# Classical Electrodynamics: Introducing Tensors in Flat Spacetime (Part 1)

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*"In the beginning, God said, 'Let the four-dimensional divergence of an antisymmetric second rank tensor be equal to zero.' And there was light".*
–Michio Kaku (about what a UC Berkeley T-shirt picture says)

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**The following** is an abridged, incomplete and *impromptu* version an introductory part of my lectures recently delivered to a set of MSc second semester students at the Department of Physics, University of Lucknow. Any feedback is welcome at my email.

**An advice** on how to read this article, e-resources or the text books referred to at the end: Never feel disheartened if you don't understand the text very well to begin with. Don't go line by line. To all of us, understanding comes in steps, and never at one go. So skip over a few hard things, and come back to them later. This may be repeated a few times. As you go deeper and deeper, mainly by solving problems given here (marked with the symbol $\oplus$), or during my lectures or elsewhere, various connections are made, and it starts making sense. Then, you can enjoy it all along. So, *Bon Voyage!*

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## 0.1 Building up on the Lorentz Transformations

Please recall that the Poincaré group is a ten parameter group which includes three boosts (spacetime rotations), three space rotations $SO(3)$, and four translations. Among them, the "orthochronous" and "anti-chronous" elements are those with the time-time components squared greater than or equal to zero.

The set of four quantities defined in four-dimensional spacetime, that change as the infinitesimal *differentials* of coordinates [1] of an event do under the Lorentz Transformations (*including the boosts*), is termed as a four-vector.

[1] If you use Cartesian coordinates, then $x^i$, and not just $dx^i$, also change as four-vectors.

You know that the transformed Cartesian coordinates are given as, [2]

$$x'^i = \Lambda^i_j x^j, \tag{1}$$

that is, $x'^0 = (x^0 - \beta x^1)\gamma; \quad x'^1 = (x^1 - \beta x^0)\gamma; \quad x'^2 = x^2; \quad x'^3 = x^3$ in a special case for $S \longrightarrow S'$ (inertial frames moving along $X-$axis).

By the same token, you may think of a **contravariant four-vector** as

$$A'^i = \Lambda^i_j A^j, \tag{2}$$

with $A'^0 = (A^0 - \beta A^1)\gamma; \quad A'^1 = (A^1 - \beta A^0)\gamma; \quad A'^2 = A^2; \quad A'^3 = A^3$ for the transformation $S \longrightarrow S'$.

$P(x^0, x^1, x^2, x^3); (A^0, A^1, A^2, A^3) \longrightarrow (x'^0, x'^1, x'^2, x'^3); (A'^0, A'^1, A'^2, A'^3)$ for the same event (spacetime point) $P$.

Thus, the statement $(x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 = (x'^0)^2 - (x'^1)^2 - (x'^2)^2 - (x'^3)^2$ would imply for the four-vector components that, $(A^0)^2 - (A^1)^2 - (A^2)^2 - (A^3)^2 = (A'^0)^2 - (A'^1)^2 - (A'^2)^2 - (A'^3)^2$ which holds good for the relative motion of the frames along an *arbitrary* direction.

## 0.2 Is it a *flat* spacetime or *curved* spacetime?

How would you check whether a given spacetime is flat or curved? We know that the square of the four-interval between two infinitesimally close events is given by $ds^2 = g_{ik} dx^i dx^k$. If you are able to write $g_{ik}$ like $(+1, -1, -1, -1)$ (in Cartesian coordinates; while in other coordinate systems these components would not be all constants), that is, as a diagonalized matrix made up of all constant terms by *any* number of attempts (i.e., by any coordinate transformations whatsoever), then it is a *flat* spacetime. And if you find it

[2]

(i) As a special case, you may write (1) as rotations in $x^1 - x^0$ plane, where $\tanh\alpha = \frac{v}{c}$ using rapidity. It is something like 3$-$ space rotations where $\Lambda^i_j \equiv (\Lambda) \equiv \frac{\partial x'^i}{\partial x^j}$. $\oplus$ Express the later in terms of angle $\theta$ to write the transformation matrix $R$.

(ii) Maths tells us that $A^i$ behaves as the components of a tangent vector $A^i \equiv \frac{dx^i}{d\lambda}$ to a curve parametrized by $\lambda$, as shown in my lectures. $\oplus$ Is it a coincidence?

impossible to do so, then it must be a *curved* spacetime. In this case, $g_{ik}$ can never be written in the Euclidean form at *all* spacetime points. It is a very important property of gravitation $\Rightarrow$ to *curve* spacetime (not just space, not just time but both together!).
The good thing is that we don't have to worry about all this difference at the present level of this Course (which sticks only to special relativity [3] or flat spacetime), while the bad news is the classical electrodynamics is not complete unless you take into account the field of gravity lurking in the background. So, the things, as we are doing here and now, are not entirely correct. This rather subtle but elegant issue can be discussed elsewhere in detail.

## 0.3 Four-vectors

The four-interval between any two infinitesimally close events is given by

$$ds = (c^2dt^2 - dx^2 - dy^2 - dz^2)^{1/2}. \tag{3}$$

So, the square of the length of vector (four-radius vector) is $(x^0)^2-(x^1)^2-(x^2)^2-(x^3)^2$, where $x^0 = ct$; $\quad x^1 = x$; $\quad x^2 = y$; $\quad x^3 = z$.

This four-dimensional geometry is essentially the *Pseudo-Euclidean* or *Minkowski* spacetime geometry introduced by Hermann Minkowski (with strange properties such as $\int ds$ stays maximum along the straight world line, compared to others describing uniform motion or even acceleration: *yes, special relativity can deal with accelerated motions as well*). The purpose of introducing four-tensors is to know what changes (and how) and what does not, when the Lorentzian spacetime transformations take place. Such quantities, obviously, are then attached with the 'geometry' of that spacetime and not just with the coordinate effects.

**Signature of spacetime** $\Rightarrow$ $(+,-,-,-)$ = number of $(+)$ coefficients $-$ number of $(-)$ coefficients $= -2$.

**Basic Definitions :**

[3]There exist several approaches; for example see the most recent reference at the time of this writing: W. N. Mathews, *Seven Formulations of the Kinematics of Special Relativity,* American Journal of Physics 88(4), 269-278, April 2020.

***Four-scalar :*** A quantity that doesn't change under the transformation of coordinates.[4].

$$\phi(x^i) = \phi[x^i(x'^k)] = \phi'(x'^k). \tag{4}$$

The form of function may change but the value remains invariant.

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$\oplus$ Could you verify if the following quantities are four-scalars: coordinate time, relativistic and rest mass, frequency, energy of light and its speed in vacuum or any other medium?

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***Four-vector :*** Any four components that transform like $x^i$ under the coordinate transformation (see the footnote 1).

***Contravariant four-vector :*** Suppose a curve is parameterized by $\lambda$, then the coordinates on this curve are $x^i = x^i(\lambda)$ and direction to the tangent at any point is given by a vector with 4- components [5]

$$A^i \equiv \frac{dx^i}{d\lambda}. \tag{5}$$

Direction of tangent is an invariant concept, though the components of $A^i$ change under coordinate transformation as $A'^i \equiv \frac{dx'^i}{d\lambda}$. Clearly,

$$A'^i \equiv \frac{dx'^i}{dx^k}\frac{dx^k}{d\lambda} = \frac{dx'^i}{dx^k}A^k \tag{6}$$

as $x'^i = \Lambda^i_k x^k$, $A'^i = \Lambda^i_k A^k$ follows as a ***linear transformation***

$$A'^i = \frac{dx'^i}{dx^k}A^k. \tag{7}$$

Any vector as above is a ***contravariant*** four-vector.

***Covariant four-vector :*** Consider a 3-dimensional hypersurface described by the equation $\phi(x^k) =$ constant. The *normal* has the direction

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[4]Of course, in special relativity, and therefore for the purpose of the present lectures, we consider only the transformations spanned by the Lorentz group, and not the arbitrary ones, which are left to general relativity.

[5]In physics, we can choose this parameter as proper time $\tau$ or line element to define quantities like four-velocity and four-momentum.

fixed by four components given by $B_i = \frac{d\phi}{dx^i}$. Under the coordinate transformation, its direction remains fixed but the components change as $B'_i = \frac{d\phi}{dx'^i}$. Thus

$$B'_i = \frac{d\phi}{dx'^i} = \frac{dx^k}{dx'^i}\frac{d\phi}{dx^k} = \frac{dx^k}{dx'^i}B_k. \tag{8}$$

Hence, the linear transformation is followed.

$$B'_i = \frac{dx^k}{dx'^i}B_k \tag{9}$$

The quantities which change as above under transformation of the coordinates form a covariant four-vector. They are defined in *dual space*.

- **Examples:**

(i) A curve parameterized by $x^0$ =constant, $x^1$ =constant, $x^2 = \lambda$, $x^3 = \lambda^2$ has tangent specified by contravariant vector components. Therefore, $A^0 = 0, \quad A^1 = 0, \quad A^2 = 1, \quad A^3 = 2\lambda$.

(ii) A unit sphere

$$\phi = (x^1)^2 + (x^2)^2 + (x^3)^2 = 1 \tag{10}$$

has the normal specified by covariant vector components $B'_i = \frac{d\phi}{dx'^i}$. Thus, $B_0 = 0, \quad B_1 = 2x^1, \quad B_2 = 2x^2, \quad B_3 = 2x^3$.

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$\oplus$ Should the dimensions of all components of a four-vector be the same?

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**Definition: Scalar Product or Inner Product:**

$$A_iB^i = A^iB_i = A^0B_0 + A^1B_1 + A^2B_2 + A^3B_3 \tag{11}$$

remains invariant. Similarly, $A_iA^i$ or $g_{ik}A^iA^k$ remain invariant for any four-vector $A^i = (A^0, \mathbf{A})$ just as does the length of a radius vector $x^i = (ct, \mathbf{r})$.

**Conventions followed:**

(i) Einstein's convention of summation over *dummy indices*

$$A_iA^i = \sum A_iA^i \tag{12}$$

holding for the same upper and lower indices only.

(ii) Latin indices for $0, 1, 2, 3, ...$ $(i, k, l, ...)$ and Greek indices for $1, 2, 3...$ $(\lambda, \mu, \nu, ...)$.

(iii) Signature of metric $(+, -, -, -)$, $ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$.

**The square of the magnitude (or norm)** of $A^i$ is given by

$$A^iA_i = A^0A_0 + A^1A_1 + A^2A_2 + A^3A_3 = (A^0)^2 - (A^1)^2 - (A^2)^2 - (A^3)^2. \tag{13}$$

It is a Lorentz scalar quantity being a tensor of zero rank, and so remains invariant under the transformations contained in the Lorentz group.

**Nature of four-vector** The squared length of a three vector is always positive. However, it is not so for four-vectors. The following criteria (with our chosen signature) apply:

(i) **Timelike :**$(A^0)^2 > (A^1)^2 + (A^2)^2 + (A^3)^2$, positive square of length.

(ii) **Spacelike :** Negative square length.

(iii) **Lightlike or Null:** Zero length.

We also call the *surfaces* as timelike, spacelike or lightlike when a covariant four-vector $B_i$ (say, the normal at an event $x^k$) orthogonal to it is spacelike, timelike or lightlike, respectively.

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⊕ Show that if you add or subtract two four-vectors, the resultant vector can have any of the above stated types (timelike, spacelike or null) independent of the nature of those two four-vectors.

⊕ Show that a lightlike four-vector is orthogonal to itself.

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## 0.4 Four-tensors:

Generalization of a four-vector with the following transformation rules –

Contravariant tensor of rank 2:

$$T'^{ik} = \frac{\partial x'^i}{\partial x^m}\frac{\partial x'^k}{\partial x^n}T^{mn}. \tag{14}$$

Covariant tensor of rank 2:

$$T'_{ik} = \frac{\partial x^m}{\partial x'^i}\frac{\partial x^n}{\partial x'^k}T_{mn} \tag{15}$$

Mixed tensor of rank 2:

$$T'^i{}_k = \frac{\partial x'^i}{\partial x^m}\frac{\partial x^n}{\partial x'^k}T^m{}_n. \tag{16}$$

(Don't put index $k$ below $i$, or $n$ below $m$ unless the *original* tensor with the *similar* indices is symmetric as defined below). They transform like product of two four-vectors. Higher rank tensors are likewise constructed by product of as many four-vectors.

**Raising or Lowering of Indices :** Under our convention, if this operation is done for time index, no change occurs in sign. However, for each space index $(1, 2, 3)$, sign changes on every operation. This result arises from the fact that we have *chosen* the spacetime signature as $(+, -, -, -)$, *i.e.*, $-2$. If we choose $(-, +, +, +)$, the above conclusion would be reverse.

**Symmetric Tensor:** If $A^{ik} = A^{ki}$ then $A^{ik}$ is a symmetric tensor.

As an example, consider $A^{32} = A^{23}$. Also see that $A^{32} = -A^3{}_2$ and $A^{23} = -A^2{}_3$. Then $A^3{}_2 = A^2{}_3$.

Thus, we can justifiably write for a symmetric tensor $A^{ik}$ :

$$A^i{}_k = A^k{}_i. \tag{17}$$

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⊕ Can you claim that the form given by equation (17) is symmetric? [6]

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**Note :** • The above second rank symmetric tensor has $\frac{N^2-N}{2} + N = \frac{N(N+1)}{2}$ independent components in $N$-dimensional spacetime.

• Since for a symmetric tensor $A^{ik}$, you have $A^3{}_2 = A_2{}^3$, you may write it as $A^3_2$. More generally, for a symmetric tensor, $A^i{}_k = A_k{}^i = A^i_k$.

**Antisymmetric (or skewsymmetric) Tensor :**

(1) If $A^{ik} = -A^{ki}$. Now, for $A^{01} = -A^0{}_1$ & $A^{10} = -A_1{}^0$ and hence $A^{01} = -A^{10}$. Therefore, $A^0{}_1 = -A_1{}^0$. An antisymmetric tensor is

[6]**Hint:** The equality $A^i{}_k = A^k{}_i$ for an *arbitrary* tensor may hold in one frame, while it may not in another frame under the Lorentz group of transformations. Check it.

also called a bivector. If $A^{ik} = a_i b_k - a_k b_i$ then it is a 'simple' or 'decomposable' bivector.

**Caution:** *If $A^{ik}$ is symmetric or antisymmetric (*remembering the fact that an arbitrary tensor may be neither symmetric nor antisymmetric*)it will remain so under Lorentz transformations.*

Thus, for an antisymmetric tensor,

$$A^i{}_k = -A^k{}_i \neq A^k_i. \tag{18}$$

Antisymmetric tensor of rank two has $\frac{N(N-1)}{2}$ independent components.

(2) $A^{00} = -A^{00}, \quad A^{11} = -A^{11}, \quad A^{22} = -A^{22}, \quad A^{33} = -A^{33}$ for antisymmetric tensor so $A^{00} = A^{11} = A^{22} = A^{33} = 0$.

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⊕ Show that the trace of an antisymmetric tensor is zero.

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## 0.5 More Problems:

⊕ 1. Show that every four-vector orthogonal to a lightlike four-vector must be spacelike (or lightlike, in trivial case). Further, show that the propagation vector of light $k^i$ is lightlike and hence, orthogonal to itself. (*By the way, this is the origin of the word "lightlike".*)

⊕ 2. Construct two orthogonal four-vectors by doing some simple algebraic operation (addition, subtraction or multiplication) between *any* two vectors of equal norm.

⊕3. A tachyonic particle flies off at time $t_1$ with speed $v > c$ and returns at time $t_2$ to the same point. Sketch its world line on light cone diagram. Is the time ordering absolute?

⊕4. Show that $\Lambda^0{}_0$ cannot lie between $+1$ & $-1$. Construct the orthochronous Lorentz group from the elements $\Lambda^\uparrow_+, \quad \Lambda^\downarrow_-, \quad \Lambda^\downarrow_+, \quad \Lambda^\uparrow_-$. Under which of these transformations are the laws of physics invariant in special relativity?

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• The resources mentioned ahead are by no means the substitute of, but are in addition to, my classroom lectures which are based on the current syllabus of PHYC- 203 course of MSc (Physics) second semester at the University of Lucknow. E-text version of some of these resources is also available on the web.

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# Bibliography


[1] D. Fleisch, *A Student's Guide to Vectors and Tensors* (Cambridge University Press, 2012) (This a well written guide to learn this subject, as good as others in this series; Freely available online: www4.danfleisch.com/sgvt). Also see *Tensors: A Guide for Undergraduate Students*, American Journal of Physics 81(7), 498-511, July 2013.

[2] D. C. Kay, *Schaum's Outline of Tensor Calculus* (McGraw-Hill, 2011) (It contains a large number of examples and problems).

[3] L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields, Course of Theoretical Physics*, Vol 2 (Pergamon Press, 1975). (Chap. 1 is nice for basic matter-of-factly concepts in special relativity, along with the other references given below and several online resources. Notice the difference in definitions of four-velocity and its derivatives from our those used in our discussion).

[4] J. D. Jackson, *Classical Electrodynamics* (Third ed., Wiley, 2005). (Chapter 11 aptly summs up mathematical aspects of special relativity, Lorentz group, four-tensors leading to concepts of covariance, etc.).

[5] D. J. Griffiths, *Introduction to Electrodynamics* (Fourth ed., Pearson, 2015). (Chap. 12 for Lorentz Transformation and four-vectors. However, signature used here $(-,+,+,+)$ is different from that used in my lectures).

[6] R. d'Inverno, *Introducing Einstein's Relativity* (Clarendon Press, Oxford, 1992).

[7] R. Resnick, *Introduction to Special Relativity* (Wiley 1995). (In Supplementary Topic A, it describes the light cones pictures of two inertial frames and the underlying hyperbolic geometry).

[8] W. N. Mathews, *Seven Formulations of the Kinematics of Special Relativity* , American Journal of Physics 88(4), 269-278, April 2020. (The connection between the speed of light and various formulations of relativity are also explored here).

[9] A. W. Joshi, *Matrices and Tensors in Physics* (Third ed., New Age, 1995). (For elementary cocepts on matrices and tensors, as conjugate $g_{ik}$ from $g^{ik}$ Chap. 17,18).

[10] J. A. Schouten, *Tensor Analysis for Physicists* (Dover, 2011).

[11] H. Stephani, *General Relativity* (Second ed., Cambridge University Press) (It takes you to tensors in non-Euclidean geometries).