Homotopy

Def - If f and f' are continuous map of the space X into the space Y, then f is said to be homotopic to f if there is a continuous map $F: X \times I \longrightarrow Y$ such that

$$F(x,0) = f(x) \quad \text{and} \quad F(x,1) = f'(x). \quad \forall x \in X$$

Then F is said a homotopy between f and f' and it is denoted by $f \simeq f'$.

Def. If $f \simeq f'$ and f' is a constant map then f is said to be null homotopic.

If t is a parameter representing time then homotopy F representing continuous deforming of f to f' as t goes from 0 to 1

Def. If f is a continuous map i.e $f: I \rightarrow X$ such that

$$f(0) = x_0 \quad \text{and} \quad f(1) = x_1$$

Then f is a path in X from x_0 to x_1 where x_0 and x_1 are initial and finial points of the path

Def If f and f' are two paths in X then f, f' are said to be path homotopic if f and f' have same initial and final points and if there is a continuous map $F: I \times I \longrightarrow X$ such that

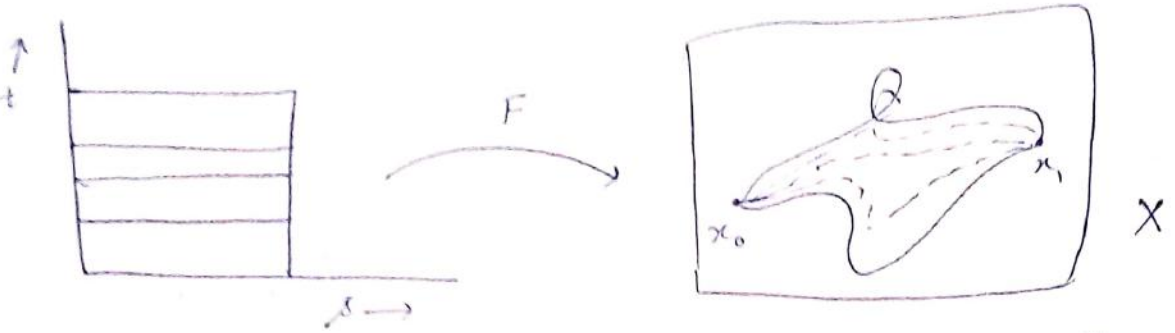
$$F(s,0) = f(s) \quad \text{and} \quad F(s,1) = f'(s)$$

$$F(0,t) = x_0 \quad \text{and} \quad F(1,t) = x_1. \quad \forall s \in I$$

and each $t \in I$.

F is said to be Path homotopy between f and f'

If f is path homotopic to f' then $f \simeq_p f'$.



In path homotopy the first condition says that F represents the continuous way of deforming the path f to path f' and second condition says that the end points of the path remain fixed during the deformation.

Equivalence Relation-

The relations ~~$\simeq$~~ and $\simeq_p$ are path hor equivalence relation. If f is a path then its path homotopy class denoted as $[f]$.

Reflexive – for any map $f : X \to Y$, define $F : X \times I \to Y$ by $F(x, t) = f(x)$ $(0 \leq t \leq 1)$

$F(x, 0) = f(x) = F(x, 1)$ $\forall x \in X$

F is continuous and if f is path then F is a path

$\Rightarrow$ F is required homotopy.

homotopy

Symmetric – Let $f \simeq g$ then $\exists$ a homotopy $F : X \times I \to Y$ such that $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$. $\forall x \in X$

Now Define a ~~relation~~ function

$$G(x, t) = F(x, 1 - t)$$
$$G(x, 0) = F(x, 1) = g(x)$$
$$G(x, 1) = F(x, 0) = f(x)$$

is a homotopy between g and f. $\because$ F is a continuous function then G is also continuous hence $g \simeq f$.

If F is path homotopy between f & g then G is a path homotopy bet g & f.

Transitive. If $f \simeq g$ and $g \simeq h$. Then $\exists$ two homotopies F_1 & F_2 as

$$F_1(x,0) = f(x) \quad \text{and} \quad F_1(x,1) = g(x)$$
$$F_2(x,0) = g(x) \quad \text{and} \quad F_2(x,1) = h(x)$$

ie $\quad F_1(x,1) = g(x) = F_2(x,0)$.

Now to prove that $f \simeq h$.

$\boxed{G : X \times I \to Y}$

Define G be a function defined as

by the equation, $G(x,t) = \begin{cases} F_1(x,2t) & , \ 0 \le t \le \frac{1}{2} \\ F_2(x, 2t-1) & , \ \frac{1}{2} \le t \le 1. \end{cases}$

The map G is well defined since $t = \frac{1}{2}$

Now $\quad F_1(x,1) = g(x) = G(x, \frac{1}{2})$

$\quad\quad F_2(x,0) = g(x) = G(x, \frac{1}{2})$

Also
$G(x,0) = F_1(x,0) = f(x)$
$G(x,1) = F_2(x,1) = h(x)$
$\therefore f \simeq h$

Also G is continuous by pasting lemma.

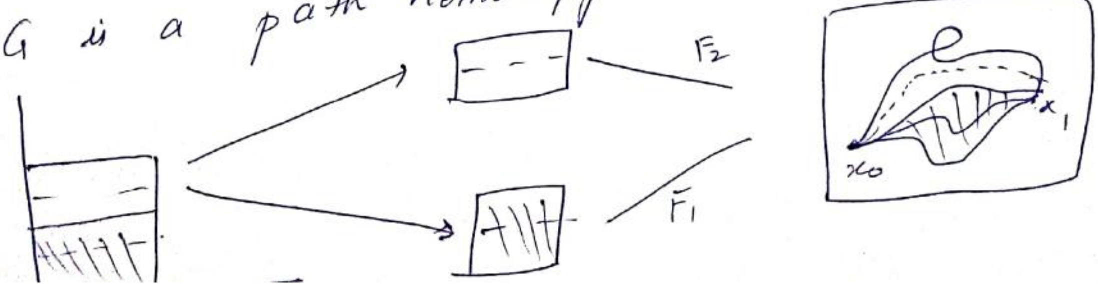
$\because$ $X \times [0, \frac{1}{2}]$ and $X \times [\frac{1}{2}, 1]$ are two closed subsets of $X \times I$ and $\frac{1}{2} \in (X \times [0, \frac{1}{2}]) \cap (X \times [\frac{1}{2}, 1])$

and $F_1(x,1) = F_2(x,0) = g(x)$.

Then G is continuous.

$\Rightarrow$ G is a homotopy between f & h

If ~~f and~~, g, h are the paths in X then G is a path homotopy between f & h.



Pasting Lemma : Let $X = A \cup B$, where A and B are closed in X. Let $f : A \to Y$ and $g : B \to Y$ be continuous. If $f(x) = g(x)$ for every $x \in A \cap B$ then f and g combine to give a continuous function $h : X \to Y$, defined by setting $h(x) = f(x)$ if $x \in A$ and $h(x) = g(x)$ if $x \in B$. ie $h(x) = \begin{cases} f(x) & \forall\ x \in A \\ g(x) & \forall\ x \in B \end{cases}$

Proof - Let C be the closed subset of Y. Now

$$h^{-1}(C) = f^{-1}(C) \cup g^{-1}(C)$$

Since f is continuous $\Rightarrow f^{-1}(C)$ is closed in A and

$\Rightarrow$ closed in X

Similarly g is continuous $\Rightarrow g^{-1}(C)$ is closed B $\Rightarrow$ closed in X

$\Rightarrow h^{-1}(C)$ is closed in X.

$\Rightarrow$ h is a continuous function

Note → The homotopy relation $\simeq$ between mappings of a space X into a space Y is an equivalence relation on the space Y^X of all continuous functions.

or, The homotopy relation $\simeq$ gives rise to a partion of the function space Y^X into the disjoint equivalent classes called homotopy class.

Now we define an operation between two disjoint classes of Y^X say $[f] * [g] = [f * g]$.

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Examples on Homotopy -

(1) Let f and g be two maps of a space X into R^2. Then $f \simeq g$ if the map F, is defined by the relation

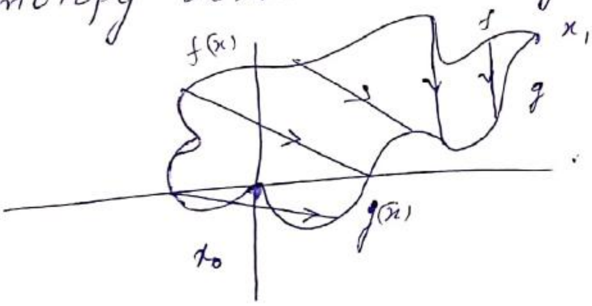
$$F(x,t) = (1-t) f(x) + t\, g(x)$$

is a homotopy between f and g.

$\therefore \quad F(x,0) = f(x), \quad F(x,1) = g(x)$

This homotopy is called straight line homotopy because it moves the point $f(x)$ to the point $g(x)$ along the straight line segment joining them.

If f and g are paths from x_0 to x_1, then F will be path homotopy between f and g. ie $f \simeq_p g$.



(2) Let A be a convex subspace of R^n, ie for $a, b \in A$, the st line joining a and b contained in A. Then any two paths f & g in A from x_0 to x_1 are path homotopic in A, for the st line homotopy F between f and g has image set in A

(3) Let $X = R^2 - \{0\}$, a punctured plane. The path f and g in X are defined by the equation

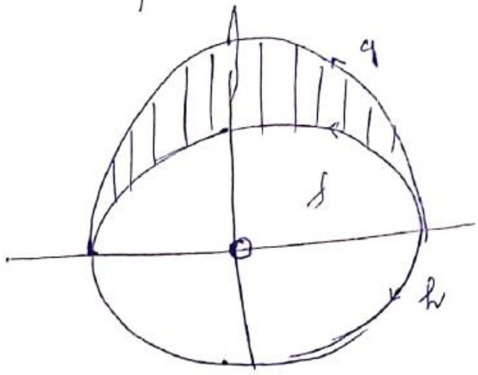
$$f(s) = (\cos \pi s, \sin \pi s), \quad g(s) = (\cos \pi s, 2 \sin \pi s)$$

ie $f \simeq_p g$ The straight line homotopy between f and g is acceptable.

But if $h(s) = (\cos \pi s, -\sin \pi s)$ is also a path in X then there exists no path homotopy between f and h.

$\because$ one can not "deform f past the hole at 0" without introducing a discontinuity.

The paths f and g would be path homotopic if they were paths in R^2.



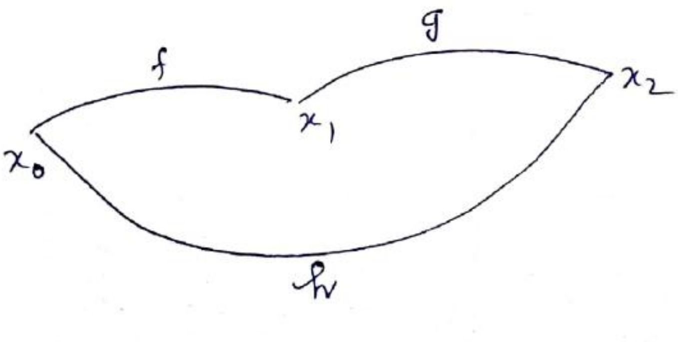
Product of two paths.

If f is a path in X from x_0 to x_1 and g is a path in X from x_1 to x_2 then the product $f * g$ of f and g is defined by path h by the equation.

$$h(s) = \begin{cases} f(2s) & \text{for } s \in [0, \frac{1}{2}] \\ g(2s-1) & \text{for } s \in [\frac{1}{2}, 1] \end{cases}$$

The function h is well defined and continuous by pasting lemma. It is a path in X from x_0 to x_2

$\therefore$ $h(0) = f(0) = x_0$, $h(1) = g(1) = x_2$



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The product operation on paths induces a well defined operation on path homotopy classes defined by the equation

$$[f] * [g] = [f * g].$$

Let F be a path homotopy between f and f' and G be a path homotopy between g and g' then

Define $$H(s,t) = \begin{cases} F(2s, t) & s \in [0, 1/2] \\ G(2s-1, t) & s \in [1/2, 1]. \end{cases}$$

Then to prove that H is a path homotopy between $f * g$ and $f' * g'$.

Now if $F : I \times I \to X$ then it is defined by

$F(s,0) = f(x)$ and $F(s,1) = f'(x)$

$F(0,t) = x_0$,, $F(1,t) = x_1$

And $G : I \times I \to X$ is a path homotopy between g & g'

i.e. $G(s,0) = g(x)$ and $G(s,1) = g'(x)$

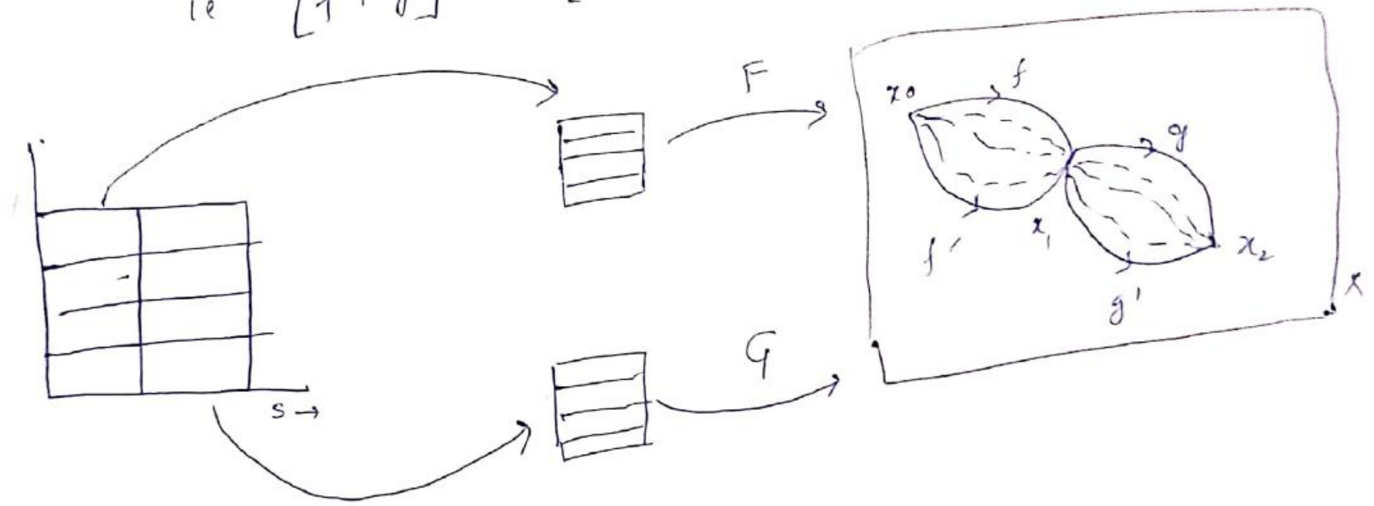
$G(0,t) = x_1$ and $G(1,t) = x_2$

$\Rightarrow F(1,t) = x_1 = G(0,t)$.

Now map H is well defined and is continuous by pasting lemma. Hence H is a path homotopy between $f * g$ and $f' * g'$ ie $f * g \simeq f' * g'$

or $[f] * [g] = [f'] * [g']$.

ie $[f * g] = [f' * g']$.



Groupoid properties of '$*$'.

The properties of '$*$' are called groupoid properties if for the pairs $[f]$, $[g]$ the relation holds $f(1) = g(0)$.

(i) If $[f] * ([g] * [h])$ is defined so is $([f] * [g]) * [h]$, and $([f] * [g]) * [h]$.

$\therefore \quad [f] * ([g] * [h]) = ([f] * [g]) * [h]$

(ii) $\forall\, x \in X$, let e_x denote the constant path $e_x : I \to X$ carrying all of I to the point x. If f is a path in X from x_0 to x_1, then

$$[f] * [e_{x_1}] = [f]$$

and $$[e_{x_0}] * [f] = [f].$$

(iii) Given the path f in X from x_0 to x_1. Let $\bar{f}$ be the path defined by $\bar{f}(s) = f(1-s)$

It is called reverse of f. Then

$$[f] * [\bar{f}] = [e_{x_0}]$$

$$[\bar{f}] * [f] = [e_{x_1}].$$

Def Let X be a space and $x_0 \in X$. A path in X is said to loop if it begins and ends at x_0. Path is based at x_0.

The set of path homotopy class of loops based at x_0 with operation $*$ is called fundamental group of X relative to the base point x_0. It is denoted by $\pi_1(X, x_0)$.

Def The homotopy relation $\simeq$ give rise to a partition of the function space Y^I into disjoint equivalent classes called homotopy classes.

If $C(X, x_0)$ be the collection of closed paths then $\simeq_{x_0}$ is an equivalence relatn

Theorem. - Show that $\pi_1(X, x_0)$ is a group.

(i) Let $[f], [g] \in \pi_1(X, x_0)$ define the operation 'o'

by $[f] \circ [g] = [f * g]$.

We first show that 'o' is well defined.

ie the operation 'o' does not depend upon the representatives f and g of $[f]$ and $[g]$ used.

Suppose $f_1 \underset{x_0}{\simeq} f_2$ and $g_1 \underset{x_0}{\simeq} g_2$ then $\exists$ homotopies h_1 & h_2

such that
$$h_1(s,0) = f_1(s) \qquad h_1(s,1) = f_2(s)$$
$$h_2(s,0) = g_1(s) \qquad h_2(s,1) = g_2(s) \qquad \forall\ s \in I.$$

and
$$h_1(0,t) = h_1(1,t) = x_0 \qquad \forall\ t \in I$$
$$h_2(0,t) = h_2(1,t) = x_0$$

Define
$$h(s,t) = \begin{cases} h_1(2s, t) & \text{if } 0 \le s \le \frac{1}{2} \\ h_2(2s-1, t) & \frac{1}{2} \le s \le 1 \end{cases}$$

Then
$$h_1\left(\tfrac{1}{2}, t\right) = h_1(1,t) = x_0$$
$$h_2\left(\tfrac{1}{2}, t\right) = h_2(0,t) = x_0$$

Hence the map h is well defined and continuous

Also
$$h(s,0) = \begin{cases} h_1(2s,0) = f_1(2s) & s \in [0,\frac{1}{2}] \\ h_2(2s-1,0) = g_1(2s-1) & s \in [\frac{1}{2},1], \end{cases}$$

Thus $h(s,0) = (f_1 * g_1)(s)$

Again
$$h(s,1) = \begin{cases} h_1(2s,1) = f_2(2s) & s \in [0,\frac{1}{2}] \\ h_2(2s-1,1) = g_2(2s-1) & s \in [\frac{1}{2},1] \end{cases}$$

$\therefore \quad h(s,1) = (f_2 * g_2)(s)$

Also $h(0,t) = h_1(0,t) = x_0$

and $h(1,t) = h_2(1,t) = x_0$

$\Rightarrow \quad f_1 * g_1 \simeq_{x_0} f_2 * g_2$.

ie if $f_1 \simeq_{x_0} f_2$ and $g_1 \simeq x_0\ g_2$ then

$f_1 * g_1 \simeq_{x_0} f_2 * g_2$ and 'o' is well defined.

ie $f * g \in \Pi_1(X, x_0)$ ie $[f] \circ [g] \in \Pi_1(X, x_0)$

ie $\Pi_1(X, x_0)$ is closed w.r.t. binary operation 'o'

(ii) let $[f], [g], [k] \in \Pi_1(X, x_0)$. Then

$$[(f*g)*k](s) = \begin{cases} (f*g)(2s) & s \in [0,\frac{1}{2}] \\ k(2s-1) & s \in [\frac{1}{2},1] \end{cases}$$

If $0 \leq s \leq \frac{1}{2}$ ie $0 \leq 2s \leq 1$ so that

$$(f*g)(2s) = \begin{cases} f(2 \cdot 2s) & 0 \leq 2s \leq \frac{1}{2} \\ g(2 \cdot (2s-1)) & \frac{1}{2} \leq 2s \leq 1 \end{cases}$$

$$= \begin{cases} f(4s) & 0 \leq s \leq \frac{1}{4} \\ g(4s-1) & \frac{1}{4} \leq s \leq \frac{1}{2} \end{cases}$$

$$\therefore \quad ((f * g) * k)\, s = \begin{cases} f\,(4s) & s \in [0, \tfrac{1}{4}] \\ g\,(4s-1) & s \in [\tfrac{1}{4}, \tfrac{1}{2}] \\ h\,(2s-1) & s \in [\tfrac{1}{2}, 1] \end{cases}$$

Similarly $$(f * (g * k))\, s = \begin{cases} f\,(2s) & s \in [0, \tfrac{1}{2}] \\ (g * k)(2s-1) & s \in [\tfrac{1}{2}, 1] \end{cases}$$

If $\frac{1}{2} \leq s \leq 1$ then $0 \leq 2s - 1 \leq 1$

$$\therefore \quad (g * k)(2s-1) = \begin{cases} g\,(2 \cdot (2s-1)) & 0 \leq 2s-1 \leq \tfrac{1}{2} \\ k\,(2 \cdot (2s-1) - 1) & \tfrac{1}{2} \leq 2s-1 \leq 1 \end{cases}$$

$$= \begin{cases} g\,(4s-2) & \tfrac{1}{2} \leq s \leq \tfrac{3}{4} \\ k\,(4s-3) & \tfrac{3}{4} \leq s \leq 1 \end{cases}$$

$$\Rightarrow (f * (g * k))(s) = \begin{cases} f\,(2s) & s \in [0, \tfrac{1}{2}] \\ g\,(4s-2) & g \in [\tfrac{1}{2}, \tfrac{3}{4}] \\ k\,(4s-3) & k \in [\tfrac{3}{4}, 1] \end{cases}$$

Define $h : I \times I \to X$ by

$$h\,(s,t) = \begin{cases} f\left(\dfrac{4s}{t+1}\right) & t \geq 4s-1 \\ g\,(4s - t - \cancel{1}) & 4s-1 \geq t \geq 4s-2 \\ k\left(\dfrac{4s-t-2}{2-t}\right) & 4s-2 \geq t. \end{cases}$$

Then by pasting lemma function is continuous

$$h\,(s,0) = \begin{cases} f\,(4s) \\ g\,(4s-1) \\ k\,(2s-1) \end{cases}$$

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$$h(s,1) = \begin{cases} f(2s) \\ g(4s-2) \\ k(4s-3) \end{cases}$$

ie $h(s,0) = ((f * g) * k)(s)$

$h(s,1) = (f * (g * k))(s)$

Also $h(0,t) = f(0) = x_0$

$h(1,t) = k(1) = x_0$

$\Rightarrow (f * g) * k \simeq_{x_0} f * (g * k)$

$\Rightarrow ([f] \circ [g]) \circ [k] = [f] \circ ([g] \circ [k])$.

(iii) Let $e_{x_0} : I \to \{x_0\}$ be a constant map. To show that $[e_{x_0}]$ is the identity element of $\pi_1(X, x_0)$.

Let $[f] \in \pi_1(X, x_0)$. Then $f(1) = x_0$.

Define $H : I \times I \to X$ by

$$H(s,t) = \begin{cases} f\left(\frac{2s}{1+t}\right), & t \geq 2s-1 \\ x_0 & t \leq 2s-1 \end{cases}$$

Then H is continuous for when $t = 2s-1$

$H(s, 2s-1) = f(1) = x_0$

$H(s, 2s-1) = x_0$

Also $H(s,0) = \begin{cases} f(2s) & s \in [0, \frac{1}{2}] \\ x_0 & s \in [\frac{1}{2}, 1] \end{cases}$

But $e_{x_0}(2s-1) = x_0$ by def of e_{x_0}

$$H(s,0) = \begin{cases} f(2s) & s \in [0, \frac{1}{2}] \\ e_{x_0}(2s-1) & s \in [\frac{1}{2}, 1] \end{cases}$$

$\therefore \quad H(s,0) = (f * e_{x_0})(s)$

~~Similarly~~ Again $H(s,1) = f(s), \quad 0 \le s \le 1$

Also $\quad H(0,t) = f(0) = x_0, \quad H(1,t) = x_0$

So that $\quad f * e_{x_0} \simeq_{x_0} f$

or $\quad [f] \circ [e_{x_0}] = [f]$

Similarly $\quad [e_{x_0}] \circ [f] = [f]$.

$\Rightarrow \quad e_{x_0}$ is the identity element of $\pi_1(X, x_0)$

(IV) Let $[f] \in \pi_1(X, x_0)$. Define a map $\bar{f}$ by $\bar{f}(s) = f(1-s)$

Then $\quad \bar{f}(0) = f(1) = x_0$ and $\bar{f}(1) = f(0) = x_0$.

Hence $\quad \bar{f} \in C(X, x_0)$ the set of closed paths.

Now $$(f * \bar{f})(s) = \begin{cases} f(2s) & s \in [0, \frac{1}{2}] \\ \bar{f}(2s-1) & s \in [\frac{1}{2}, 1] \end{cases}$$

$$= \begin{cases} f(2s) & s \in [0, \frac{1}{2}] \\ f(2-2s) & s \in [\frac{1}{2}, 1]. \end{cases}$$

Now define a map $H : I \times I \longrightarrow X$ by

$$H(s,t) = \begin{cases} x_0 & 0 \le s \le \frac{t}{2} \\ f(2s-t) & \frac{t}{2} \le s \le \frac{1}{2} \\ \bar{f}(2s+t-1) & \frac{1}{2} \le s \le 1-\frac{t}{2} \\ x_0 & 1-\frac{t}{2} \le s \le 1. \end{cases}$$

$\forall \; s,t \in I$. Then by continuity of H follows from the continuity of f and $\bar{f}$.

$\therefore \quad H(s,0) = \begin{cases} f(2s) & 0 \le s \le \frac{1}{2} \\ \bar{f}(2s-1) & \frac{1}{2} \le s \le 1 \end{cases}$

and $\quad H(s,1) = x_0 \quad$ if $0 \le s \le 1$.

Therefor $\quad H(s,0) = (f * \bar{f})(s)$

$H(s,1) = e_{x_0}(s) = x_0$.

Also $\quad H(0,t) = x_0 = H(1,t)$

$\Rightarrow \quad f * \bar{f} \simeq_{x_0} e_{x_0}$

Similarly $\quad \bar{f} * f \simeq_{x_0} e_{x_0}$

ie $\bar{f}$ is the inverse of f

Hence $\Pi_1(X, x_0)$ is a group.

$\Pi_1(X, x_0), \circ\}$ is called first homotopy group or Poincare gr.

examples. (1) If R^n is euclidean n-space. Then $\Pi_1(R^n, x_0)$ is a trivial group (the group consisting the identity alone)

For, if f is a loop in R^n based at x_0 the straight line homotopy is a path homotopy between f and the constant path x_0.

If X is any convex set of R^n then $\Pi_1(X, x_0)$ is a trivial group.

If B^n is a unit ball in R^n where

$$B^n = \{x : x_1^2 + x_2^2 + \dots + x_n^2 \le 1\}$$

has a fundamental trivial group.

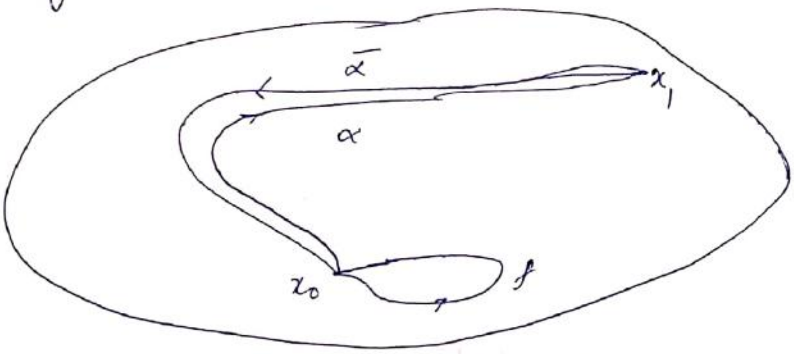
Def. Let α be a path in X from x_0 to x_1. Define a map

$$\hat{\alpha} : \pi_1(X, x_0) \longrightarrow \pi_1(X, x_1)$$

by the equation

$$\hat{\alpha}([f]) = [\bar{\alpha}] * [f] * [\alpha].$$

The map $\hat{\alpha}$ is well defined because '$*$' is well defined. If f is loop based at x_0 then $\bar{\alpha} * (f * \alpha)$ is a loop based at x_1. Hence $\hat{\alpha}$ maps $\pi_1(X, x_0)$ into $\pi_1(X, x_1)$. $\hat{\alpha}$ only depends on the path homotopy class of α.



Theorem $\hat{\alpha}$ is a group isomorphism.

Proof To show that $\hat{\alpha}$ is a homomorphism, to compute

$$\hat{\alpha}([f]) * \hat{\alpha}([g]) = \left([\bar{\alpha}] * [f] * [\alpha]\right)\left([\bar{\alpha}] * [g] * [\alpha]\right)$$
$$= [\bar{\alpha}] * [f] * [e_{x_0}] * [g] * [\alpha]$$
$$= [\bar{\alpha}] * [f] * [g] * [\alpha]$$
$$= [\bar{\alpha}] * [f * g] * [\alpha]$$
$$= \hat{\alpha}([f * g]) = \hat{\alpha}([f] * [g]).$$

To show that $\hat{\alpha}$ is an isomorphism, we want to show that if β denotes the path $\bar{\alpha}$, which is the reverse of α then $\hat{\beta}$ is an inverse for $\hat{\alpha}$. So to compute $\forall\ [h] \in \pi_1(X, x_0)$

$$\hat{\beta}[h] = [\bar{\beta}] * [h] * [\beta] = [\alpha] * [h] * [\bar{\alpha}]$$
$$\hat{\alpha}(\hat{\beta}([h])) = \bar{\alpha} * [\alpha] * [h] * [\bar{\alpha}] * [\alpha] = [h].$$

Similarly to show that

$$\hat{\beta}\left(\hat{\alpha}([f])\right) = [\bar{\beta}] * [\bar{\alpha}] * [f] * [\alpha] * [\beta]$$
$$= [\alpha] * [\bar{\alpha}] * [f] * [\alpha] * [\bar{\alpha}] = [f].$$

$\forall \; [f] \in \pi_1(X, x_0)$.

Hence $\hat{\beta}$ is the inverse of $\hat{\alpha}$

$\Rightarrow \hat{\alpha}$ is a group isomorphism.

Path connected - Let X be a path top space. $\forall \; x, y \in X$ in X is said to be path connected if there is a path from x to y

ex 1. Let $X = \{0, 1\}$ and $\tau = \{X, \phi, \{0\}\}$ then X is path connected if $f : I \to X$ is defined by

$$f(t) = \begin{cases} 0 & \text{if } t \in [0, 1) \\ 1 & \text{if } t = 1 \end{cases}$$

2. Indiscrete space is path connected.

3. Every path connected space is connected.

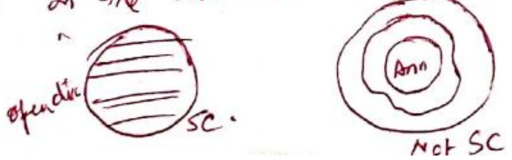
Corollary. If X is path connected and x_0 and x_1 are two points of X then $\pi_1(X, x_0)$ is isomorphic to $\pi_1(X, x_1)$.

Definition - A space X is said to be simply connected if it is a path connected space and if $\pi_1(X, x_0)$ is the trivial one element group for some $x_0 \in X$, and hence for every $x_0 \in X$.

$\pi_1(X, x_0)$ is the trivial group by writing $\pi_1(X, x_0) = 0\,(e_{x_0})$

Def - A topological space X is said to be simply connected if every closed path in X is contractible to a point.

ex In the Euclidean Plane, R^2, an open disc is simply conn. but annulus is not since there are closed curves which are not contractible to a point.



Lemma. In a simply connected space X, any two paths having same initial and final points are path homotopic.

Proof - Let α, β be two paths from x_0 to x_1 in X. Then $\alpha * \bar{\beta}$ is defined and a loop in X based at x_0.

Since X is simply connected, this loop is path homotopic to constant loop at x_0.

$$\text{i.e} \quad [\alpha * \bar{\beta}] = [e_{x_0}]$$

$$[\alpha * \bar{\beta}] * [\beta] = [e_{x_0}] * [\beta]$$

$$\Rightarrow \quad [\alpha] * [\bar{\beta}] * [\beta] = e_{x_0} * [\beta]$$

$$[\alpha] = [\beta] \quad \Rightarrow \quad \alpha \simeq \beta.$$

Def Let $h : (X, x_0) \longrightarrow (Y, y_0)$ be a continuous map

Define $h_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0)$

by the equation $h_*([f]) \Rightarrow [h \circ f]$.

The map h_* is well defined, for if F is path homotopy between f & f' then $h \circ F$ is a path homotopy between the paths $h \circ f$ & $h \circ f'$.

$$F(s,0) = f(s), \quad F(s,1) = f'(s)$$

$$F(0,t) = x_0 \quad F(1,t) = x_1$$

and $(h \circ F)(s,0) = (h \circ f)(s)$
$(h \circ F)(s,1) = (h \circ f')(s)$
$(h \circ F)(0,t) = x_0$
$(h \circ F)(1,t) = x_1$

where f and f' are paths in X from x_0 to x_1.

$\therefore$ h_* is a homomorphism follows from the equation

$$(h \circ f) * (h \circ g) = h \circ (f * g).$$

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h_* depends not only on the map $h: X \to Y$ but also on the choice of the base point x_0. (Once x_0 is chosen, y_0 is determined by h).

If x_0 and x_1 are two different base points of X then it is not possible to use h_* for two different homomorphisms; since one having the domain $\pi_1(X, x_0)$ and other having domain $\pi_1(Y, x_1)$

If X is path connected then groups are isomorphic they are still not same group. In that case,

$$(h_{x_0})_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0)$$

for first homomorphism

and $(h_{x_1})_*$ for the second homomorphism

If there is only one ~~homomorphism~~ base point then omit the base point and using h_*.

Properties of Induced homomorphisms.

are called Functional properties.

<u>Theorem</u>. If $h : (X, x_0) \longrightarrow (Y, y_0)$ and $k : (Y, y_0) \longrightarrow (Z, z_0)$.

are continuous maps. then

$$(k \circ h)_* = k_* \circ h_*.$$

If $i : (X, x_0) \rightarrow (X, x_0)$ is a identity map then i_* is the identity homomorphism.

Proof.

We know that

$$I \times I \xrightarrow[\text{Cont}]{f} X \xrightarrow[\text{Cont}]{h} Y \quad \text{then} \quad I \times I \xrightarrow[\text{Cont.}]{h \circ f} Y$$

then $(h \circ f)(0) = h(f(0)) = h(x_0) = y_0$

$(h \circ f)(1) = h(f(1)) = h(x_0) = y_0$

ie $h \circ f$ forms a loop based at y_0

Similarly

$I \times I \xrightarrow[\text{Con}]{g} Y \xrightarrow[\text{Cont}]{k} Z$ then $I \times I \xrightarrow[\text{Cont}]{k \circ g} Z$

$(k \circ g)(0) = k(g(0)) = k(y_0) = z_0$

$(k \circ g)(1) = k(g(1)) = k(y_0) = z_0$

Now $(k \circ h)(x_0) = (k \circ h)(x_0) = k(h(x_0)) = k(y_0) = z_0$

Let $[f] \in \pi_1(X, x_0)$

Then $(k \circ h)_* [f] = [(k \circ h) \circ f] = [k \circ (h \circ f)]$

$$(k_* \circ h_*)[f] = [k_*(h_*([f]))]$$
$$= k_*([h \circ f])$$
$$= [k \circ (h \circ f)]$$

Hence $(k \circ h)_* [f] = (k_* \circ h_*)[f]$.

(ii) If i_* is the identity homomorphism

then $i_*[f] = [i \circ f] = [f]$

Therefore i_* is induced homomorphism

Corollary- If $h : (X, x_0) \to (Y, y_0)$ is a homeomorphism of X and Y then h_* is an isomorphism of $\pi_1(X, x_0)$ with $\pi_1(Y, y_0)$

Proof - Let $k : (Y, y_0) \to (X, x_0)$ be the inverse of h.

Then $(k_* \circ h_*)[f] = (k \circ h)_* [f] = i_* [f]$ $\forall [f] \in \pi_{(X,x_0)}$

where i is the identity map. ie $i : (X, x_0) \to (X, x_0)$.

Similarly if $j : (Y, y_0) \to (Y, y_0)$ is an identity map

and $(h_* \circ k_*) [g] = (h \circ k) [g] = j_* [g]$

ie j_* is also an identity homomorphism.

Since i_* and j_* are the identity homomorphisms of the groups $\pi_1 (X, x_0)$ and $\pi_1 (Y, y_0)$ respectively

ie k_* is the inverse of h_*.

$\Rightarrow \pi_1 (X, x_0)$ and $\pi_1 (Y, y_0)$ are isomorphic under h_*.

Other Statement of the above Cono -.

Two topologically equivalent spaces have isomorphic fundamental groups.

Def. A topological space is said to be contractible if $\exists$ $p \in X$ such that the identity map $i : X \to X$ is homotopic to the constant map. $c : X \to [p]$. ie $i \simeq c$.

Def. A subset A of a top space X is called the retract of X iff $\exists$ a continuous function $f : X \to A$ such that $f(x) = x \quad \forall \; x \in A$.

Ex R is contractible

$\forall \; x \in R$ define $h : R \times I \to R$ by

$$h(x, t) = (1-t) x$$

Then h is a homotopy

$$\because \begin{array}{l} h(x, 0) = x = i(x) \\ h(x, 1) = 0 = c(x) \end{array}$$

$\Rightarrow \quad i \simeq c \quad \Rightarrow$ contractible.