Embedding and Metrization

A topological space X is said to be embedded in a top space Y if X is homeomorphic to the subspace of Y.

Def. — A cartesian product of closed unit intervals [0, 1] with product topology is called a cube.

A cube is then the set Q^A of all functions on a set A to the closed unit interval Q, with the topology of point wise, or coordinate wise, convergence.

<u>Evaluation Map</u> - Let F be a family of functions such that each member f of F is on a top space X to a space Y_f. $f : X \to Y_f$

A map e is called evaluation map if the map

$$e : X \longrightarrow \Pi \{ Y_f : f \in F \}$$

is defined by point $x \in X$ into the member of product whose f^{th} coord is $f(x)$

$$e(x)_f = f(x)$$

where $f(x)$ is the f^{th} coordinate of $\Pi \{ Y_f : f \in F \}$, $x \in X$.

The range be different for different members of the family.

So that e is continuous if the members of F are continuous and e is homeomorphism if F contains enough function.

<u>Def</u> - A family F of functions on X distinguishes points iff for each pair of distinct points x and y there is f in F such that $f(x) \neq f(y)$

ie if $x \neq y \in X$ and $f \in F$ then $f(x) \neq f(y)$

<u>Def</u>- The family F distinguishes points and closed sets iff for each closed ~~set~~ subset A of X and each member x of $X \sim A$ there is f in F such that $f(x) \notin \overline{f[A]}$.

ie $A \subset X$ and $x \in X \sim A \Rightarrow f(x) \notin \overline{f[A]} \quad \forall f \in F$.

Embedding Lemma - Let F be the family of continuous functions $\forall f \in F$, being on a topological space X to a top space Y_f.

Then :

(a) The evaluation map e is continuous function on X to $\Pi\{Y_f : f \in F\}$

(b) The function e is an open map of X onto $e[X]$ if F distinguishes points and closed sets.

(c) The function e is one to one iff F distinguishes points.

Proof- Let P_f be the projection function on Y to Y_f where

$Y = \Pi\{Y_f : f \in F\}$ be a product space. and the

evaluation map $e : X \longrightarrow \Pi(Y_f : f \in F)$

Then $P_f \circ e : X \longrightarrow Y_f \quad \forall f \in F, \quad x \in X$

is defined by $\quad P_f \circ e(x) = P_f(e(x)) = e(x)_f = f(x)$

ie

$$\begin{tikzcd} X \arrow[r, "e"] \arrow[rr, bend right, "P_f o e"'] & Y \arrow[r, "P_f"] & Y_f \end{tikzcd}$$

$$\begin{tikzcd} X \arrow[r, "e"] \arrow[dr] & Y \arrow[d, "P_f"] \\ & Y_f \end{tikzcd}$$

$\Rightarrow \quad P_f \circ e \equiv f.$

Now using the result, "[A function f on a topological space to a product space is continuous iff $P_a \circ f$ is continuous for each projection P_a.]"

$\because$ F is family of continuous functions

$\therefore$ f is continous.

$\Rightarrow$ $P_f \circ e$ is continuous

$\Rightarrow$ e is continuous.

Let $e : X \to Y$ be evaluation map. let G be open in X ~~that~~ $e(x) \in e(G)$ so that $x \in G$. Then $x \notin X \sim G$ and $X \sim G$ is a closed set in X. Since F separates pts & cl. sets then there is some function $f \in F$ s.t $f(x) \notin \overline{f(X \sim G)}$, whence $f(x) \in Y_f - \overline{f(X \sim G)}$, $Y = \pi\{Y_f\}$

Now $V = Y_f - \overline{f(X \sim G)}$ is open in Y_f and $f(x) \in V$.

Also $f = P_f \circ e \Rightarrow f(x) = P_f(e(x))$

Since $f(x) \in V \Rightarrow P_f(e(x)) \in V \Rightarrow e(x) \in P_f^{-1}(V)$

$\Rightarrow P_f^{-1}(V)$ is open in Y $\because$ V is open in Y_f and P_f is continuous

Put $W = P_f^{-1}(V) \cap e(X)$.

Then W is open in $e[X]$

Also $e(x) \in P_f^{-1}(V)$ and $e(x) \in e(X) \Rightarrow e(x) \in W$. —— ①

Now $W \subseteq e(G)$ —— ②

For, $e(t) \in W \Rightarrow e(t) \in P_f^{-1}(V) \Rightarrow P_f(e(t)) \in V$

$\Rightarrow P_f(e(t)) \in P_f(P_f^{-1})(V)$

$\Rightarrow (P_f \circ e)(t) \in V \Rightarrow f(t) \in V$

$\Rightarrow f(t) \notin \overline{f(X \sim G)}$

$\Rightarrow t \in G$ since F separates points and closed sets in X

$\Rightarrow e(t) \in e(G)$,

Hence from ① and ② $e(x) \in W \subseteq e(G)$, where V is open in $e(X)$. Therefore $e(G)$ is open in W, whence e is open map

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It is sufficient to show that the image under e of an open nbd U of a point x contains intersection of $e[X]$ and the nbd of $e(x)$ in the product.

$\therefore$ $e[G]$ is an open subset of $e[X]$. Hence e is open map of X onto $e[X]$.

(iii) Let e be one - one and x and y be two distinct points of X. Then $x \neq y \Rightarrow e(x) \neq e(y)$

$\Rightarrow$ ~~$e(x)(x) \neq f(y)$~~

$\Rightarrow \exists\, f \in F, \quad (e(x))_f \neq (e(y))_f$

$$\Rightarrow f(x) \neq f(y)$$

Hence F distinguishes points.

Conversely, let F distinguishes points and x and y are two distinct points of X such that $x \neq y$

$\therefore \quad f(x) \neq f(y)$

$\Rightarrow \quad (e(x))_f \neq (e(y))_f$

$\Rightarrow \quad e(x) \neq e(y)$

$\Rightarrow \quad e$ is one - one.

We can summarise the whole result as

Let F be a family of continuous functions. A map $f \in F$ from X to Y_f is continuous. If F distinguishes points and closed sets then evaluation map $e : X \rightarrow Y = \Pi\{Y_f : f \in F\}$ is a homeomorphism of X onto the subspace $e[X]$ of Y.

ie the evaluation map e is an embedding.

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Theorem :- A Tychonoff cube is Tychonoff space.

Proof. We know that the unit interval $I = [0,1]$ is a metric space with defined (metric) by $d(x,y) = |x-y|$ for all $x, y \in I$. Since every metric space is normal

Also I is $T_2 \Rightarrow I$ is $T_1 \Rightarrow I$ is T_1 + normal $\Rightarrow T_4$

$\because$ every T_4 is CR. $\Rightarrow I$ is $CR + T_1 \Rightarrow$ Tychonoff sp.

$\Rightarrow I^F$ is also Tychonoff space.

where $I^F = I \times I \times I \cdots$ into F times, I^F is called Tychonoff cube.
the number of factors equals to the cardinality of F

Embedding Theorem —

A topological space is a Tychonoff space iff X is embedded in a Tychonoff cube.

Proof- Suppose that X is a Tychonoff space. Let F be the family of continuous functions from X to I. Let $e : X \to e(X)$ be the evaluation map. Subspace of Y

where $e[X] \subseteq I^F$ be the evaluation map.

Let $x \in X$ and F be a closed set in X with $x \notin F$.

Since X is CR then $\exists$ a continuous function $f : X \to I$

Such that $f(x) = 0$ and $f(F) = \{1\}$.

$\Rightarrow f(x) \notin \overline{f(F)}$ i.e F distinguishes points and closed sets in X. By Embedding lemma e is an open map.

Let $x \neq y$ in X. Since X is T_1 the singleton set $\{y\}$ is closed and $x \notin \{y\}$.

But X is CR so that $\exists$ a continuous function $g : X \to I$

Such that $g(x) = 0$ and $g[\{y\}] = \{1\}$.

i.e $g(x) = 0$ and $g(y) = \{1\}$.

$\Rightarrow g(x) \neq g(y) \Rightarrow F$ ~~sepe~~ distinguishes points in X

$\Rightarrow e$ is one - one.

Also e is continuous since $f \in F$ is continuous by embedding lemma.

Therefore e is a homeomorphism of X onto $e[X]$ which is a subspace of a Tychonoff cube I^F

$\Rightarrow$ X is embedded into a Tychonoff cube.

Conversely, suppose X is embedded in a Tychonoff cube I^F. Then there is a homeomorphism h of X onto the subspace $h[X]$ of I^F.

But I^F is a Tychonoff space. Also every subspace of Tychonoff space is Tychonoff space.

$\Rightarrow$ $h[X]$ is Tychonoff space

we know that product of Tychonoff space is Tychonoff $\Rightarrow$ X is Tychonoff.

Theorem - Product of Tychonoff space is Tychonoff.

Proof - For convenience, let us agree to say that, a continuous function f on a top X to a $[0,1]$ is a pair (x, U) iff x is a point and U is a nbd of x such that

$$f(x) = 0 \quad \text{and} \quad f(X \sim U) = 1.$$

If $f_1, f_2, \ldots, f_n$ are the functions for the pair $(x_1, U_1), (x_2, U_2), \ldots, (x_n, U_n)$, where n is a positive integer, and if

$$g(x) = \sup\{f_i(x) : i = 1, 2, \ldots, n\}$$

then g is a function for $(x, \cap\{U_i : i = 1, 2, \ldots n\})$ as $g(x) = 0$ and $g(X \sim \{\cap U_i : i = 1, 2, \ldots, n\}) = 1$

$\Rightarrow g$ is a continuous function for $(x_i, \{\cap U_i : i = 1, 2, \ldots n\})$.

ie the space is completely regular if $\forall x$ and for each nbd U of x belonging to some ~~some~~ sub base for the topology, there is a function for (x, U).

Let $\{X_a : a \in A\}$ be coordinate spaces of $T_{3\frac{1}{2}}^{CR}$ - spaces and $X = \prod\{X_a : a \in A\}$ be a product space. Let $x \in X$ and U_a be any nbd of $x_a \in X_a$. If f is a function ~~from~~ for (x_a, U_a), then $f \circ P_a$ is a function for $(x, P_a^{-1}[U_a])$ where P_a is the projection into a^{th} coordinate space.

The family of sets of the form $P_a^{-1}[U_a]$ is a subbase for the product topology. Hence X is CR. Since Product of T_1 space is T_1 space. $\therefore$ Product of Tychonoff space is Tychonoff.

Theorem - A Tychonoff space can be embedded into a Cube.

Proof - we shall show that each Tychonoff space is homeomorphic to a subspace of a Cube.

Let X be a Tychonoff space. Consider F to be a family of all continuous functions defined on X to $[0,1]$.

Then by Embedding lemma, we shall prove that The evaluation map e of X into a cube $[0,1]^F$ is homeomorphic to a subspace of a cube Q^F

By Embedding lemma; e is continuous. To show that e is one-one.

Let $x, y \in X$ Such that $x \neq y$.

Consider x and $X \sim \{y\}$, is a nbd of x.

$\Rightarrow \exists$ $f \in F$ such that $f(x) = 0$ and $f(y) = 1$

$\Rightarrow$ $f(x) \neq f(y)$.

ie F distinguish points $\Rightarrow$ e is one-one

Let $x \in X$ and A be any closed set Such that $x \notin A$.

$\Rightarrow$ $X \sim A$ is a nbd of x

$\Rightarrow$ $\exists$ $f \in F$ such that $f(x) = 0$ and $f(A) = 1$

or $f(\overline{A}) = \overline{\{1\}} = \{1\}$.

$\Rightarrow$ $f(x) \notin f(\overline{A})$

$\Rightarrow$ F distinguishes points and closed sets.

Hence E is an open map of X onto $e(X) \subseteq Q^F$

Hence X is homeomorphic to a subspace $e(X)$ of Q^F.

Metrizable

A topological space is said to be metrizable if $\exists$ a metric d for X such that metric topology $\mathcal{T}_d$ is identical with the topology $\mathcal{T}$. The metric d is called an admissible metric for $(X, \mathcal{T})$. A space which is not metrizable is called a non metrizable space.

Some metrizable spaces.

(1) Every discrete space X is metrizable since the trivial metric d on X defined by $d(x,y) = \begin{cases} 1, & x \neq y \\ 0 & x = y. \end{cases}$

induces the descrete top on X.

(2) The real line R with usual topology is metrizable since the usual metric on R induces the usual top on R

(3) Consider the Sierpinski space $X = \{0,1\}$ with topology $\mathcal{T} = \{\phi, X, \{0\}\}$.

The space is non-metrizable space because $\mathcal{T}_d \neq \mathcal{T}$.

Properties -

(i) Metrizability is hereditary

(ii) Metrizability is a topological invariant.

Metrization Problem -

In topology the problem of finding necessary and sufficient conditions (topological) for a topological space to be metrizable is known as a metrization problem.

Urysohn's Metrization Theorem –

Let X be a second countable T_3 - space. Then X can be embedded in a Tychonoff cube and hence X is metrizable.

If X is C_{II} normal space then $\exists$ a homeomorphism f of X onto a subspace of R^∞, and X is therefore metrizable.

Proof – Since X is C_{II} $\Rightarrow$ Lindelöff. If X is T_3 $\Rightarrow T_1$ + regular
$\Rightarrow$ X is regular Lindelöff space $\Rightarrow$ X is normal.

Again, that X is C_{II} $\Rightarrow$ there is a countable base $\mathcal{B}$ for its topology. Put

$$\alpha = \{ (U, V) : \bar{U} \subseteq V, \quad U, V \in \mathcal{B} \}.$$

Then α is countable. Also $\bar{U} \subseteq V \Rightarrow \bar{U} \cap (X \setminus V) = \phi$

Since V is open, then $X \setminus V$ is closed.

Thus $\bar{U}$ and $X \setminus V$ are disjoint closed subsets of a normal space X

Applying Urysohn Lemma,

$\exists$ a continuous function $f : X \to [0, 1]$ such that

$$f(\bar{U}) = \{0\} \text{ and } f(X \setminus V) = \{1\}.$$

Denote F the family of such Urysohn's functions as (U, V) varies over α.

Since α is countable F is also countable. Put $I = [0, 1]$ then we get a Tychonoff cube I^F which is metrizable

Using the theorem –

Let (X_n, d_n), $(n = 1, 2, \cdots)$ be a countable number of metric spaces each of diameter at most 1. Let $X = \prod_{n=1}^{\infty} X_n$. For each $x = \{x_n\}$, $y = \{y_n\}$

in X, let $d(x,y) = \sum_{n=1}^{\infty} \frac{d_n(x_n, y_n)}{2^n}$.

Then (X, d) is a metric space.

Also the metric topology $\mathcal{T}_d$ is identical with the product topology $\mathcal{T}$ on X.

Let $x \in X$ and A be closed set in X s.t. $x \notin A$. Then $X \setminus A$ is open nbd of x. So that $\exists$ some $v \in \mathcal{B}$ s.t. (ii)

$x \in v \subseteq X \setminus A$ whence $A \subseteq X \setminus v$.

But X is regular. Hence we can choose a basic open set u such that $x \in u$ and $\bar{u} \subseteq v$, $(u, v) \in \alpha$.

Then $\exists$ Urysohn's function $f \in F$ s.t. $f(\bar{u}) = \{0\}$ and $f(X \setminus v) = \{1\}$.

But $x \in u \Rightarrow f(x) = 0$.

Also $A \subseteq X \setminus v \Rightarrow f(A) = \{1\}$.

Hence $\overline{f(A)} = \overline{\{1\}} \Rightarrow f(x) \notin \overline{f(A)}$.

ie F separates points and closed sets in X.

Next let $x \neq y$ in X. Let $A = \{y\}$. Since X is T_1 - space $\Rightarrow A$ is closed in X also $x \notin A$. Since $x \neq y$. Hence $\exists$ a Urysohn function f such that

$f(x) = \{0\}$ and $f(y) = f(A) = \{1\}$.

$\Rightarrow f(x) \neq f(y)$. $\Rightarrow F$ separates points ~~and~~ in X.

Again let $e : X \to e[X] \subseteq I^F$ be the evaluation map.

Then by applying embedding lemma, e is a homeomorphism of X onto $e[X]$. Metrizability is hereditary property

Therefore metrizability of $I^F \Rightarrow$ metrizability of its subspace $e[X]$