Tychonoff Product Topology

Def. Let Λ be any index set. Let $\{X_\alpha : \alpha \in \Lambda\}$ be a non empty collection of the sets X_α
Consider the set X of all functions

$$x : \Lambda \longrightarrow \cup\{X_\alpha : \alpha \Lambda\}$$

such that $x(\alpha) \in X_\alpha \ \forall \ \alpha \in \Lambda$.
Then X is called the ~~indexed set of~~ Cartesian product of indexed collection of $\{X_\alpha : \alpha \in \Lambda\}$.

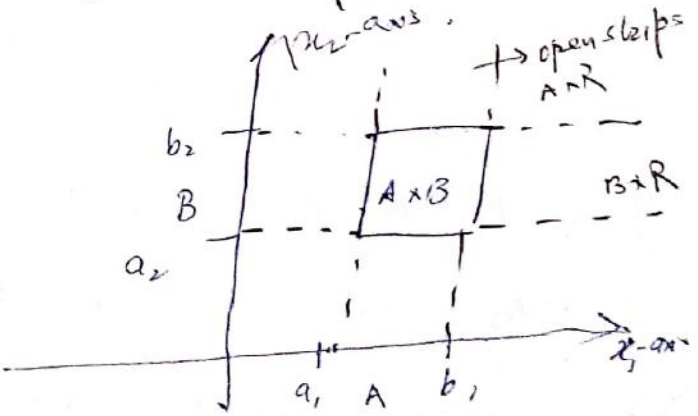
i.e $\quad X = \Pi\{X_\alpha : \alpha \in \Lambda\}$ or ΠX_α

The set X_α is ~~called~~ the α^{th}- coordinate of the product. If $x \in X$ then x is called the point in the product set X $\forall \ \alpha \in \Lambda$ $x(\alpha)$ is called the α^{th}- coordinate of x i.e $x = x_\alpha = x(\alpha)$,

The product set X is empty iff $X_\alpha = \phi$ for some $\alpha \in \Lambda$.

The map $P_\alpha : X \longrightarrow X_\alpha$ defined by $P_\alpha(x) = x_\alpha \ \forall \ x \in X$.
is called the α^{th}- projection map.

If each coordinate set in the product space X is a topological space, then there is a standard way of defining a topology on X



Let $R^2 = R \times R$ is an Euclidean Plane
Let $A = (a_1, b_1)$, $B = (a_2, b_2)$ be bounded open intervals on the axis x_1 & x_2 respectively. Then

$$A \times B = \{(x_1, x_2) : x_1 \in A, \ x_2 \in B\}$$

$$= \{(x_1, x_2) : a_1 < x_1 < b_1, \quad a_2 < x_2 < b_2$$

Now $A \times B$ is called an open rectangle in R^2.

Sets of the form

$$A \times R = \{ (x_1, x_2) : a_1 < x_1 < b_1 , x_2 \in R \text{ arbitrary} \}$$

$$R \times B = \{ (x_1, x_2) : x_1 \in R , a_2 < x_2 < b_2 , \}.$$

are called open strips in R^2.

Since $A \times B = (A \times R) \cap (R \times B)$

every open rectangle in R^2 is the intersection of two open strips.

Now the collection of $\mathcal{L}$ of all open strips in R^2 forms a subbase for a top $\mathfrak{T}$ on R^2. and the collection $\mathcal{B}$ of all open rectangles in R^2 forms a base for this topology on R^2. The open sets in R^2 are the union of these open rectangles.

Now let us take two topological spaces (X_1, τ_1) & (X_2, τ_2)

Let $X = X_1 \times X_2$ be the product of X_1 & X_2

& $\mathcal{L}$ be the collection of all subsets of X of the form $G_1 \times X_2$ and $X_1 \times G_2$ where $G_1 \in \tau_1$, $G_2 \in \tau_2$.

Then topology on X generated by $\mathcal{L}$ is called the product topology $\mathcal{I}$ on X. and for this topology $\mathcal{I}$, $\mathcal{L}$ is a subbase. The base generated by $\mathcal{B}$ is the family of all sets of the form $G_1 \times G_2$; ie open sets in X are union of these sets.

Theorem. Let (X_1, τ_1) & (X_2, τ_2) be two top space. Then the collection

$$\mathcal{B} = \{ G_1 \times G_2 : G_1 \in \tau_1, G_2 \in \tau_2 \}$$

is a base for some topology for $X_1 \times X_2$.

Proof. (i) $X_1 \in \tau_1$, $X_2 \in \tau_2 \Rightarrow X_1 \times X_2 \in \mathcal{B}$

$\therefore X_1 \times X_2$ is the union of the members of $\mathcal{B}$

(ii) Let $G_1 \times G_2$ and $H_1 \times H_2$ be two member of $\mathcal{B}$. then

$$(G_1 \times G_2) \cap (H_1 \times H_2) = (G_1 \cap H_1) \times (G_2 \cap H_2) \in \mathcal{B}$$

$\because G_1 \cap H_1 \in \tau_1$ and $G_2 \cap H_2 \in \tau_2$

$\Rightarrow \mathcal{B}$ is a base for the topology on $X_1 \times X_2$

Projection Mapping

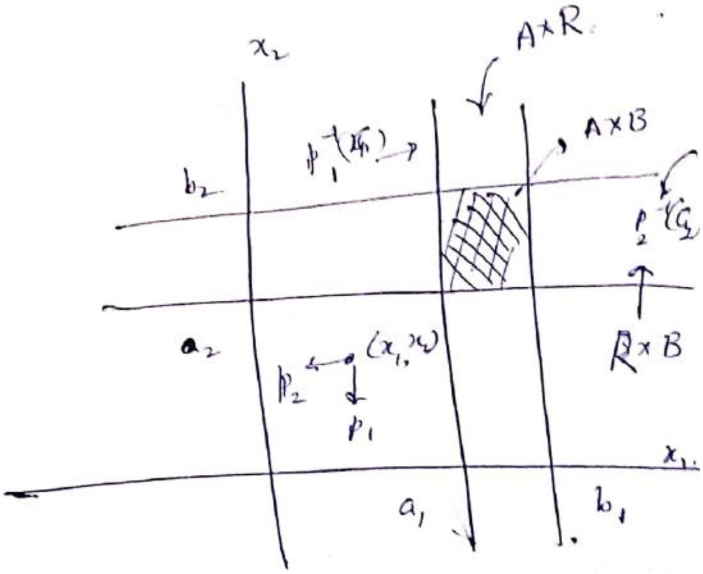
Define a mapping.

$P_1 : X \to X_1$ where $X = X_1 \times X_2$

by $p_1(x_1, x_2) = x_1$

and $P_2 : X \to X_2$ by

$p_2(x_1, x_2) = x_2$



These two mappings are called projection map.

These two mappings are continuous.

To show that Projection mapping are

For $G_1 \in \tau_1$ we have

$$p_1^{-1}(G_1) = \{ (x_1, x_2) \in X : p_1(x_1, x_2) \in G_1 \}.$$

$$= \{ (x_1, x_2) \in X : x_1 \in G_1 \}$$

$= G_1 \times X_2$ is $\mathcal{T}$-open in X_1. This's show that inverse image of τ_1 open subset of X_1 is a $\mathcal{T}$-open subset of X.

Similarly $p_2^{-1}(G_2) = X_1 \times G_2$, $G_2 \in \tau_2$.

$\Rightarrow p_1$ & p_2 are Continuous maps.

i.e. The product topology $\mathfrak{I}$ on X is a topology on the product space X with respect to which two projections are continuous.

Theorem

To show that $\mathfrak{I}$ is a weakest topology of on X.

Proof — Let $\mathfrak{I}^*$ be the other top for X for which p_1 and p_2 are continuous. To show that every $\mathfrak{I}$-open set is $\mathfrak{I}^*$-open.

Let A by any arbitrary $\mathfrak{I}$-open set so that A can be expressible as the union of some members of $\mathcal{B}$ for $\mathfrak{I}$.

i.e. $A = \cup \{ G_1 \times G_2 : G_1 \in \tau_1 , G_2 \in \tau_2 \}$

$= \cup \{ (G_1 \cap X_1) \times (G_2 \cap X_2) : G_1 \in \tau_1 , G_2 \in \tau_2 \}$

$= \cup \{ (G_1 \times X_2) \cap (X_1 \times G_2) : G_1 \in \tau_1 , G_2 \in \tau_2 \}$

$= \cup \{ p_1^{-1}(G_1) \cap p_2^{-1}(G_2) \} \in \mathfrak{I}^*$

$\because$ p_1 & p_2 are continuous therefore $p_1^{-1}(G_1)$ and $p_2^{-1}(G_2)$ is open ($\mathfrak{I}^*$) and their union is therefore $\mathfrak{I}^*$-open.

Hence every $\mathfrak{I}$-open set is $\mathfrak{I}^*$-open.

Consequently $\mathfrak{I}$ is the smallest top for X

Theorem. Show that P_1 and P_2 are open maps.

Proof — Let A be any arbitrary subset of X. $(X = X_1 \times X_2)$

Then A can be expressible as the union of some members of $\mathcal{B}$, base for $\mathfrak{I}$.

i.e. $A = \cup \{ G_1 \times G_2 : G_1 \in \tau_1 , G_2 \in \tau_2 \}$

and $G_1 \times G_2 \in \mathcal{B}^* \subseteq \mathcal{B}$

$\therefore \quad P_1(A) = p_1[\cup (G_1 \times G_2) : G_1 \in \tau_1 \ \& \ G_2 \in \tau_2]$

$= \cup \{ p_1 [G_1 \times G_2) : G_1 \in \tau_1 , G_2 \in \tau_2 \}$

$= \cup \{ G_1 : G_1 \in \tau_1 \}$

which is τ_1 - open

Hence every image under p_1 of every $\mathcal{T}$- open subset of X is $\cancel{\tau_1}$- open sub set of X_1.

$\Rightarrow$ p_1 is open map.

Similarly we can show that p_2 is open map

<u>Theorem</u>. Let $X = X_1 \times X_2$ and $\mathcal{B}_1$ & $\mathcal{B}_2$ be the bases for the tops τ_1 & τ_2 respectively then

$\mathcal{B}^* = \{ B_1 \times B_2 : B_1 \in \mathcal{B}_1 , B_2 \in \mathcal{B}_2 \}$

is a base for the product topology $\mathcal{T}$ on X.

Proof let $\mathcal{B} = \{ G_1 \times G_2 : G_1 \in \tau_1 , G_2 \in \tau_2 \}$

be a base for the top $\mathcal{T}$ on X

But any $(x_1, x_2) \in G \in \mathcal{T} \Rightarrow \exists \ G_1 \times G_2 \in \mathcal{B}$ s.t.

$(x_1, x_2) \in G_1 \times G_2 \subseteq G$.

$\Rightarrow \quad x_1 \in G_1 \in \tau_1$ and $x_2 \in G_2 \in \tau_2$

Then by def of base

$\forall . \ B_1 \in \mathcal{B}_1$

$x_1 \in B_1 \subseteq G_1$

$\forall \ B_2 \in \mathcal{B}_2$

and $x_2 \in B_2 \subseteq G_2$

But $\mathcal{B}^* = B_1 \times B_2$ then

$(x_1, x_2) \in B_1 \times B_2 \subseteq G_1 \times G_2 \subseteq G$.

$\Rightarrow$ $\mathcal{B}$ is a base for top $\mathcal{T}$ on X.

ie $\mathcal{B}$ and $\mathcal{B}^*$ are bases for same top $\mathcal{T}$ on X.

ex let $\tau_1 = \{\varphi, \{p\}, X_1\}$, $X_1 = \{p, q, r\}$

$\tau_2 = \{\varphi, X_2, \{a\}, \{b\}, \{a, b\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}\}$

$X_2 = \{a, b, c, d\}$.

Find the base for topology $\mathcal{J}$.

Let $\mathcal{B}_1$ & $\mathcal{B}_2$ be the bases for τ_1 & τ_2 respectively

$\therefore \mathcal{B} = \{B_1 \times B_2 : B_1 \in \mathcal{B}_1 \ \& \ B_2 \in \mathcal{B}_2\}$.

let $\mathcal{B}_1 = \{\{p\}, X_1\}$, $\mathcal{B}_2 = \{\{a\} \{b\}, \{c, d\}\}$.

The collection of $\mathcal{B}$ are

$\{p\} \times \{a\}$, $\{p\} \times \{b\}$, $\{p\} \times \{c, d\}$, $X_1 \times \{a\}$, $X_1 \times \{b\}$, $X_1 \times \{c, d\}$

ie $\mathcal{B} = \{(p, a)$ (p, b) (p, c) (p, d)
$\{(p, a)$ (q, a) $(r, a)\}$; $\{(p, b), (q, b), (r, b)\}$ $\{(p, c)$ (q, c)
$(r, c)\}$ $(\{p, d\}, (q, d), (r, d)\}$.

Product Topology for Cartesian Product of an arbitrary family of Topological Spaces.

Suppose that $\{X_\alpha, \tau_\alpha : \alpha \in \Lambda\}$ is a family of topological spaces let $X = \prod\{X_\alpha : \alpha \in \Lambda\}$ be the product of

Let $p_\alpha : X \to X_\alpha$ be the α^{th} projection map defined by $p_\alpha(x) = x_\alpha$ $\forall x \in X$, $x_\alpha \in X_\alpha$

Consider the collection

$$\mathcal{J} = \{p_\alpha^{-1}(G_\alpha) : G_\alpha \in \tau_\alpha\}$$

Then $\mathcal{J}$ is a sub base for a topology $\mathcal{J}$ on X This topology is called the Tychonoff product Topology for X and the space $(X, \mathcal{J})$ is called product space.

Def. The class of subsets of a product space X of the form

$$p_\beta^{-1}(G_\beta) = \Pi\{X_\alpha : \alpha \neq \beta\} \times G_\beta$$

where G_β is an open subset of the coordinate space X_β is a sub base, and is called defining sub base for the product topology.

Def. The class of the subsets of the product space $X = \Pi\{X_\alpha : \alpha \in \Lambda\}$ of the form

$$p_{\beta_1}^{-1}(G_{\beta_1}) \cap \cdots \cap p_{\beta_m}^{-1}(G_{\beta_m}) = \Pi\{X_\alpha : \alpha \neq \beta_1 \ldots \beta_m\} \times G_{\beta_1} \times \cdots \times G_{\beta_m}$$

where G_{β_K} is an open subset of the coordinate space X_{β_K} is a base and is called the defining base for the product topology.

Ex → $\tau_1 = \{X, \phi, \{a\}, \{b, c\}\}$, $X_1 = \{a, b, c\}$

$\tau_2 = \{X_2, \phi, u\}$, $X_2 = \{u, v\}$

Find the defining subbase and defining base for the product topology.

Let $X = (X_1 \times X_2) = \{\{a, u\}, \{b, u\}, \{c, u\}, \{a, v\}, \{b, v\}, \{c, v\}\}$ be a product set on which the product topology $\mathcal{T}$ is defined.

Now defining subbase for $\mathcal{T}$ is the class of inverse sets $p_{x_1}^{-1}(G)$ and $p_{x_2}^{-1}(H)$, where G & H are open sets in X_1 & τ_2.

Now $P_1 \equiv p_{x_1}^{-1}[X_1] = X_1 \times X_2 = p_{x_2}^{-1}[X_2]$

$P_2 \equiv p_{x_1}^{-1}[\phi] = \phi = p_{x}^{-1}[\phi]$

$P_3 \equiv p_{x_1}^{-1}[\{a\}] = \{(a, u), (a, v)\}$

Base

$P_1 \cap P_2$, $P_1 \cap P_4$, $P_1 \cap P_5$

$P_3 \cap P_4$, $P_3 \cap P_5$, $P_4 \cap P_5$

$\therefore \mathcal{B} = \{P_1, P_2, P_3, P_4, P_5, \{a, u\}, \{(b, u), (c, u)\}\}$

$P_4 \equiv p_{x_1}^{-1}[\{b, c\}] = \{(b,u), (b,v), (c,u), (c,v)\}$.

$P_5 \equiv p_{x_2}^{-1}[\{u\}] = \{(a,u), (b,u), (c,u)\}$.

Hence $P_1, P_2, P_3, P_4, P_5, P_6$ forms a Subbase S. Base consist of finite intersections of members of subbase.

<u>Theorem</u> For each fixed $\alpha \in \Lambda$ the projection map P_α is continuous

Proof. We have $P_\alpha : X \to X_\alpha$

Let G_α be any open subset in X_α. Then by def of Product topology $P_\alpha^{-1}(G_\alpha)$ is a subbase member in X and hence open in X_α. Therefore P_α is Continuous.

Theorem - For each fixed $\alpha \in \Lambda$, the projection map P_α is open map

Proof - Let Q be the defining base for the product topology $\mathcal{T}$ on X. Let $B \in Q$ implies that $B = \bigcap_{i=1}^{n} S_i$ where each $S_i \in S$ the defining subbase for $\mathcal{T}$.

But $S_i = p_{\alpha_i}^{-1}(G_{\alpha_i})$, $i = 1, 2, \ldots n$, $G_{\alpha_i} \in \mathcal{T}_{\alpha_i}$.

Therefor $B = \bigcap p_{\alpha_i}^{-1}(G_{\alpha_i})$

$\therefore x \in B \iff x \in p_{\alpha_i}^{-1}(G_{\alpha_i}) \iff p_{\alpha_i}(x) \in G_{\alpha_i} \iff x_{\alpha_i} \in G_{\alpha_i}$

whence $B = G_{\alpha_1} \times G_{\alpha_2} \times \cdots G_{\alpha_n} \times \Pi\{X_\alpha : \alpha \in \Lambda, \alpha \neq \alpha_1, \alpha_2 \cdots \alpha_n\}$

Now $P_\alpha(B) = \begin{cases} G_\alpha & \text{if } \alpha = \alpha_i \\ X_\alpha & \text{otherwise} \end{cases}$

$\because G_{i_\alpha}$ is open in X_{α_i} and X_α is open for each fixed α, then $P_\alpha(B)$ is open set in X_α.

Theorem Product of T_1 space is T_1 space

Proof - Let $\{X_\alpha : \alpha \in \Lambda\}$ be any coordinate space. Let it be T_1

Let $X = \Pi\{X_\alpha : \alpha \in \Lambda\}$ be product space.

Let $x_\alpha \in X_\alpha$ and $x \in X$. Since X_α is T_1 then $\{x_\alpha\}$ is closed $\forall \alpha \in \Lambda$.

$\because$ each projection map p_α is continuous then $p_\alpha^{-1}\{x_\alpha\}$ is closed in X $\forall$ $\alpha \in \Lambda$.

$\therefore$ $\bigcap_\alpha p_\alpha^{-1}[\{x_\alpha\}]$ is closed in X

but $\bigcap_\alpha p_\alpha^{-1}[\{x_\alpha\}] = \{x\}$

$\Rightarrow$ every singleton subset of X is closed

$\Rightarrow$ X is T_1- space.

Theorem Product of T_2-space is T_2.

Proof - Let $\{X_\alpha : \alpha \in \Lambda\}$ be a T_2- coordinate space. Let $X = \Pi X_\alpha$ have the product topology. Let $x = (x_\alpha)_{\alpha \in \Lambda}$ and $y = (y_\alpha)_{\alpha \in \Lambda}$ be two distinct points in X. Then $x_\alpha \neq y_\alpha$ in X_α for some $\alpha \in \Lambda$. Since X_α is Hausdorff then $\exists$ open nbds U_α of x_α and V_α of y_α in X_α such that $U_\alpha \cap V_\alpha = \varphi$.

Since the projection map $p_\alpha : X \longrightarrow X_\alpha$ is continuous

Put $G = p_\alpha^{-1}[(U_\alpha)]$ and $H = p_\alpha^{-1}[V_\alpha]$

Since U_α and V_α are open in X_α, by continuity of p_α, G and H are open in X

Now if $G \cap H \neq \varphi$ Then $\exists$ $z = (z_\alpha)_{\alpha \in \Lambda} \in G \cap H$. But

$p_\alpha(z) = z_\alpha \Rightarrow z_\alpha \in U_\alpha \cap V_\alpha$

This gives a contradiction $\because$ $U_\alpha \cap V_\alpha = \varphi$.

Hence $G \cap H = \varphi$.

$\Rightarrow$ the distinct points $x, y \in X$ have disjoint nbds G & H $\Rightarrow$ X is Hausdorff.

Hence the image under p_α of each basic member of X is open in X_α.

Now let G be open in X. Then G is the union of the family $\mathcal{V}$ of base members of the form B.

So, $$p_\alpha(G) = \cup\{p_\alpha(B) : B \in \mathcal{V} \subseteq B\},$$

which is union of open sets in X_α and hence is open in X_α.

$\Rightarrow$ The projection map is open map.

Theorem – A function f on a topological space Y to a product space X is continuous iff the composition $p_\alpha \circ f$ is continuous for each $\alpha \in \Lambda$

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[rr, bend right, "p_\alpha \circ f"'] & X \arrow[r, "p_\alpha"] & X_\alpha \end{tikzcd}$$

Proof – Let f be continuous and p_α is continuous using the theorem $\forall$ fixed $\alpha \in \Lambda$, p_α is continuous $\Rightarrow p_\alpha \circ f$ is continuous

Conversely. Let $p_\alpha \circ f : Y \to X_\alpha$ is continuous for each $\alpha \in \Lambda$

If G_α is any open set in X_α then $(p_\alpha \circ f)^{-1}(G_\alpha)$ is open in Y.

ie $(f^{-1} \circ p_\alpha^{-1})(G_\alpha)$ is open in Y

ie $f^{-1}(p_\alpha^{-1}(G_\alpha))$ is open in Y.

But $G_\alpha \in \mathcal{T}_\alpha$ and so that $p_\alpha^{-1}(G_\alpha)$ is a sub base member for the product topology $\mathcal{T}$ on X. Hence by using theorem f is continuous iff $f^{-1}(S)$ is open in X, where S is a sub base member of Y and $f : X \to Y$.

$\Rightarrow f$ is continuous.

Theorem - A net S in a product space converges to a point s iff its projection in each coordinate space converges to the projection of s.

Proof - Let $\{S_n : n \in D\}$ be a net in X. Since it converges to a point s. Each projection P_a is continuous function

$\therefore \{P_a(S_n) : n \in D\}$ converges to s_a $\forall a \in A$.

Conversely, let $\{P_a(S_n) : n \in D\}$ converge to s_a $\forall a \in A$.

where $s_a = P_a(s)$.

Let U_a be any nbd of s_a. Then

$\{P_a(S_n) : n \in D\}$ is eventually in U_a.

$\exists$ $n \in D$ such that $\forall$ $n \geq m$, $P_a(S_n) \in U_a$

or $\quad S_n \in P_a^{-1}[U_a]$

If $\{S_n : n \in D\}$ is eventually in U and V then it is also eventually in $U \cap V$.

$\exists$ $m_1 \in D$ such that $n \geq m_1$, $S_n \in U$

$\exists$ $m_2 \in D$ such that $n \geq m_2$, $S_n \in V$.

Since $m_1, m_2 \in D$, $\exists$ $m \in D$ such that $m \geq m_1$, $m \geq m_2$

we have $\exists$ $m \in V$ such that $\forall$ $n \geq m$, $S_n \in U \cap V$.

$\Rightarrow$ The net is eventually in $P_a^{-1}[U_a]$

$\Rightarrow$ The net $\{S_n : n \in D\}$ is eventually in finite intersections of the net of the form $P_a^{-1}[U_a]$ where U_a is a nbd of s_a, $\forall a \in A$. Such set is form a base for the nbd system of s. Hence the net $\{S_n : n \in D\}$ is eventually in each nbd of s.

Generalized Heine-Borel Theorem.

Every closed and bounded subspace of R^n is compact.

Proof A closed rectangle is defined as a product of n closed intervals as

$$\prod_{i=1}^{n}[a_i, b_i] = \{(x_1, x_2, \dots x_n) : a_i \le x_i \le b_i \text{ for each } i\}$$

where $[a_i, b_i]$ is a bounded closed interval on the real line.

Now a closed and bounded subspace of R^n is a closed subspace of some closed rectangle. Then using the theorem "Any closed subspace of a compact space is compact".
ie it suffices to show that any closed rectangle is compact as a subspace of R^n.

Let $X = \prod [a_i, b_i]$ be a closed rectangle in R^n.
Each coordinate space $[a_i, b_i]$ is compact by using the theorem "Every closed and bounded subspace on the real line is compact".
Also by Tychonoff Theorem "Product of non empty class of compact space is compact".
Then it is suffices to show that the product topology on X is same as its relative topology as a subspace of R^n.
$\because$ The open rectangle in R^n form an ~~subspace~~ open base for its usual topology ie for its metric topology.
$\Rightarrow$ product topology on R^n is same as usual topology
$\because$ the relative topology on X is the weak topology ~~on X~~ generated by its n projections onto coordinate spaces $[a_i, b_i]$, but this is the product topology on X.
This completes our Proof.

✓ Tychonoff Theorem — Product of any non empty class of compact space is compact.

Proof. Let $\{X_a : a \in \Lambda\}$ be a non empty class of compact spaces and let $X = \Pi\{X_a : a \in \Lambda\}$ be its product spaces. Let $\{F_j\}$ be the non empty subclass of the defining closed sub base for the product topology on X. ie each F_j is the product of the form $F_j = \Pi F_{a_j}$ where F_{a_j} is a closed subset of X_a. which equals X_a for all a's but one.

Assume that the class $\{F_j\}$ has a finite intersection prop.

Now using the theorem

A top space is compact iff every class of closed ~~subsets~~ sub with the finite intersection property has a non empty intersection

So that if we prove that $\cap \{F_j\} \neq \phi$ then the product X is compact.

For a given fixed a, $\{F_{a_j}\}$ is the class of closed subsets of of X_a with the finite intersection property.

and since X_a is compact then there is a point $x_a \in X_a$ which belongs to $\cap F_{a_j}$ ie $x_a \in \cap F_{a_j}$

If we do this for each a, we obtain a point $x = \{x_a\}$ in X which is in $\cap F_{a_j}$

ie $\cap F_{a_j}$ is non empty.

Hence X is compact.

Theorem Product of non empty class of connected space is connected.

Proof – Let $\{X_i\}$ be the non empty class of connected spaces and

$X = \Pi\{X_i\}$ be the product space.

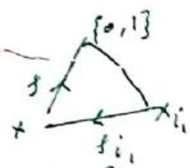
Assume that X is disconnected and to prove a contradiction as f is onto $X \to \{0,1\}$

By the theorem " $\exists$ a continuous map f of X onto the discrete two-point space $\{0,1\}$ ",

Let $a = \{a_i\}$ be the fixed point in X.

Now consider a particular index i_1. Define a map f_{i_1} of X_{i_1} into X ie $f_{i_1} : X_{i_1} \to X$

by $f_{i_1}(x_{i_1}) = y_i$, ~~where $y_i = a_i$~~.



where $y_i = a_i$, $i \neq i_1$

$y_{i_1} = x_{i_1}$,

$f : X \to \{0,1\}$

$f_{i_1} : X_{i_1} \to X$

The map f_{i_1} is continuous.

Now $f f_{i_1}$ is a continuous map of X_{i_1} into $\{0,1\}$. Since X_{i_1} is connected then using the above theorem

$f f_{i_1}$ is a constant

ie $$(f f_{i_1})(x_{i_1}) = f(a)$$

for every point x_{i_1} in X_{i_1}.

$\Rightarrow f(x) = f(a)$ for all x's in X which equal a in all coordinate spaces except X_{i_1}. By repeating this process with an other index i_2 again we find that $f(x) = f(a)$, for all x's in X which equal a in

all but a finite number of coordinate spaces.
The set of all x's of this kind is a dense
Subset of X Now using the result
If X is a top sp. and Y is a metric sp and $A \subseteq X$
then $\exists$ a continuous map of A into Y has at most one
continuous extension to a map of $\bar{A}$ into Y.

$\Rightarrow$ f is continuous map
This contradicts our assumption that f map X onto $\{0,1\}$.