{i\phi} - re^{i\theta}|^2} \leq \frac{R + r}{R - r}$$

Thus we get

$$\frac{R - r}{R + r} \cdot \frac{1}{2\pi}\int_0^{2\pi} u(a + Re^{i\phi})\mathrm{d}\phi \leq u(a + re^{i\theta}) \leq \frac{R + r}{R - r} \cdot \frac{1}{2\pi}\int_0^{2\pi} u(a + Re^{i\phi})\mathrm{d}\phi$$

which by Mean Value Property proves the result. ■

7 Order of an entire (integral) functions

An integral function $f(z)$ is said to be of finite order if there is a positive number A such that, as $|z| = r \to \infty$

$$f(z) = O\left(e^{r^A}\right) \tag{19}$$

or

$$|f(z)| < e^{r^A}.$$

The lower bound ρ of numbers A, for which (19) holds, is called the order of $f(z)$. Hence, if $M(r)$ is the maximum modulus of $f(z)$ on the circle $|z| = r$, then the order ρ of $f(z)$ is given by

$$\rho = \inf\left\{A \geq 0 : M(r) \leq e^{r^A} \text{ for large value of } r\right\}$$

or

$$\rho = \lim_{r\to\infty} \sup \frac{\log\log M(r)}{\log r}. \tag{20}$$

Problem 12 *Find the order of following functions:*

(i) $p(z) = a_0 + a_1 z + a_2 z^2 + ... + a_n z^n, a_n \neq 0$

(ii) $f(z) = exp\,(az)\,, a \neq 0$

(iii) $f(z) = sinz$

(iv) $f(z) = cosz$

Solution 1 *We first find maximum modulus of each of the functions and then apply directly the formula (20).*
(i) On $|z| = r$,

$$\begin{aligned} |p(z)| &= \left|a_0 + a_1 z + a_2 z^2 + ... + a_n z^n\right| \\ &= |a_n z^n| \left|\frac{a_0}{a_n z^n} + \frac{a_1}{a_n z^{n-1}} + \frac{a_2}{a_n z^{n-2}} + ... + 1\right| \\ &\leq |a_n|\, r^n := M(r), \end{aligned}$$

for large value of r*. We get*

$$\begin{aligned} \lim_{r\to\infty} \frac{\log\log M(r)}{\log r} &= \lim_{r\to\infty} \frac{\log\left(\log |a_n|\, r^n\right)}{\log r} \quad \left(\frac{\infty}{\infty}\ form\right) \\ &= \lim_{r\to\infty} \frac{1}{\log |a_n|\, r^n} \frac{1/\left(|a_n|\, r^n\right)}{1/r} n\, |a_n|\, r^{n-1} \\ &= \lim_{r\to\infty} \frac{n}{\log |a_n|\, r^n} = 0 \end{aligned}$$

Hence, order of $p(z)$ *is* 0.

(ii) Here, $M(r) = e^{|a|r}$ *and*

$$\begin{aligned}\lim_{r\to\infty} \frac{\log\log M(r)}{\log r} &= \lim_{r\to\infty} \frac{\log(|a|\,r)}{\log r} \quad \left(\frac{\infty}{\infty} \textit{ form}\right)\\ &= \lim_{r\to\infty} \frac{1}{|a|\,r} \frac{|a|}{1/r} = 1.\end{aligned}$$

(iii) We have

$$|\sin z| = \left|z - \frac{z^3}{3!} + ...\right| \leq r + \frac{r^3}{3!} + ... = \frac{e^r - e^{-r}}{2} := M(r).$$

Hence,

$$\begin{aligned}\log M(r) &= \log e^r \left(\frac{1-e^{-2r}}{2}\right)\\ &= r + \log\left(\frac{1-e^{-2r}}{2}\right)\\ &= r\left(1 + \frac{1}{r}\log\left(\frac{1-e^{-2r}}{2}\right)\right)\end{aligned}$$

$$\lim_{r\to\infty} \frac{\log\log M(r)}{\log r} = 1 + \lim_{r\to\infty} \frac{\log\left(1 + \frac{1}{r}\log\left(\frac{1-e^{-2r}}{2}\right)\right)}{\log r} = 1.$$

(iv) Similarly, we may find the order of $cosz$ *is also* 1.

8 Canonical products

If $f(z)$ is an entire function of order ρ, then for all values of r,

$$\log\left|f(re^{i\theta})\right| < Kr^{\rho+\epsilon},$$

where K depends only on ϵ and from Jensen's formula (8):

$$\int_0^r \frac{n(x)}{x} \mathrm{d}x = \frac{1}{2\pi} \int_0^{2\pi} \log\left|f(re^{i\theta})\right| \mathrm{d}\theta - \log|f(0)|\,, \tag{21}$$

we see that

$$\int_0^{2r} \frac{n(x)}{x} \mathrm{d}x < Kr^{\rho+\epsilon}.$$

But, since $n(r)$ is non-decreasing,

$$\int_r^{2r} \frac{n(x)}{x} \mathrm{d}x \geq n(r) \int_r^{2r} \frac{\mathrm{d}x}{x} = n(r)\log 2.$$

Hence,

$$n(r) \leq \frac{1}{\log 2} \int_0^{2r} \frac{n(x)}{x} \mathrm{d}x < Kr^{\rho+\epsilon}$$

or

$$n(r) = O\left(r^{\rho+\epsilon}\right)$$

which says that the higher the order of a function is, the more zeros it may have in a given region.

Theorem 13 *If $r_1, r_2, \ldots$ are the moduli of the zeros of $f(z)$, then the series $\sum r_n^{-\alpha}$ is convergent if $\alpha > \rho$.*

Proof. Let $\alpha > \rho$. Then we will show that the series $\sum r_n^{-\alpha}$ is convergent. If $\alpha > \beta > \rho$, then $n(r) < Ar^{\beta}$. Taking $r = r_n$, we get $n < Ar_n^{\beta}$ or $r_n^{-\alpha} < An^{-\alpha/\beta}$ which proves the result. ■

The lower bound of positive number α for which $\sum r_n^{-\alpha}$ is convergent is called the exponent of convergence of the series, and is denoted by ρ_1 and $\rho_1 \leq \rho$.

If $f(z)$ is of finite order, then there is an integer p, independent of n, such that the product

$$\prod_{n=1}^{\infty} E\left(\frac{z}{z_n}, p\right) \tag{22}$$

is convergent for all values of z, which is possible if

$$\sum \left(\frac{r}{r_n}\right)^{p+1} \tag{23}$$

is convergent that is if $p + 1 > \rho_1$ and so certainly if $p + 1 > \rho$.

If p is the smallest integer for which (23) is convergent, the product (22) is called the *canonical product* formed with the zeros of $f(z)$ and p is called its *genus.*

Theorem 14 (Hadamard's factorization theorem) *If f(z) is an integral function of order ρ, with zeros $z_1, z_2, \ldots$ $(f(0) \neq 0)$, then*

$$f(z) = e^{Q(z)}P(z),$$

where $P(z)$ is the canonical product formed with the zeros of $f(z)$, and $Q(z)$ is the polynomial of degree not greater than ρ.

This theorem follows from the Weierstrass's Factorization theorem. Only it needs to show that the entire function $g(z)$ is a polynomial $Q(z)$ and the product $P(z)$ is the canonical product.