

FIGURE 6 A directed graph.

$$\begin{array}{c}
 \begin{array}{ccccccc}
 e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7
 \end{array} \\
 \begin{array}{l}
 v_1 \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \\
 v_2 \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\
 v_3 \begin{bmatrix} 1 & -1 & 1 & 0 & 1 & 0 & 0 \end{bmatrix} \\
 v_4 \begin{bmatrix} 0 & 0 & -1 & -1 & 0 & -1 & 0 \end{bmatrix} \\
 v_5 \begin{bmatrix} 0 & 0 & 0 & 1 & -1 & 0 & 1 \end{bmatrix}
 \end{array}
 \end{array}$$

From the definition of A_c , it should be clear that each column of this matrix has exactly two nonzero entries, one +1 and one -1; therefore, we can obtain any row of A_c from the remaining rows. Thus,

$$\text{rank}(A_c) \leq n - 1$$

An $(n - 1)$ -rowed submatrix of A_c is referred to as an *incidence matrix* of G . The vertex which corresponds to the row which is not in A_c is called the *reference vertex* of A .

2. Cut Matrix

Consider a cut (V_a, V_b) in a connected directed graph G with n vertices and m edges. Recall that (V_a, V_b) consists of all those edges connecting vertices in V_a to V_b . This cut may be assigned an orientation from V_a to V_b or from V_b to V_a . Suppose the orientation of (V_a, V_b) is from V_a to V_b . Then the orientation of an edge (v_i, v_j) is said to agree with the cut orientation if $v_i \in V_a$, and $v_j \in V_b$.

The *cut matrix* $Q_c = [q_{ij}]$ of G has m columns, one for each edge, and has one row for each cut. The element q_{ij} is defined as follows:

$$q_{ij} = \begin{cases} 1, & \text{if the } j\text{th edge is in the } i\text{th cut and its} \\ & \text{orientation agrees with the cut orientation} \\ -1, & \text{if the } j\text{th edge is in the } i\text{th cut and its} \\ & \text{orientation does not agree} \\ & \text{with the cut orientation} \\ 0, & \text{if the } j\text{th edge is not in the } i\text{th cut} \end{cases}$$

Each row of Q_c is called the *cut vector*. The edges incident on a vertex form a cut. Thus it follows that the

matrix A_c is a submatrix of Q_c . Next we identify another important submatrix of Q_c .

Recall that each branch of a spanning tree T of connected graph G defines a fundamental cutset. The submatrix of Q_c corresponding to the $n - 1$ fundamental cutsets defined by T is called the *fundamental cutset matrix* Q_f of G with respect to T .

Let b_1, b_2, \dots, b_{n-1} denote the branches of T . Let us assume that the orientation of a fundamental cutset is chosen so as to agree with that of the defining branch. Suppose we arrange the rows and the columns of Q_f so that the i th column corresponds to the fundamental cutset defined by b_i . Then the matrix Q_f can be displayed in a convenient form as follows:

$$Q_f = [U \mid Q_{fc}]$$

where U is the unit matrix of order $n - 1$, and its columns correspond to the branches of T . As an example, the fundamental cutset matrix of the graph in Fig. 6 with respect to the spanning tree $T = (e_1, e_2, e_5, e_6)$ is given below:

$$\begin{array}{c}
 \begin{array}{ccccccc}
 e_1 & e_2 & e_5 & e_6 & e_3 & e_4 & e_7
 \end{array} \\
 \begin{array}{l}
 e_1 \begin{bmatrix} 1 & 0 & 0 & 0 & -1 & -1 & -1 \end{bmatrix} \\
 e_2 \begin{bmatrix} 0 & 1 & 0 & 0 & -1 & -1 & -1 \end{bmatrix} \\
 e_5 \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & -1 & -1 \end{bmatrix} \\
 e_6 \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}
 \end{array}
 \end{array}$$

It is clear that the rank of Q_f is $n - 1$. Hence,

$$\text{rank}(Q_c) \geq n - 1$$

3. Circuit Matrix

Consider a circuit C in a connected directed graph G with n vertices and m edges. This circuit can be traversed in one of two directions, clockwise or counterclockwise. The direction we choose for traversing C is called the *orientation* of C . If an edge $e = (v_i, v_j)$ directed from v_i to v_j is in C and if v_i appears before v_j as we traverse C in the direction specified by the orientation of C , then we say that the orientation agrees with the orientation of e .

The *circuit matrix* $B_c = [b_{ij}]$ of G has m columns, one for each edge, and has one row for each circuit in G . The element b_{ij} is defined as follows:

$$b_{ij} = \begin{cases} 1, & \text{if the } j\text{th edge is in the } i\text{th circuit} \\ & \text{and its orientation agrees} \\ & \text{with the circuit orientation} \\ -1, & \text{if the } j\text{th edge is in the } i\text{th circuit} \\ & \text{and its orientation does not agree} \\ & \text{with circuit orientation} \\ 0, & \text{if the } j\text{th edge is not in the } i\text{th circuit} \end{cases}$$

The submatrix of B_c corresponding to the fundamental circuits defined by the chords of a spanning tree T is called

the fundamental circuit matrix B_f of G with respect to the spanning tree T .

Let $c_1, c_2, c_3, \dots, c_{m-n+1}$ denote the chords of T . Suppose we arrange the columns and the rows of B_f so that the i th row corresponds to the fundamental circuit defined by the chord c_i and the i th column corresponds to the chord c_i . If, in addition, we choose the orientation of a fundamental circuit to agree with the orientation of the defining chord, we can write B_f as:

$$B_f = [U \perp B_{ft}]$$

where U is the unit matrix of order $m - n + 1$, and its columns correspond to the chords of T .

As an example, the fundamental circuit matrix of the graph shown in Fig. 6 with respect to the tree $T = (e_1, e_2, e_5, e_6)$ is given below:

$$B_f = \begin{matrix} & e_3 & e_4 & e_7 & e_1 & e_2 & e_5 & e_6 \\ \begin{matrix} e_3 \\ e_4 \\ e_7 \end{matrix} & \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 & 1 & 1 & -1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{bmatrix} \end{matrix}$$

It is clear from the above that the rank of B_f is $m - n + 1$; hence,

$$\text{rank}(B_c) \geq m - n + 1.$$

The following results constitute the foundation of the graph theoretic application to electrical circuit analysis.

Theorem 4 (orthogonality relationship)

1. A circuit and a cutset in a connected graph have an even number of common edges.
2. If circuit and a cutset in a directed graph have $2k$ common edges, then k of these edges have the same relative orientation in the circuit and the cutset, and the remaining k edges have one orientation in the circuit and the opposite orientation in the cutset.

Theorem 5

If the columns of the circuit matrix B_c and the columns of the cut matrix Q_c are arranged in the same edge order, then

$$B_c Q_c = 0$$

Theorem 6

$\text{Rank}(B_c) = m - n + 1$ and $\text{rank}(Q_c) = n - 1$.

Note that it follows from the above theorem that the rank of the circuit matrix is equal to the nullity of the graph, and the rank of the cut matrix is equal to the rank of the graph. This result, in fact, motivated the definitions of the rank and nullity of a graph

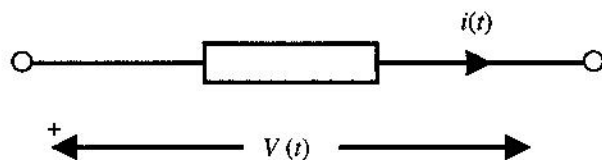


FIGURE 7 A network element with reference convention.

II. GRAPHS AND ELECTRICAL NETWORKS

An electrical network is an interconnection of electrical network elements such as resistances, capacitances, inductances, voltage and current sources, etc. Each network element is associated with two variables, the voltage variable, $v(t)$ and the current variable $i(t)$. We also assign reference directions to the network elements (see Fig. 7) so that $i(t)$ is positive whenever the current is in the direction of the arrow, and $v(t)$ is positive whenever the voltage drop in the network element is in the direction of the arrow. Replacing each element and its associated reference direction by a directed edge results in the directed graph representing the network. For example, a simple electrical network and the corresponding directed graph are shown in Fig. 8.

The physical relationship between the current and voltage variables of network elements is specified by Ohm's law. For voltage and current sources, the voltage and current variables are required to have specified values. The linear dependence among the voltage variables in the network and the linear dependence among the current variables are governed by Kirchhoff's voltage and current laws.

Kirchhoff's Voltage Law (KVL): The algebraic sum of the voltages around any circuit is equal to zero.

Kirchhoff's Current Law (KCL): The algebraic sum of the currents flowing out of a node is equal to zero.

As an example, the KVL equation for the circuit 1, 3, 5 and the KCL equation for vertex b in the graph of Fig. 8 and

$$\begin{array}{ll} \text{Circuit 1, 3, 5} & v_1 + v_3 + v_5 = 0 \\ \text{Vertex } b & -i_1 + i_2 + i_3 = 0 \end{array}$$

It can be easily seen that KVL and KCL equations for an electrical network N can be conveniently written as:

$$A_c I_e = 0$$

and

$$B_c V_e = 0$$

where A_c and B_c are, respectively, the incidence and circuit matrices of the directed graph representing N ; I_e and V_e are, respectively, the column vectors of element currents and voltages in N . Because each row in the cut matrix Q_c can be expressed as a linear combination of the rows of

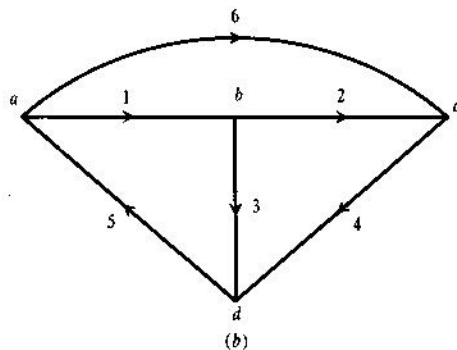
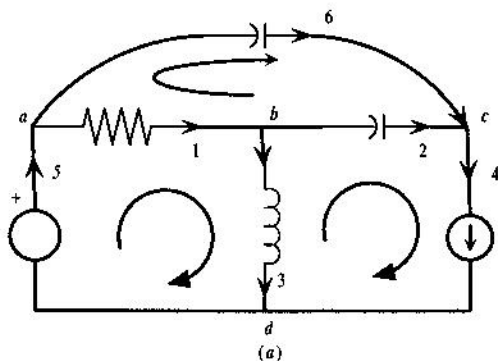


FIGURE 8 (a) An electrical network N ; (b) directed graph representation of N .

the matrix, in the above we can replace A_c by Q_c ; thus, we have

$$\text{KCL: } Q_c I_e = 0$$

$$\text{KVL: } B_c V_e = 0$$

Thus, KCL can also be stated as: The algebraic sum of the currents in any cut of N is equal to zero.

If a network N has n vertices and m elements and its graph is connected, then there are only $(n - 1)$ linearly independent cuts and only $(m - n + 1)$ linearly independent circuits. Thus, in writing KVL and KCL equations we need to use only B_f , a fundamental circuit matrix, and Q_f , a fundamental circuit matrix, respectively. Thus, we have

$$\text{KCL: } Q_f I_e = 0$$

$$\text{KVL: } B_f V_e = 0$$

We note that the KCL and KVL equations depend only on the way network elements are interconnected and not on the nature of the network elements. Thus, several results in electrical network theory are essentially graph theoretic in nature. Some results of interest in electrical network analysis are presented in the remainder of this chapter. In the following, a network N and its directed graph representation are both denoted by N .

Loop and Cutset Transformations

Let T be a spanning tree of an electrical network. Let I_c and V_t be the column vectors of chord currents and branch currents with respect to T .

1. Loop transformation:

$$I_e = B_f' I_c$$

2. Cutset transformation:

$$V_e = Q_f' V_t$$

If, in the cutset transformation, we replace Q_f by the reduced incidence matrix A , then we get the *node transformation* given below:

$$V_e = A' V_n$$

where the elements in the vector V_n can be interpreted as the voltages of the nodes with respect to the reference node r . (Note: the matrix A does not contain the row corresponding to the node r .)

The above transformations have been extensively employed in developing different methods of network analysis. Two of these methods are described in the following.

III. LOOP AND CUTSET SYSTEMS OF EQUATIONS

As we observed earlier, the problem of network analysis is to determine the voltages and currents associated with the elements of an electrical network. These voltages and currents can be determined from Kirchhoff's equations and the element voltage-current (in short, $v - i$) relations given by Ohm's law. However, these equations involve a large number of variables. As can be seen from the loop and cutset transformations, not all these variables are independent. Furthermore, in place of KCL equations we can use the loop transformation which involves only chord currents as variables. Similarly, KVL equations can be replaced by the cutset transformation which involves only branch voltage variables. We can take advantage of these transformations to establish different systems of network equations known as the loop and cutset systems.

In deriving the loop system we use the loop transformation in place of KCL, and in this case the loop variables (chord currents) will serve as independent variables. In deriving the cutset system we use the cutset transformation in place of KVL, and the cutset variables (tree branch voltages) will serve as the independent variables in this case. Consider a connected electrical network N . We assume that N consists of only resistances (R), capacitances (C), inductances (L) (referred to collectively as RCL) including mutual inductances, and independent

voltage and current sources. We also assume that all initial inductor currents and initial capacitor voltages have been replaced by appropriate sources. Further, the voltage and current variables are all Laplace transforms of the complex frequency variable s . In N there can be no circuit consisting of only independent voltage sources, for, if such a circuit of sources were present, then by KVL there would be a linear relationship among the corresponding voltages, violating the independence of the voltage sources. For the same reason, in N there can be no cutset consisting of only independent current sources. So there exists in N a spanning tree containing all the voltage sources but not current sources. Such a tree is the starting point for the development of both the loop and cutset systems of equations.

Let T be a spanning tree of the given network such that T contains all the voltage sources but no current sources. Let us partition the element voltage V_e and the element current vector I_e as follows:

$$V_e = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} \quad \text{and} \quad I_e = \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix}$$

where the subscripts 1, 2, and 3 refer to the vectors corresponding to the current sources, RCL elements, and voltage sources, respectively. Let B_f be the fundamental circuit matrix of N , and Q_f the fundamental cutset matrix of N with respect to T . Then the KVL and the KCL equations can be written as follows:

$$\text{KVL: } B_f V_e = \begin{bmatrix} U & B_{12} & B_{13} \\ 0 & B_{22} & B_{23} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = 0$$

$$\text{KCL: } Q_f I_e = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{21} & Q_{22} & U \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = 0$$

A. Loop Method of Network Analysis

• Step 1: Solve the following for the vector I_1 (note that I_1 is the vector of currents in the nonsource chords of T).

$$Z_1 I_1 = -B_{23} V_3 - B_{22} Z_2 B_{12} I_1 \quad (1)$$

where Z_2 is the impedance matrix of RCL elements and $Z_1 = B_{22} Z_2 B_{22}$. Equation (1) is called the *loop system of equations*.

• Step 2: Calculate I_2 using:

$$I_2 = B_{12} I_1 + B_{22} I_1 \quad (2)$$

then,

$$V_2 = Z_2 I_2 \quad (3)$$

• Step 3: Determine V_1 and I_3 using the following:

$$V_1 = -B_{12} V_2 - B_{13} V_3 \quad (4)$$

$$I_3 = B_{13} I_1 + B_{23} I_1 \quad (5)$$

Note that I_1 and V_3 have specified values, since they correspond to current and voltage sources, respectively.

B. Cutset Method of Network Analysis

• Step 1: Solve the following for the vector V_b (note that V_b is the vector of voltages in the nonsource branches of T):

$$Y_b V_b = -Q_{11} I_1 - Q_{12} Y_2 Q_{22} V_b \quad (6)$$

where Y_2 is the admittance matrix of RLC elements and $Y_b = Q_{12} Y_2 Q_{12}$. Equation (6) is called the *cutset system of equations*.

• Step 2: Calculate V_2 using:

$$V_2 = Q_{12} V_b + Q_{22} V_3 \quad (7)$$

then,

$$I_2 = Y_2 V_2 \quad (8)$$

• Step 3: Determine V_1 and I_3 using the following:

$$V_1 = Q_{21} V_b + Q_{21} V_3 \quad (9)$$

$$I_3 = -Q_{21} I_1 - Q_{22} I_2 \quad (10)$$

Note that I_1 and V_3 have specified values, since they correspond to current and voltage sources.

This completes the cutset method of network analysis. Next we illustrate the loop and cutset methods of analysis on the network shown in Fig. 9. The graph of the network is shown Fig. 9b. We choose the spanning tree T consisting of edges 4, 5, and 6. Note that T contains the voltage source and has no current source. The fundamental circuit and the fundamental cutset matrices with respect to T are given below in the required partitioned form:

$$B_f = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 0 & 0 & -1 & -1 & 1 \\ 1 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 & -1 & 0 \end{bmatrix}$$

$$Q_f = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & -1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

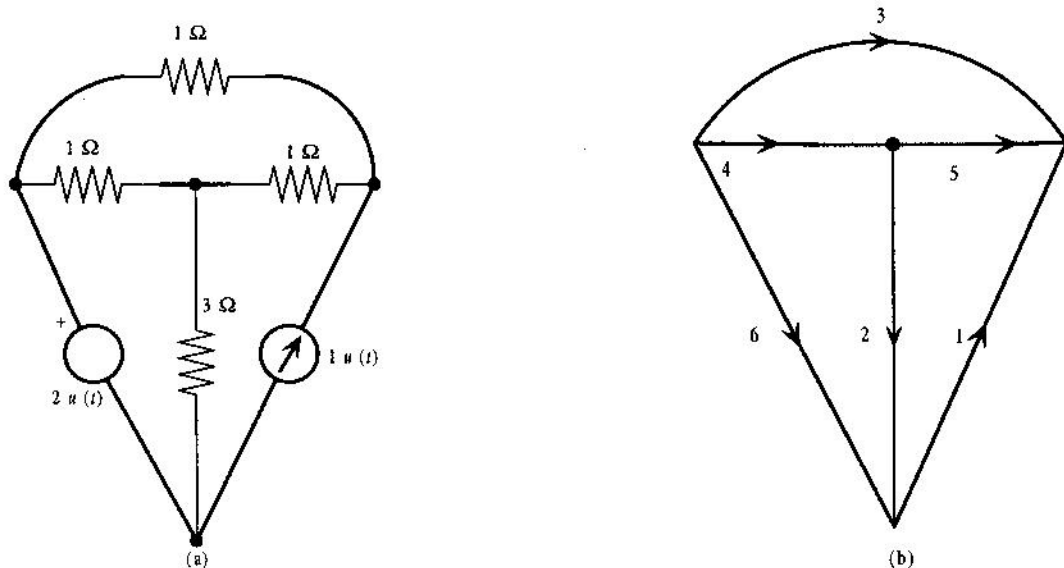


FIGURE 9 A network and its graph.

From these matrices we get:

$$B_{12} = \begin{bmatrix} 0 & 0 & -1 & -1 \end{bmatrix}$$

$$B_{13} = [1]$$

$$B_{22} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 \end{bmatrix}$$

$$B_{23} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$Q_{11} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$Q_{12} = \begin{bmatrix} -1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

$$Q_{21} = [-1]$$

$$Q_{22} = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$$

We also have

$$Z_2 = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$Y_2 = \begin{bmatrix} 1/3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$v_6 = 2 \text{ volts}$$

$$i_1 = 1 \text{ ampere}$$

C. Loop Method Example

Edges 2 and 3 are nonsource chords. So,

$$I_l = \begin{bmatrix} i_2 \\ i_3 \end{bmatrix}$$

Substituting

$$Z_l = B_{22} Z_2 B_{22}' = \begin{bmatrix} 4 & -1 \\ -1 & 3 \end{bmatrix}$$

in Eq. (1), we get the following loop system of equations:

$$\begin{bmatrix} 4 & -1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} i_2 \\ i_3 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

Solving for i_2 and i_3 , we get:

$$I_l = \begin{bmatrix} i_2 \\ i_3 \end{bmatrix} = 1/11 \begin{bmatrix} 7 \\ -5 \end{bmatrix}$$

Using Eq. (2), we get:

$$I_2 = \begin{bmatrix} i_2 \\ i_3 \\ i_4 \\ i_5 \end{bmatrix} = 1/11 \begin{bmatrix} 7 \\ -5 \\ 1 \\ -6 \end{bmatrix}$$

Then, using $V_2 = Z_2 I_2$, we get:

$$V_2 = \begin{bmatrix} v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix} = 1/11 \begin{bmatrix} 21 \\ -5 \\ 1 \\ -6 \end{bmatrix}$$

Finally, from Eqs. (4) and (5) we get:

$$V_1 = [v_1] = -27/11$$

$$I_3 = [i_6] = 4/11$$

D. Cutset Method Example

Edges 4 and 5 are the nonsource branches, so:

$$V_b = \begin{bmatrix} v_4 \\ v_5 \end{bmatrix}$$

Substituting

$$Y_b = \begin{bmatrix} 7/3 & 1 \\ 1 & 2 \end{bmatrix}$$

in Eq. (6), we get the following cutset system of equations:

$$Y_b V_b = \begin{bmatrix} 7/3 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} v_4 \\ v_5 \end{bmatrix} = \begin{bmatrix} -1/3 \\ -1 \end{bmatrix}$$

Solving for V_b ,

$$V_b = \begin{bmatrix} v_4 \\ v_5 \end{bmatrix} = 1/11 \begin{bmatrix} 1 \\ -6 \end{bmatrix}$$

From Eq. (7) we get:

$$V_2 = \begin{bmatrix} v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix} = 1/11 \begin{bmatrix} 21 \\ -5 \\ 1 \\ -6 \end{bmatrix}$$

Using

$$I_2 = Y_2 V_2$$

we get:

$$I_2 = \begin{bmatrix} i_2 \\ i_3 \\ i_4 \\ i_5 \end{bmatrix} = 1/11 \begin{bmatrix} 7 \\ -5 \\ 1 \\ -6 \end{bmatrix}$$

Finally, using Eqs. (9) and (10),

$$V_1 = [v_1] = -27/11$$

$$I_3 = [i_6] = 4/11$$

This completes our illustration of the loop and cutset methods of circuit analysis.

Suppose a network N has no independent voltage sources. Then a convenient description of N with the node voltages as independent variables can be obtained as follows. Let A be the incidence matrix of N with vertex v_r as reference. Let us also partition A as $A = [A_{11} \ A_{12}]$, where the columns of A_{11} and A_{12} correspond, respectively, to the RCL elements and current sources. If I_1 and I_2 denote the column vectors of RCL element currents and current source currents, then KCL equations for N become:

$$A_{11} = -A_{12}I_2$$

We also have

$$I_1 = Y_1 V_1$$

where V_1 is the column vector of voltages of RCL elements and Y_1 is the corresponding admittance matrix. Furthermore, by the node transformation we have:

$$V_1 = A_{11}' V_n$$

where V_n is the column vector of node voltages. So, we get from the KCL equations:

$$(A_{11} Y_1 A_{11}') V_n = -A_{12} I_2$$

The above equations are called *node equations*. The matrix $A_{11} Y_1 A_{11}'$ is called the *node admittance matrix* of N .

FURTHER READING

For a more comprehensive discussion of other developments in graph theoretic concepts, please consult Chen (1972, 2001), Swamy and Thulasiraman (1981), and Watanabe and Shinoda (1999). For a very good treatment of linear circuits and other related references, see Balabanian and Bickart (1981). Mitra (1974) provides a very good early work on active networks, while Chua et al. (1987) and Hasler and Neirynck (1986) are good sources for nonlinear network theory.

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