Unit 4

Digital Filters

Differential Equation for most practical cases -

$$\sum_{k=0}^{M-1} a_k . y(n-k) = \sum_{k=0}^{N-1} b_k . x(n-k)$$

$$a_0 y(n) + \sum_{k=1}^{M-1} a_k . y(n-k) = \sum_{k=0}^{N-1} b_k . x(n-k)$$

$$y(n) = -\sum_{k=1}^{M-1} a_k . y(n-k) + \sum_{k=0}^{N-1} b_k . x(n-k).$$

- Finite Impulse Response (FIR) -

$a_k = 0$ $\quad\quad 1 \leq k \leq M-1$

$$y(n) = \sum_{k=0}^{N-1} b_k . x(n-k)$$

This will be a non-recursive system because there's no feedback.

length of h(n) will be of finite duration.

This system is also called moving average system, because the present output depends on the weighted sum of inputs.

$$Y(z) = \sum_{k=0}^{N-1} b_k \cdot z^{-k} \cdot X(z)$$

$$H(z) = \sum_{k=0}^{N-1} b_k \cdot z^{-k}$$

This system is also called as All Zero System. because we get the poles at z=0, and the magnitude response will depend only on the zeroes and not on pulse. The system is always stable.

- IIR (Infinite Response) :-

This is also known as All Pole System also Auto Regressive systems.

$$y(n) = -\sum_{k=1}^{M-1} a_k \cdot y(n-k) + b_0 \cdot x(n)$$

Find its z transform;

$$Y(z) = -\sum_{k=1}^{M-1} a_k \cdot z^{-k} \cdot Y(z) + b_0 X(z)$$

$$H(z) = \frac{b_0}{1 + \sum_{k=1}^{M-1} a_k \cdot z^{-k}}$$

This IIR system is called a Recursive system.

The duration of impulse response is infinite.

This is called as Auto Regressive system because the output depends on its previous values.

This is called All pole system because poles can exist anywhere but zeroes lies only on $z=0$.

They are not necessarily stable.

- Pole zero or Auto Regressive Moving Average System -

$$y(n) = -\sum_{k=1}^{M-1} a_k \cdot y(n-k) + \sum_{k=0}^{N-1} b_k \cdot x(n-k)$$

$$Y(z) = -\sum_{k=1}^{M-1} a_k \cdot z^{-k} \cdot Y(z) + \sum_{k=0}^{N-1} b_k \ X(z)\ z^{-k}$$

$$Y(z) + \sum_{k=1}^{M-1} a_k \ z^{-k} \cdot Y(z) = \sum_{k=0}^{N-1} b_k \cdot z^{-k} \cdot X(z)$$

$$Y(z)\left[1 + \sum_{k=1}^{M-1} a_k \cdot z^{-k}\right] = \sum_{k=0}^{N-1} b_k \cdot z^{-k} \cdot X(z)$$

$$\frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{N-1} b_k \cdot z^{-k}}{\left(1 + \sum_{k=1}^{M-1} a_k \cdot z^{-k}\right)}$$

This will be a Recursive system.

FIR System :-

$$H(z) = \sum_{k=0}^{m-1} b_k . z^{-k}$$

$$H(z) = b_0 z^{-0} + b_1 z^{-1} + b_2 z^{-2} + b_3 z^{-3} + b_4 z^{-4} + \cdots + b_{m-1} z^{-(m-1)}$$

$$Y(z) = \left[b_0 z^{-0} + b_1 z^{-1} + \cdots + b_{m-1} z^{-(m-1)} \right] . X(z)$$

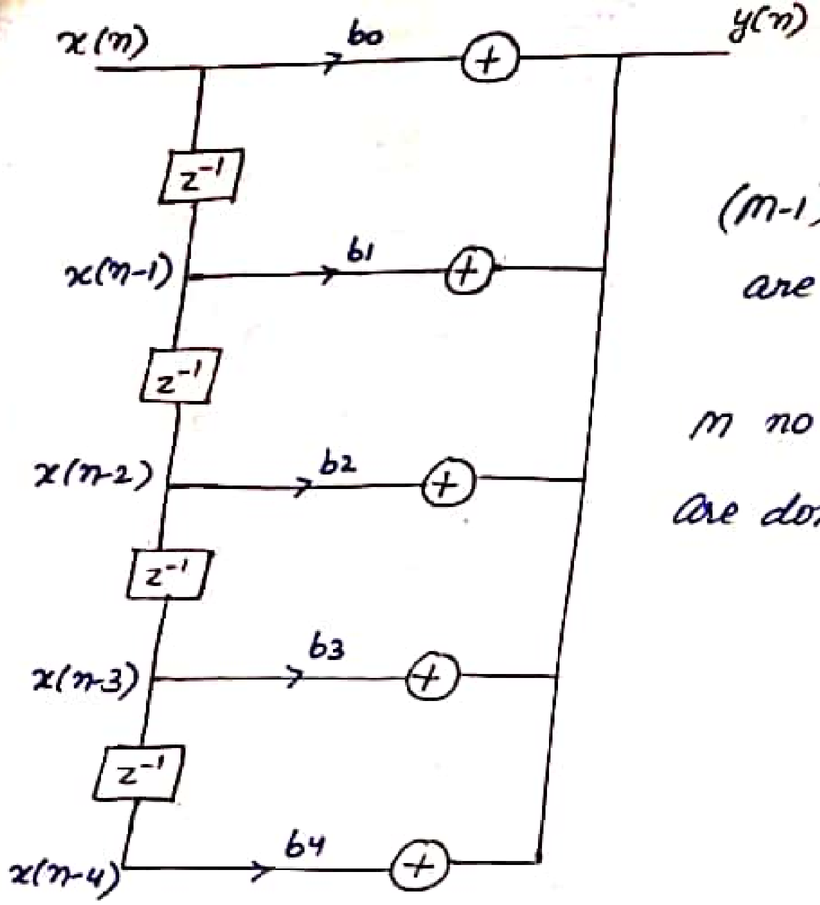
$$y(n) = b_0 x(n) + b_1 x(n-1) + b_2 . x(n-2) + \cdots + b_{m-1} x(n-(m-1))$$

length of the filter is M and the order of the filter is (m-1).

- **Direct form or Tapped Delay in line or Transversal :-**

$m = 5$

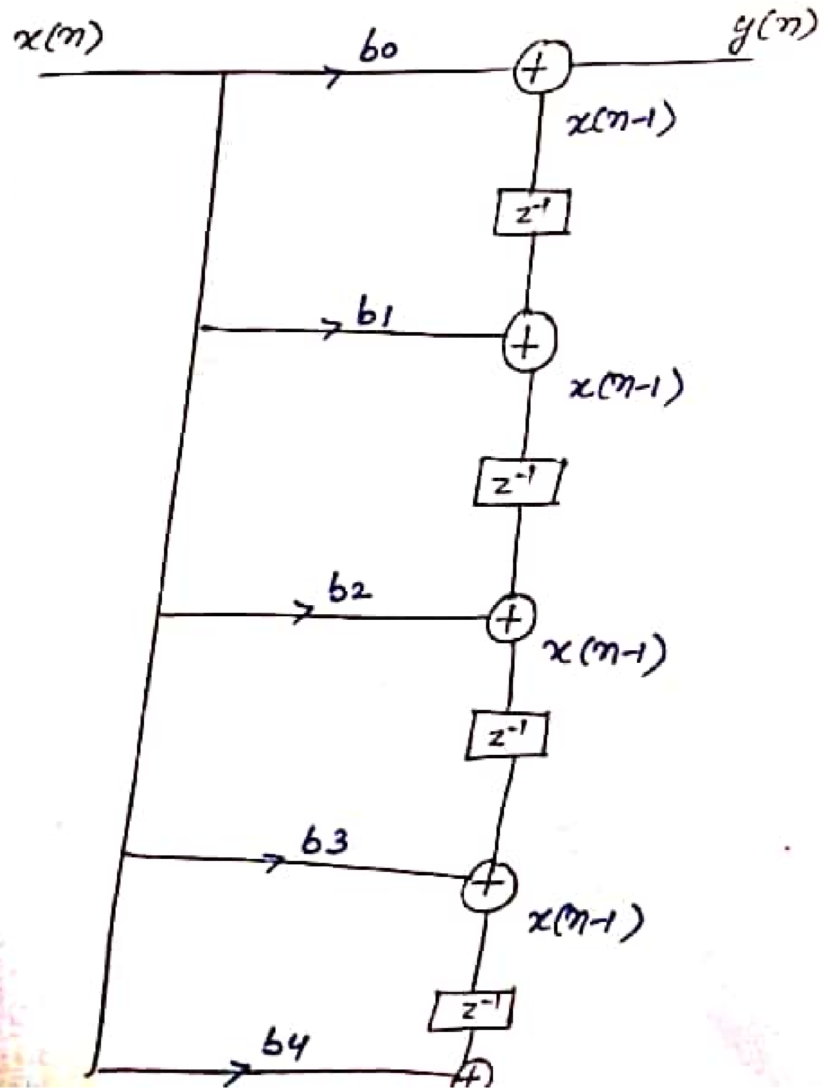
$$y(n) = b_0 x(n) + b_1 x(n-1) + b_2 . x(n-2) + \cdots + b_4 . x . (n-4)$$



(m-1) no. of delay lines are used.

m no. of multiplications are done.

- Transposed Direct Form :-



Ques. Consider the system with function,

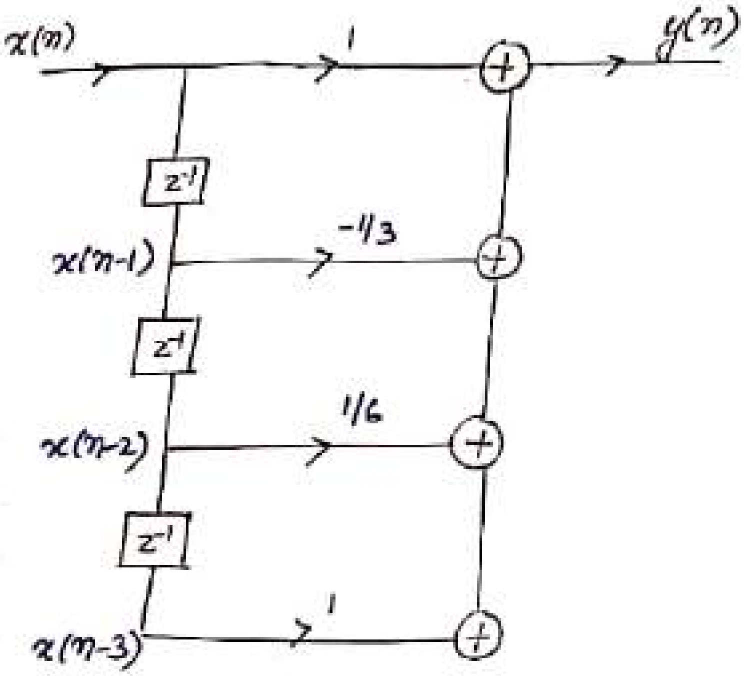
$$H(z) = 1 - \frac{1}{3}z^{-1} + \frac{1}{6}z^{-2} + z^{-3}$$

Order = 3

length of the system = 4

That means, 3 delay lines are used and 4 multipliers are needed.

$$Y(z) = \left[1 - \frac{1}{3}z^{-1} + \frac{1}{6}z^{-2} + z^{-3}\right] . X(z)$$



- **Cascade form Structure :-**

$$H(z) = b_0 + b_1 z^{-1} + b_2 z^{-2} + b_3 z^{-3} + \dots\dots + b_{m-1} . z^{-(m-1)}$$

$$H(z) = b_0\left[1 + \frac{b_1}{b_0}z^{-1} + \frac{b_2}{b_0}z^{-2} + \frac{b_3}{b_0}z^{-3} + \dots\dots + \frac{b_{m-1}}{b_0}z^{-(m-1)}\right]$$

$$H(z) = \begin{cases} b_0 \prod_{k=1}^{(m-1)/2} \left[1 + B_{1k} z^{-1} + B_{2k} z^{-2}\right] & ; m \text{ is odd} \\ b_0 (1 + b_{10} z^{-1}) \prod_{k=1}^{(m-2)/2} \left[1 + B_{1k} z^{-1} + B_{2k} z^{-2}\right] & ; m \text{ is even} \end{cases}$$

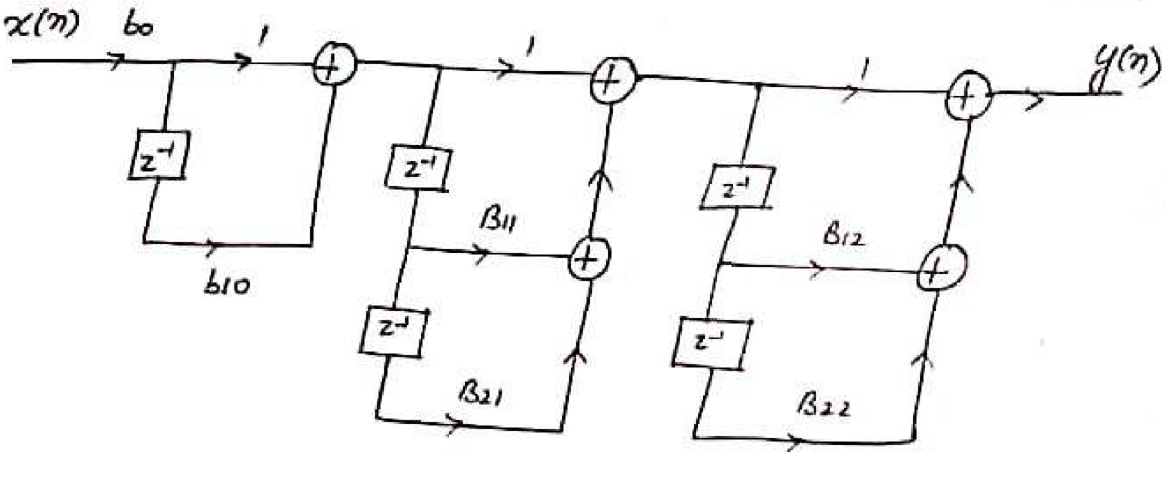
Example :-

$m = 6$

$$H(z) = b_0 . (1 + b_{10} z^{-1}) \prod_{k=1}^{(6-2)/2} \left[1 + B_{1k} z^{-1} + B_{2k} z^{-2}\right]$$

$$= b_0 (1 + b_{10} z^{-1}) \prod_{k=1}^{2} \left[1 + B_{1k} z^{-1} + B_{2k} z^{-2}\right]$$

$$H(z) = b_0 . (1 + b_{10} z^{-1}) . (1 + B_{11} z^{-1} + B_{21} z^{-2}) (1 + B_{12} z^{-1} + B_{22} z^{-2})$$



This format is used when the order of the filter is very high.

Transpose Cascaded form will also be similar to the Transpose direct form.

(only z^{-1} will be shifted)

Ques: Obtain the cascaded form of the given filter.

$$H(z) = 1 + \frac{6}{5}z^{-1} + \frac{7}{5}z^{-2} + \frac{26}{25}z^{-3} + \frac{1}{5}z^{-4}$$

$m = 5$ i.e odd

Hence,

$$H(z) = b_0 \prod_{k=1}^{(m-1)/2} \left[1 + B_{1k}z^{-1} + B_{2k}z^{-2}\right]$$

i.e. we obtain two factors

$$H(z) = b_0 \prod_{k=1}^{2} \left[1 + B_{1k}z^{-1} + B_{2k}z^{-2}\right]$$

$$= b_0 (1 + B_{11}z^{-1} + B_{21}z^{-2}) \; (1 + B_{12}z^{-1} + B_{22}z^{-1})$$

$$= (b_0 + b_0 B_{11} z^{-1} + b_0 B_{21} z^{-2}) . (1 + B_{12}z^{-1} + B_{22}z^{-1})$$

$$= b_0 + b_0 B_{12} z^{-1} + b_0 B_{22} z^{-2} + b_0 B_{11} z^{-1} + B_{11} B_{12} b_0 z^{-2}$$
$$+ \; b_0 B_{11} B_{22} z^{-3} + b_0 B_{21} z^{-2} + b_0 B_{21} B_{12} z^{-3} +$$
$$b_0 B_{21} B_{22} z^{-4}$$

$$= b_0 + (B_{11} + B_{12}) b_0 z^{-1} + (B_{22} + B_{11}B_{12} + B_{21}) b_0 . z^{-2}$$
$$+ \; (B_{11}B_{22} + B_{21}B_{12}) b_0 z^{-3} + B_{21}B_{22} b_0 z^{-4}$$

On comparing:

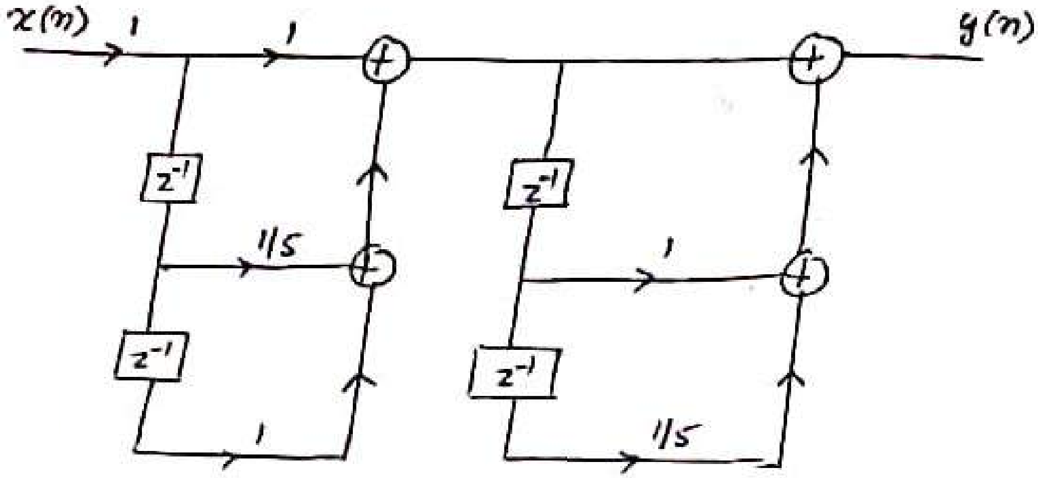
$b_0 = 1$

$B_{11} + B_{12} = \frac{6}{5}$

$B_{11}B_{22} + B_{21}B_{12} = \frac{26}{25}$

$B_{22} + B_{11}B_{12} + B_{21} = \frac{7}{5}$

$$= 1\left[1+\frac{1}{5}z^{-1}+z^{-2}\right]\cdot\left[1+z^{-1}+\frac{1}{5}z^{-2}\right]$$



- ## Linear Phase Structure :-

It is desirable for linear phase which has a linear phase response.

$$\angle H(e^{j\omega}) = \beta - \alpha\omega \qquad -\pi < \omega < \pi$$

When $\beta = 0$,

we obtain a symmetric condition,

$$h(n) = h(m-1-n)$$

When $\beta = \pm\frac{\pi}{2}$,

we obtain an asymmetric condition,

$$h(n) = -h(m-1-n)$$

If m is odd, we obtain $\left(\frac{m+1}{2}\right)$ no. of multipliers.

If m is even, the multipliers are reduced to $\frac{m}{2}$.

Eg: $M=6$ and $\beta=0$

$$H(z) = b_0 + b_1 z^{-1} + b_2 z^{-2} + b_3 z^{-3} + b_4 z^{-4} + b_5 z^{-5}$$

$$H(z) = \sum_{k=0}^{M-1} a_k \cdot z^{-k} = \sum_{k=0}^{M-1} h(n) \cdot z^{-k}$$

$$H(z) = h(0) + h(1) z^{-1} + h(2) z^{-2} + h(3) z^{-3} + h(4) z^{-4} + h(5) z^{-5}.$$

$$h(n) = h(1-M-n)$$

$$h(0) = h(5)$$

$$h(1) = h(4)$$

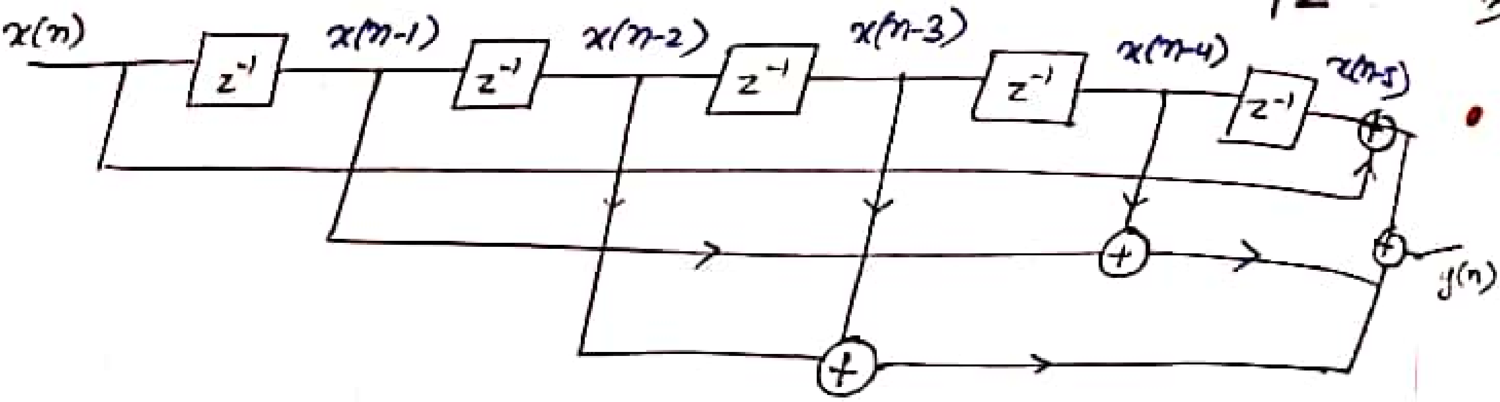
$$h(2) = h(3)$$

$\therefore$

$$H(z) = h(0) \cdot [1 + z^{-5}] + h(1) \cdot [z^{-1} + z^{-4}] + h(2) [z^{-2} + z^{-3}]$$

$$Y(z) = [b_0 (1 + z^{-5}) + b_1 (z^{-1} + z^{-4}) + b_2 (z^{-2} + z^{-3})] \cdot X(z)$$

$$y(n) = b_0 [x(n) + x(n-5)] + b_1 [x(n-1) + x(n-4)] + b_2 [x(n-2) + x(n-3)].$$



Here, instead of 6 we are using just half 3 multipliers. The circuit becomes simple.

Frequency Sampling -

$$H(z) = \sum_{n=0}^{M-1} h(n).z^{-n}$$

$$= \sum_{n=0}^{M-1} \left[\frac{1}{M} \sum_{k=0}^{M-1} H(k)\, e^{-j\frac{2\pi}{M}.kn} \right] . z^{-n}$$

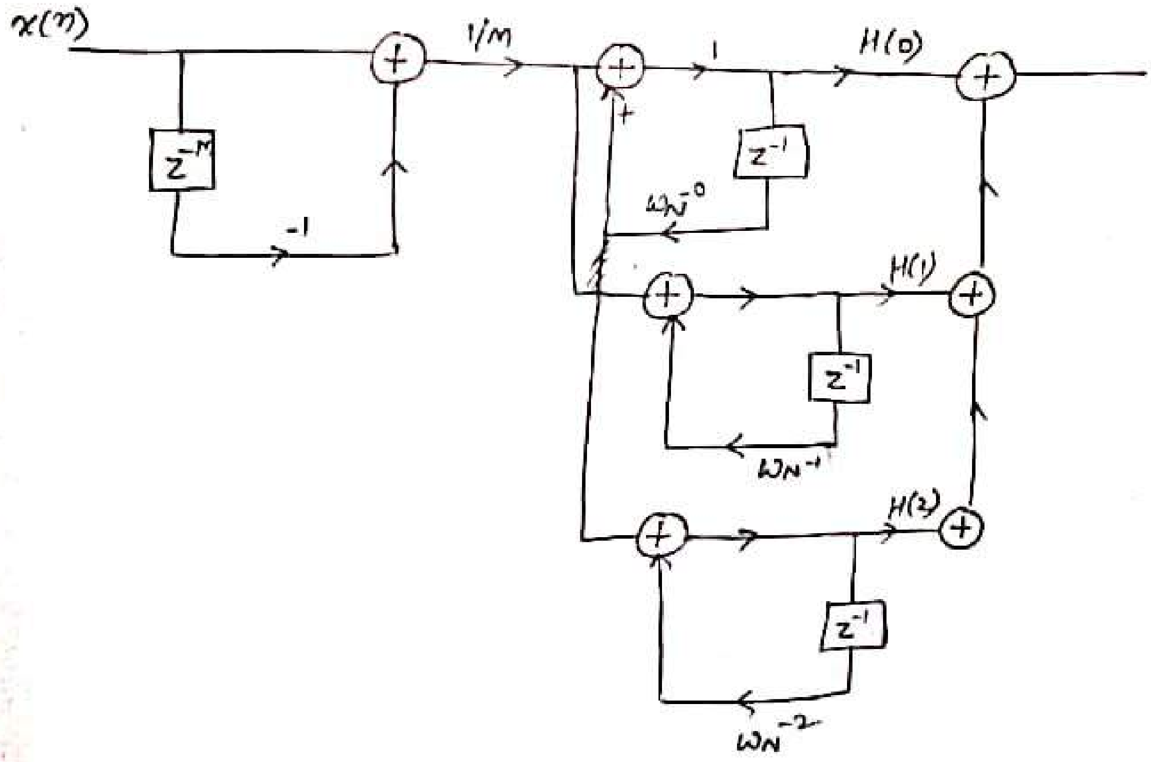
$$= \frac{1}{M} . \sum_{k=0}^{M-1} H(k). \sum_{n=0}^{M-1} e^{j\frac{2\pi}{M}.kn} . z^{-n}$$

$$= \frac{1}{M} . \sum_{k=0}^{M-1} H(k). \left[1 + e^{j\frac{2\pi}{M}k} z^{-1} + e^{j\frac{2\pi}{M}2} . z^{-2} + \dots \dots e^{j\frac{2\pi}{M}(M-1)} . z^{-(M-1)} \right].$$

= forming a finite GP

$$\underbrace{\frac{1-z^{-M}}{M}}_{H_1(z)} . \underbrace{\sum_{k=0}^{M-1} \frac{H(k)}{1-e^{j\frac{2\pi}{M}k} z^{-1}}}_{H_2(z)}$$

$$\frac{G(s)}{1-G(s).H(s)}$$



$$H(K) = H^*(M-K)$$

$$H^*(K) = H(M-K)$$

$$e^{j\frac{2\pi}{m}(m-k)} = e^{-j\frac{2\pi}{m}\cdot k}.$$

$$H(z) = \frac{1-z^{-m}}{m} \underbrace{\sum_{k=0}^{m-1} \frac{H(k)}{1-e^{j\frac{2\pi}{m}k}\cdot z^{-1}}}_{H_2(z)}$$

put $m=3$ in $H_2(z)$.

$$\sum_{k=0}^{m-1} \frac{H(K)}{1-e^{j\frac{2\pi}{m}k}\cdot z^{-1}} = \sum_{k=0}^{2} \frac{H(K)}{1-e^{j\frac{2\pi}{3}K}\cdot z^{-1}}$$

$$H_2(z) = \frac{H(0)}{1-e^{j\frac{2\pi}{3}(0)}\cdot z^{-1}} + \frac{H(1)}{1-e^{j\frac{2\pi}{3}(1)}\cdot z^{-1}} + \frac{H(2)}{1-e^{j\frac{2\pi}{3}(2)}\cdot z^{-1}}$$

for $m=3$.
$k=2$

$$H(2) = H^*(3-2)$$

$$H(2) = H^*(1)$$

Also;

$e^{j\frac{2\pi}{m}(m-k)}$ put $m=3$
$k=1$

$$= e^{-j\frac{2\pi}{m}k}$$

$$= e^{-j\frac{2\pi}{3}(1)}$$

6.

$$H_2(z) = \frac{H(0)}{1-e^{j\frac{2\pi}{3}(0)}z^{-1}} + \frac{H(1)}{1-e^{j\frac{2\pi}{3}}.z^{-1}} + \frac{H^*(1)}{1-e^{-j\frac{2\pi}{3}}.z^{-1}}$$

$$H_2(z) = \begin{cases} \sum_{k=1}^{\frac{m-1}{2}} \left[\frac{H(k)}{1-e^{j\frac{2\pi}{m}k}.z^{-1}} + \frac{H^*(k)}{1-e^{-j\frac{2\pi}{m}k}.z^{-1}}\right] + \frac{H(0)}{1-z^{-1}} & \text{for } m \text{ is odd.} \\ \sum_{k=1}^{\frac{m}{2}-1} \left[\frac{H(k)}{1-e^{j\frac{2\pi}{m}k}.z^{-1}} + \frac{H^*(k)}{1-e^{-j\frac{2\pi}{m}k}.z^{-1}}\right] + \frac{H(0)}{1-z^{-1}} + \frac{H(m/2)}{1-z^{-1}} & \text{for } m \text{ is even.} \end{cases}$$

To remove the complex terms;

$$\frac{H(k)}{1-e^{j\frac{2\pi}{m}k}.z^{-1}} + \frac{H^*(k)}{1-e^{-j\frac{2\pi}{m}.k}.z^{-1}}$$

On cross multiplying them;

$$= \frac{H(k)\left[1-e^{-j\frac{2\pi}{m}.k}.z^{-1}\right] + H^*(k)\left[1-e^{j\frac{2\pi}{m}k}.z^{-1}\right]}{\left[1-e^{j\frac{2\pi}{m}k}.z^{-1}\right].\left[1-e^{-j\frac{2\pi}{m}k}.z^{-1}\right]}$$

$$= \frac{H(k). - H(k).e^{-j\frac{2\pi}{m}.k}.z^{-1} + H^*(k) - H^*(k).e^{j\frac{2\pi}{m}k}.z^{-1}}{1-e^{-j\frac{2\pi}{m}k}.z^{-1} - e^{j\frac{2\pi}{m}k}.z^{-1} + e^{j\frac{2\pi}{m}k - j\frac{2\pi}{m}k}.z^{-2}}$$

$$= \frac{H(k)+H^*(k) - z^{-1}\left[H(k).e^{-j\frac{2\pi}{m}.k} + H^*(k).e^{-j\frac{2\pi}{m}k}\right]}{1-2\cos\left(\frac{2\pi}{m}.k\right)z^{-1} + z^{-2}}$$

When $H(k) + H^*(k)$, we obtain the number as twice of the real part of $H(k)$. 96

eg $H(k) = 2+3j$

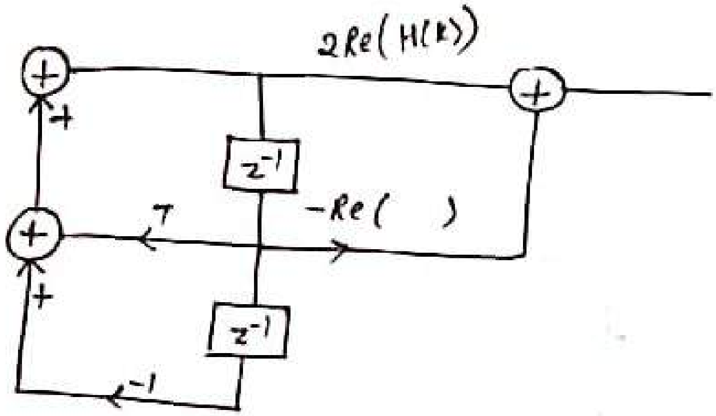
$H^*(k) = 2-3j$

$\Rightarrow H(k) + H^*(H) = 4 = 2 Re H(k)$

$\therefore$

$$H_2(z) = \frac{2 \cdot Re[H(k)] - z^{-1}\left[2 Re\left[H(k) \cdot e^{-j\frac{2\pi}{m}k}\right]\right]}{1 - 2\cos\left(\frac{2\pi}{m} \cdot k\right) \cdot z^{-1} + z^{-2}}$$

$$\frac{1}{1 - 7z^{-1} + z^{-2}}$$



Ques: Let $h(n) = \{1/2, 1, 1/2\}$
Find $H(K)$
and frequency sampling structure

$$H(K) = \{2, -0.25-0.433j, -0.25+0.433j\}$$

here, $M = 3$.

$$H(z) = \left[\frac{1-z^{-3}}{3}\right] \cdot \left[\frac{2\,Re(H(1)) - 2\cdot Re\left(H(1)\, e^{-j\frac{2\pi(1)}{3}}\right)}{1-2\cos\left(\frac{2\pi}{3}\right)\cdot z^{-1} + z^{-2}}\right] \cdot \frac{H(0)}{1-z^{-1}}$$

$$H(z) = \left[\frac{1-z^{-3}}{3}\right] \cdot \left[\frac{-0.5 - 2\cdot Re\left[(-0.25-0.433j)\cdot e^{-j\frac{2\pi}{3}}\right]}{1-2\cos\left(\frac{2\pi}{3}\right)\cdot z^{-1} + z^{-2}}\right] \cdot \frac{H(0)}{1-z^{-1}}$$

$$H(z) = \left[\frac{1-z^{-3}}{3}\right] \cdot \left[\frac{-0.5 - 2 \times 0.5 z^{-1}}{1-2\cos\left(\frac{2\pi}{3}\right) z^{-1} + z^{-2}}\right] \cdot \frac{H(0)}{1-z^{-1}}$$

$e^{-j\frac{2\pi}{3}} = \cos\frac{2\pi}{3} - j\sin\frac{2\pi}{3}$

$-0.25 + 0.433$

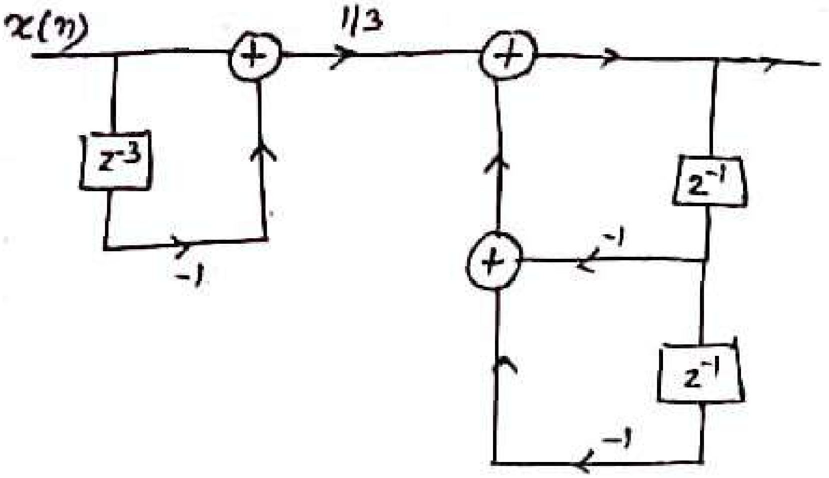
$-2\cos\left(\frac{2\pi}{3}\right) = -1$

$$H(z) = \left[\frac{1-z^{-3}}{3}\right] \cdot \left[\frac{-0.5+0.5z^{-1}}{1+z^{-1}+z^{-2}}\right] + \left[\frac{H(0)}{1-z^{-1}}\right]$$

$1+z^{-1}+z^{-2}$

$H(0) = 2.$

$$H(z) = \left[\frac{1-z^{-3}}{3}\right] \cdot \left[\frac{-0.5+0.5z^{-1}}{1+z^{-1}+z^{-2}} + \frac{2}{1-z^{-1}}\right]$$



Ques. Conversion of non-recursive structures to Recursive structures -

$$H(z) = \sum_{n=0}^{m-1} h(n) \cdot z^{-n} = \sum_{n=0}^{m-1} b_n \, z^{-n}$$

$b_0 = b_1 = b_2 = b_3 \ldots\ldots = b_n = b$

all the coefficients should be same.

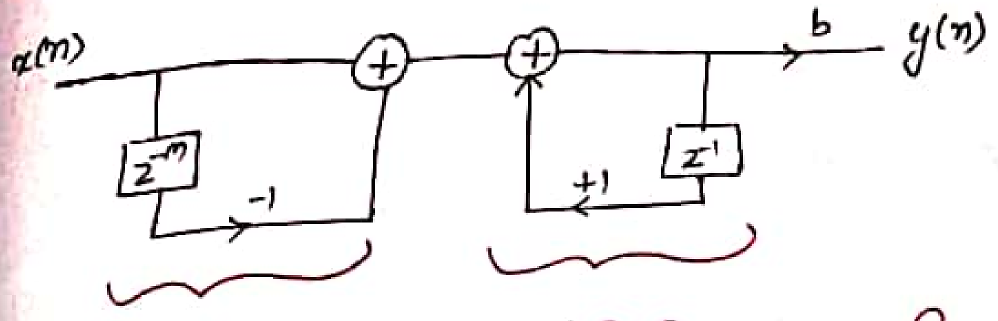
$\therefore$

$$H(z) = \sum_{n=0}^{m-1} h(n) \cdot z^{-n} = \sum_{n=0}^{m-1} b \cdot z^{-n}$$

$$= b\left[1 + z^{-1} + z^{-2} + z^{-3} + \ldots\ldots z^{-(m-1)}\right]$$

$$= b \cdot \left(\frac{1 - z^{-m}}{1 - z^{-1}}\right)$$

$$= (1 - z^{-m}) \cdot \frac{b}{(1 - z^{-1})}$$



This is called as Comp filter. There's no feedback here, hence this is the Non-recursive part.

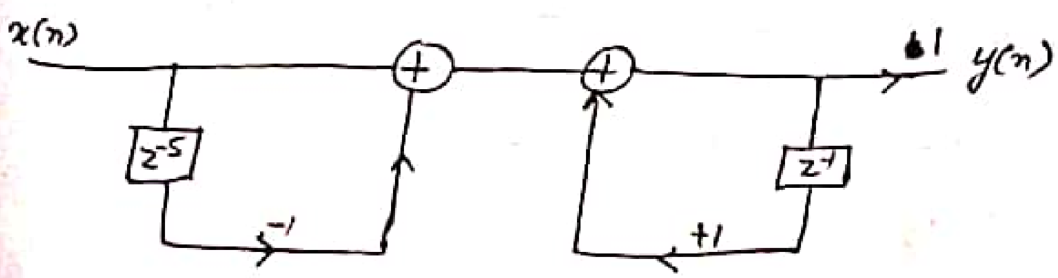
This is called Resonators filters. There are feedback hence, this is the Recursive part.

Ques. Convert the non-recursive structure of -

$$H(z) = 1 + z^{-1} + z^{-2} + z^{-3} + z^{-4}$$

All the coefficients are same hence, it can be converted into the summation of Recursive and non-recursive part.

$$H(z) = \frac{1 - z^{-5}}{1 - z^{-1}}$$



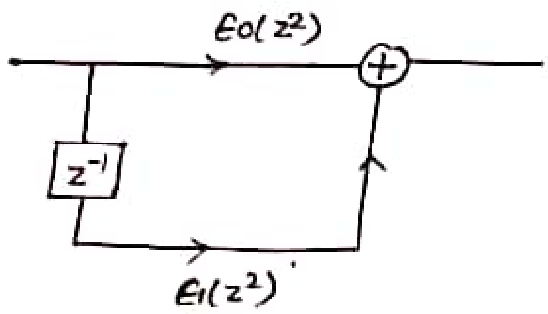
• Poly Phase Structures -

$$H(z) = \sum_{n=0}^{m-1} h(n).\, z^{-n}$$

Dividing $h(n)$ into two parts as even and odd.

$$H(z) = \sum_{n=0}^{\frac{m-1}{2}} h(2n)\, z^{-2n} + \sum_{n=0}^{\frac{m}{2}-1} h(2n+1)\, z^{-(2n+1)}$$

$$= E_0.(z^2) + z^{-1} E_1.\,(z^2)$$



Suppose $m = 9$,

$$H(z) = \sum_{n=0}^{8} h(n)\, z^{-n}$$

$$= h(0). + h(1).z^{-1} + h(2).z^{-2} + h(3)\, z^{-3} + h(4)\, z^{-4} + h(5)\, z^{-5} + h(6)\, z^{-6} + h(7)\, z^{-7} + h(8).\, z^{-8}$$

E_0 represents the Even terms;

$$E(0) = h(0) + h(2) z^{-2} + h(4) z^{-4} + h(6) z^{-6} + h(8) z^{-8}$$

E_1 represents the odd terms;

$$E(1) = h(1) z^{-1} + h(3) z^{-3} + h(5)\, z^{-5} + h(7) z^{-7}$$

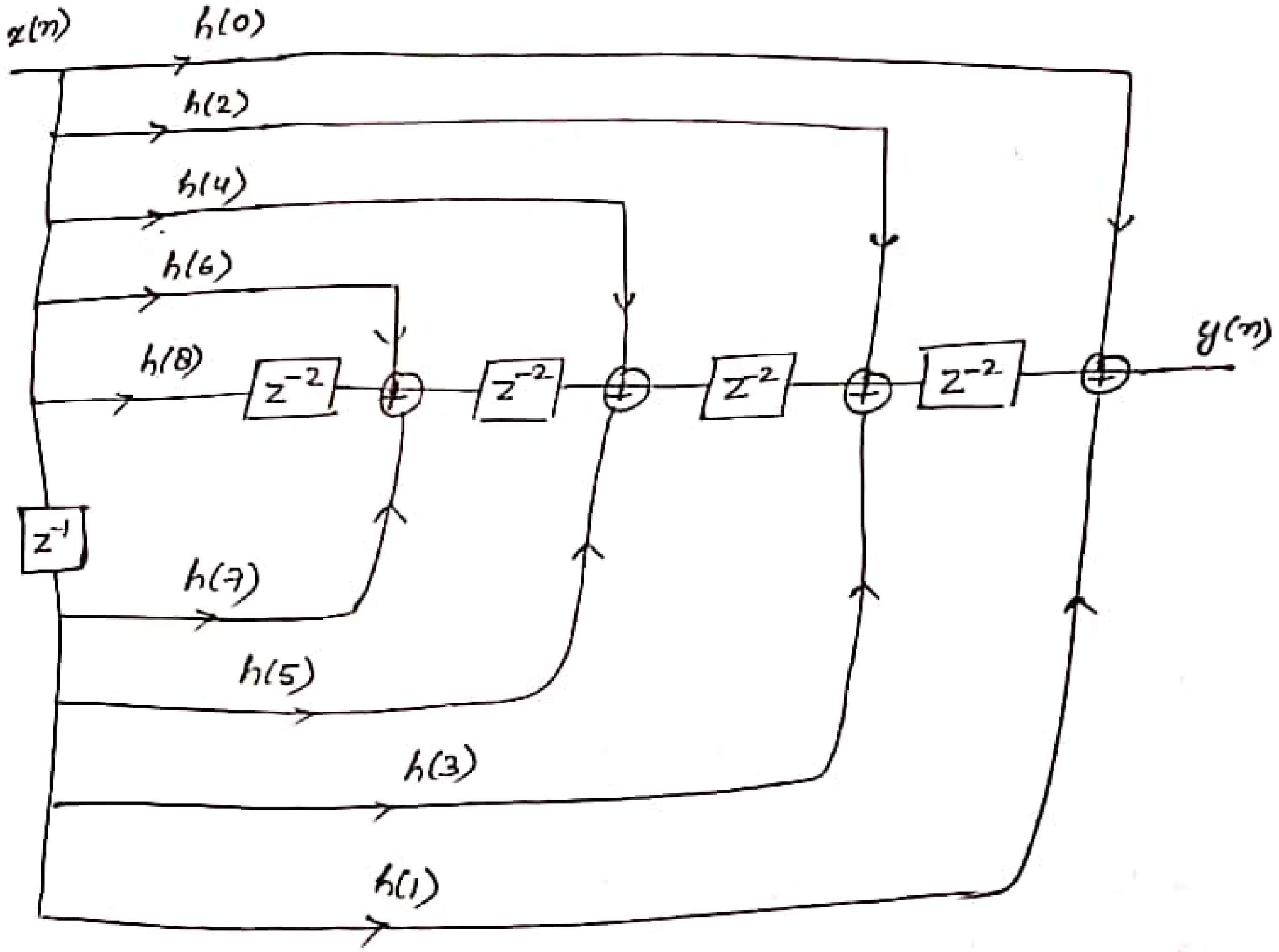
$$E(1) = z^{-1}\left[h(1) + h(3) \cdot z^{-2} + h(5) z^{-4} + h(7) z^{-6} \right]$$

All the powers in $E(0)$ and $E(1)$ are the multiples of 2.

$\therefore$

$$H(z) = E_0(z^2) + z^{-1} E_1(z^2)$$

$$H(z) = h(0) + h(2) z^{-2} + h(4) \cdot z^{-4} + h(6) z^{-6} + h(8) z^{-8} + z^{-1}\left[h(1) + h(3) \cdot z^{-2} + h(5) z^{-4} + h(7) z^{-6} \right]$$



Applications -

↳ Used in hardware utilisation.

↳ Used as multirate process.

$$H(z) = \frac{\sum_{k=0}^{m-1} b_k . z^{-k}}{1 + \sum_{k=1}^{N-1} a_k . z^{-k}} \quad \text{——— (1)}$$

There are two forms in this -

↳ Direct I form

(we consider zeroes first and then the poles)

↳ Direct II form.

(we consider poles first and then the zeroes.)

- **Direct - I form -**

$$H(z) = \frac{Y(z)}{X(z)} = \underbrace{\frac{W(z)}{X(z)}}_{\text{Zeroes}} \cdot \underbrace{\frac{Y(z)}{W(z)}}_{\text{Poles}}$$

$$\frac{W(z)}{X(z)} = \sum_{k=0}^{m-1} b_k . z^{-k} \quad \{\text{from eq}^n (1)\}$$

suppose , $m = 4$

$$= \sum_{k=0}^{3} b_k . z^{-k}$$

$$\frac{W(z)}{X(z)} = b_0 + b_1 z^{-1} + b_2 z^{-2} + b_3 z^{-3}$$

$$\Rightarrow \quad W(z) = [b_0 + b_1 z^{-1} + b_2 z^{-2} + b_3 . z^{-3}] . X(z)$$

Taking its inverse;

$$w(n) = b_0 x(n) + b_1 . x(n-1) + b_2 . x(n-2) + b_3 x(n-3) \quad \text{———} (2)$$

$$\frac{Y(z)}{W(z)} = \frac{1}{1 + \sum_{k=1}^{N-1} a_k . z^{-k}} \quad \{ \text{from eq}^n (1) \}$$

suppose, $N = 4$

$$\Rightarrow Y(z) . [1 + \sum_{k=1}^{3} a_k . z^{-k}] = W(z)$$

$$\Rightarrow Y(z) . [1 + a_1 z^{-1} + a_2 z^{-2} + a_3 z^{-3}] = W(z)$$

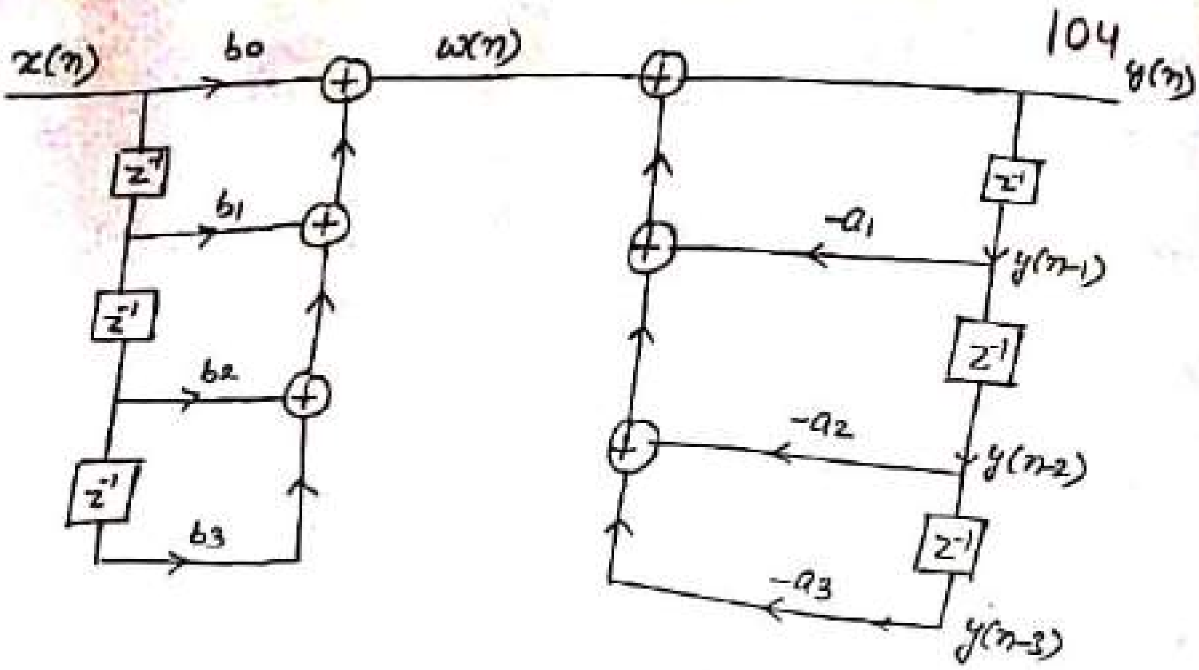
$$\Rightarrow Y(z) + a_1 z^{-1} Y(z) + a_2 z^{-2} Y(z) + a_3 z^{-3} . Y(z) = W(z)$$

Taking its inverse;

$$w(n) = y(n) + a_1 y(n-1) + a_2 y(n-2) + a_3 . y(n-3)$$

$$\Rightarrow \quad y(n) = w(n) - a_1 y(n-1) - a_2 . y(n-2) - a_3 . y(n-3) . \quad \text{———} (3)$$

IIR systems are always Recursive i.e, they always have a feedback.



No. of multipliers used in this will be = $M+N-1$

No. of delays used in this will be = $M+N-2$

No. of adders used in this will be = $M+N-2$

6th April, 19

$$H(z) = H_1(z) \cdot H_2(z)$$

$$= \underbrace{\frac{W(z)}{X(z)}}_{\text{Pole}} \cdot \underbrace{\frac{Y(z)}{W(z)}}_{\text{zero}}$$

$$= \frac{\sum_{k=0}^{M-1} b_k \cdot z^{-k}}{1 + \sum_{k=1}^{M-1} a_k \cdot z^{-k}} \qquad (M=4)$$

for pole,

$$\frac{W(z)}{X(z)} = \frac{1}{1 + a_1 z^{-1} + a_2 z^{-2} + a_3 z^{-3}}$$

$$W(z) + a_1 z^{-1} W(z) + a_2 \cdot z^{-2} W(z) + a_3 z^{-3} W(z) = X(z)$$

$$w(n) + a_1 w(n-1) + a_2 w(n-2) + a_3 w(n-3) = x(n)$$

(from time shifting property)

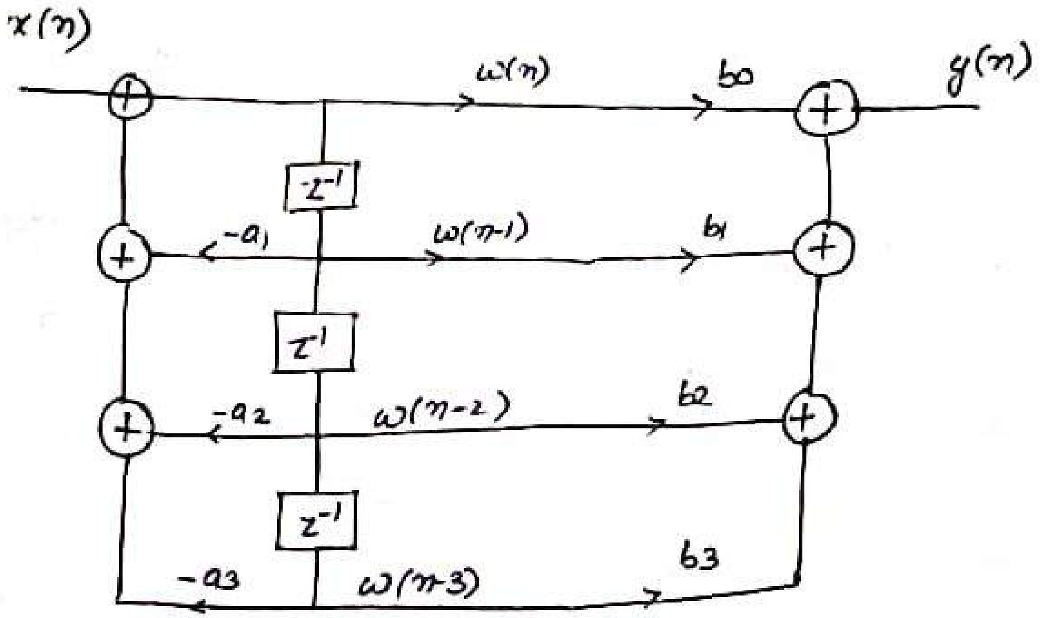
$$w(n) = x(n) - a_1 w(n-1) - a_2 w(n-2) - a_3 w(n-3) \quad \text{—— (1)}$$

for zero,

$$\frac{Y(z)}{W(z)} = \sum_{k=0}^{M-1} w_k \cdot z^{-k} \qquad (M = 4)$$

$$= b_0 + b_1 z^{-1} + b_2 \cdot z^{-2} + b_3 z^{-3}$$

$$y(n) = b_0 \cdot w(n) + b_1 \cdot w(n-1) + b_2 \cdot w(n-2) + b_3 \cdot w(n-3)$$



No. of multipliers = 7

No. of delays = max. of M-1

Ques: Obtain Direct-I form and Direct-II form for $H(z) = \dfrac{1+2z^{-1}-z^{-2}}{1+z^{-1}-z^{-2}}$ 106

Imp

- **Direct-II Form-**

$$H(z) = \frac{1+2z^{-1}-z^{-2}}{1+z^{-1}-z^{-2}} = \underbrace{\frac{W(z)}{X(z)}}_{\text{for pole}} \cdot \underbrace{\frac{Y(z)}{W(z)}}_{\text{for zero}}$$

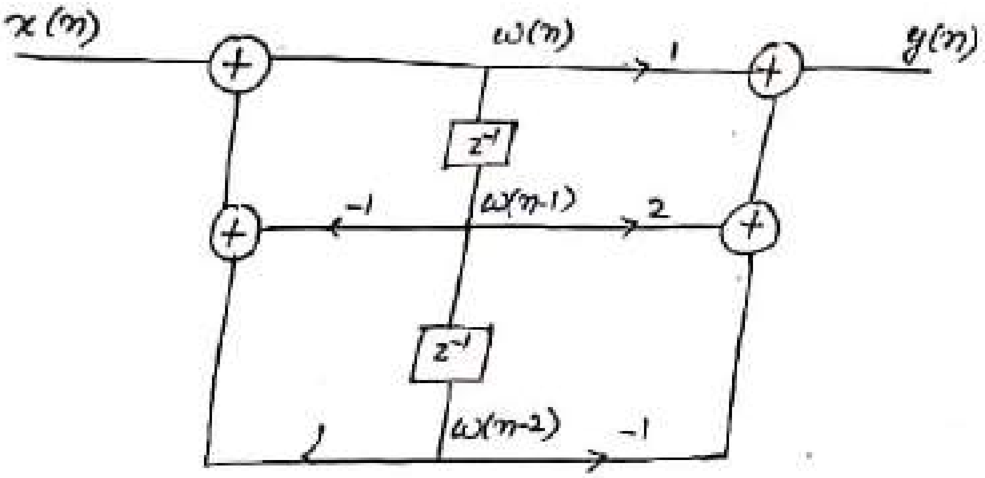
$$\frac{W(z)}{X(z)} = \frac{1}{1+z^{-1}-z^{-2}}$$

$$\frac{Y(z)}{W(z)} = 1+2z^{-1}-z^{-2}$$

$$w(n) = x(n) - w(n-1) + w(n-2)$$

and,

$$y(n) = w(n) + 2w(n-1) - w(n-2)$$



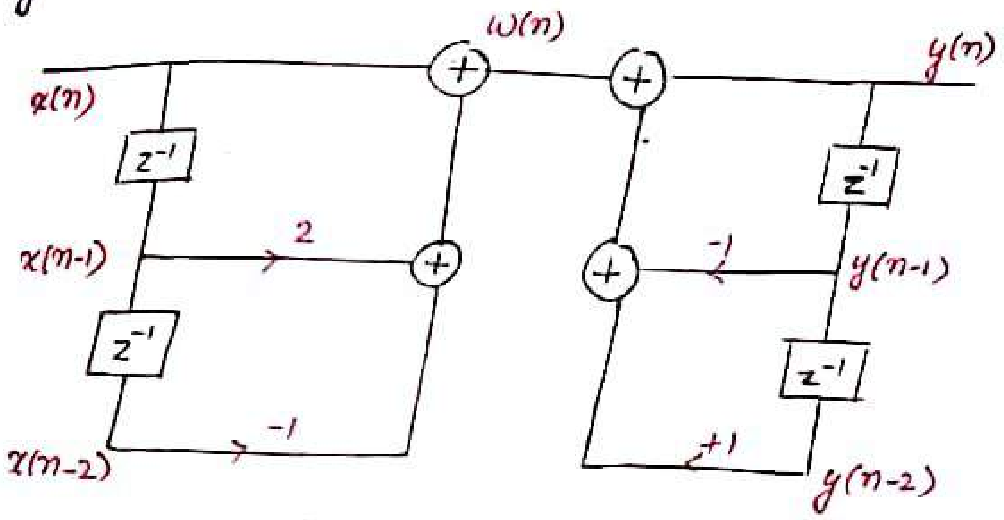
- Direct I form - (reverse of direct-II form)

$$\frac{W(z)}{X(z)} = 1 + 2z^{-1} - z^{-2}$$

$$\frac{Y(z)}{W(z)} = \frac{1}{1+z^{-1}-z^{-2}}$$

$$w(n) = x(n) + 2x(n-1) - x(n-2)$$

$$y(n) = w(n) - y(n-1) + y(n-2)$$



multiplier = 5
delay = 4

- Cascade Form -

$$H(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \dots\dots + b_{M-1} z^{-(M-1)}}{1 + a_1 z^{-1} + a_2 z^{-2} + \dots\dots + a_{N-1} z^{-(N-1)}}$$

$$b_0 \prod_{k=1}^{\frac{N-1}{2}} \frac{(1 + B_{1k} z^{-1} + B_{2k} z^{-2})}{(1 + A_{1k} z^{-1} + A_{2k} z^{-2})} \quad ; \; N \text{ is odd}$$

$$= b_0 \frac{(1 + b_{10} z^{-1})}{(1 + a_{10} z^{-1})} \prod_{k=1}^{\frac{N-2}{2}} \left(\frac{1 + B_{1k} z^{-1} + B_{2k} z^{-2}}{1 + A_{1k} z^{-1} + A_{2k} z^{-2}} \right) ; \; N \text{ is even}$$

Ques. Obtain the Cascade structure of -

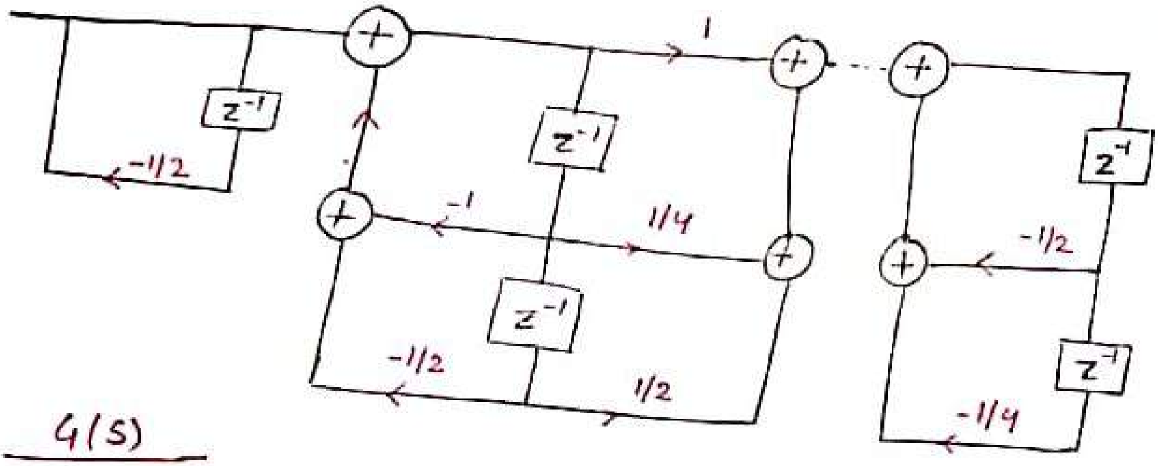
$$H(z) = \frac{1+\frac{1}{4}z^{-1}+\frac{1}{2}z^{-2}}{(1+\frac{1}{2}z^{-1})\cdot(1+z^{-1}+\frac{1}{2}z^{-2})\ (1+\frac{1}{2}z^{-1}+\frac{1}{4}z^{-2})}$$

N is even = 6

$b_0 = 1$

$b_{10} = 0$

$$H(z) = \frac{1}{(1+\frac{1}{2}z^{-1})}\cdot\frac{(1+\frac{1}{4}z^{-1}+\frac{1}{2}z^{-2})}{(1+z^{-1}+\frac{1}{2}z^{-2})}\cdot\frac{1}{(1+\frac{1}{2}z^{-1}+\frac{1}{4}z^{-2})}$$



$$\frac{G(S)}{1+G(S)\cdot H(S)}$$

$G(S) = 1$

$H(S) = \frac{1}{2}z^{-1}$

• Parallel Form Structure :-

$$H(z) = \frac{b_0+b_1z^{-1}+b_2z^{-2}+\dots\dots+b_{m-1}z^{-(m-1)}}{1+a_1z^{-1}+a_2z^{-2}+\dots\dots+a_{N-1}.z^{-(N-1)}}$$

When, $m<N$

where m is the maximum power in the numerator

N is the maximum power in the denominator.

$$H(z) = \begin{cases} \sum_{k=1}^{\frac{N-1}{2}} \frac{B_{0k} + B_{1k} z^{-1}}{1 + A_{1k} z^{-1} + A_{2k} z^{-2}} & ; N \text{ is odd.} \\ \frac{b_{10}}{1 + a_{10} z^{-1}} + \sum_{k=1}^{\frac{N-2}{2}} \frac{B_{0k} + B_{1k} z^{-1}}{1 + A_{1k} z^{-1} + A_{2k} z^{-2}} & ; N \text{ is even} \end{cases}$$

When $m > N$,

$$H(z) = \begin{cases} \sum_{k=0}^{m-N} C_k . z^{-k} + \sum_{k=1}^{\frac{N-1}{2}} \frac{B_{0k} + B_{1k} z^{-1}}{1 + A_{1k} z^{-1} + A_{2k} z^{-2}} & ; N \text{ is odd} \\ \sum_{k=0}^{m-N} C_k z^{-k} + \frac{b_{10}}{1 + a_{10} z^{-1}} \sum_{k=1}^{\frac{N-2}{2}} \frac{B_{0k} + B_{1k} z^{-1}}{1 + A_{1k} z^{-1} + A_{2k} z^{-2}} & ; N \text{ is even} \end{cases}$$

When $m = N$,

there will be a constant term or value i.e C_k.

Ques. $$H(z) = \frac{1 + \frac{1}{4} z^{-1}}{\left(1 + \frac{1}{2} z^{-1}\right)\left(1 + \frac{1}{2} z^{-1} + \frac{1}{4} z^{-2}\right)}$$ $m = 2$, $N = 4$

$$H(z) = \frac{b_{10}}{1 + a_{10} z^{-1}} + \sum_{k=1}^{\frac{N-2}{2}} \frac{B_{0k} + B_{1k} z^{-1}}{1 + A_{1k} z^{-1} + A_{2k} z^{-2}}$$

$$= \frac{b_{10}}{1 + a_{10} z^{-1}} + \frac{B_{01} + B_{11} z^{-1}}{1 + A_{11} z^{-1} + A_{21} z^{-2}}$$

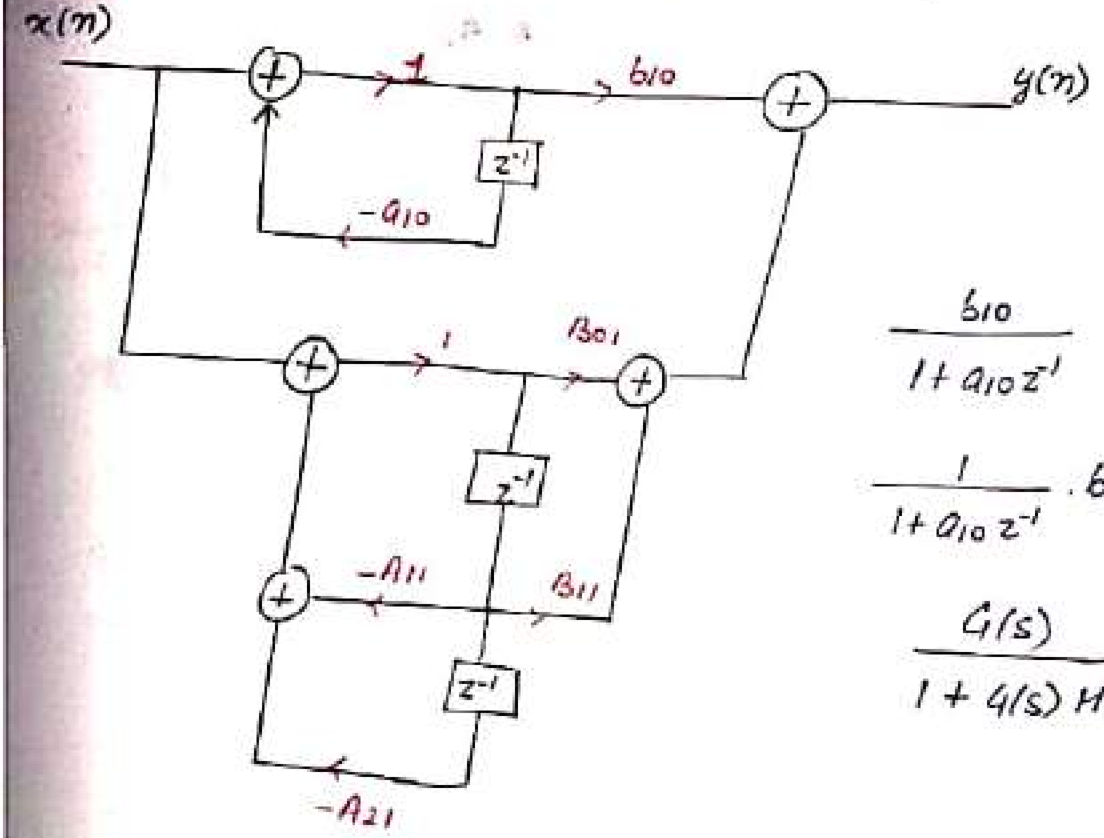
$$= \frac{b_{10}\left[1 + A_{11} z^{-1} + A_{21} z^{-2}\right] + \left[B_{01} + B_{11} z^{-1}\right].\left[1 + a_{10} z^{-1}\right]}{\left(1 + a_{10} z^{-1}\right).\left(1 + A_{11} z^{-1} + A_{21} z^{-2}\right)}$$

$$= \frac{b_{10} + b_{10} A_{11} z^{-1} + b_{10} A_{21} z^{-2} + B_{01} + B_{11} z^{-1} + B_{10} a_{10} z^{-1} + B_{11} a_{10} z^{-2}}{\left(1 + a_{10} z^{-1}\right)\left(1 + A_{11} z^{-1} + A_{21} z^{-2}\right)}$$

$A_{11} = 1/2$ $\qquad a_{11} = 1/2$

$A_{21} = 1/4$

$$= \frac{b_{10} + b_{10}\frac{1}{2}z^{-1} + b_{10}\cdot\frac{1}{4}z^{-2} + B_{01}\cdot(1) + B_{10}\cdot\frac{1}{2}z^{-1} + B_{11}z^{-1} + B_{11}\frac{1}{2}z^{-2}}{\left(1+\frac{1}{2}z^{-1}\right)\cdot\left(1+\frac{1}{2}z^{-1}+\frac{1}{4}z^{-2}\right)}$$



$$\frac{b_{10}}{1+a_{10}z^{-1}}$$

$$\frac{1}{1+a_{10}z^{-1}}\cdot b_{10}$$

$$\frac{G(s)}{1+G(s)H(s)}\cdot b_{10}$$

Polyphase Structure :-

$$H(z) = E_0(z^2) + z^{-1} E_1(z^2)$$

$E_0(z^2)$ represents the even part of $X(z)$

$z^{-1}E_1(z^2)$ represents the odd part of $X(z)$.

Ques. Obtain the polyphase structure of -

$$H(z) = \frac{1-2z^{-1}}{1+3z^{-1}}$$

$$x(t) = x_e + x_o$$

$$x_e = \frac{x(t) + x(-t)}{2}$$

$$x_o = \frac{x(t) - x(-t)}{2}$$

$$\therefore \quad E_0(z^2) = \frac{H(z) + H(-z)}{2} = \frac{\frac{1-2z^{-1}}{1+3z^{-1}} + \frac{1+2z^{-1}}{1-3z^{-1}}}{2}$$

$$= \frac{(1-2z^{-1}).(1-3z^{-1}) + (1+3z^{-1})(1+2z^{-1})}{2.(1+3z^{-1}).(1-3z^{-1})}$$

$$= \frac{1-3z^{-1}-2z^{-1}+6z^{-1}+1+2z^{-1}+3z^{-1}+6z^{-1}}{2.[(1)^2-(3z^{-1})^2]}$$

$$= \frac{2+12z^{-2}}{2.[1-9z^{-2}]} = \frac{1+6z^{-2}}{1-9z^{-2}}$$

$$z^{-1}E_1(z^2) = \frac{H(z) - H(-z)}{2}$$

$$= \frac{\dfrac{1-2z^{-1}}{1+3z^{-1}} - \dfrac{1+2z^{-1}}{1-3z^{-1}}}{2}$$

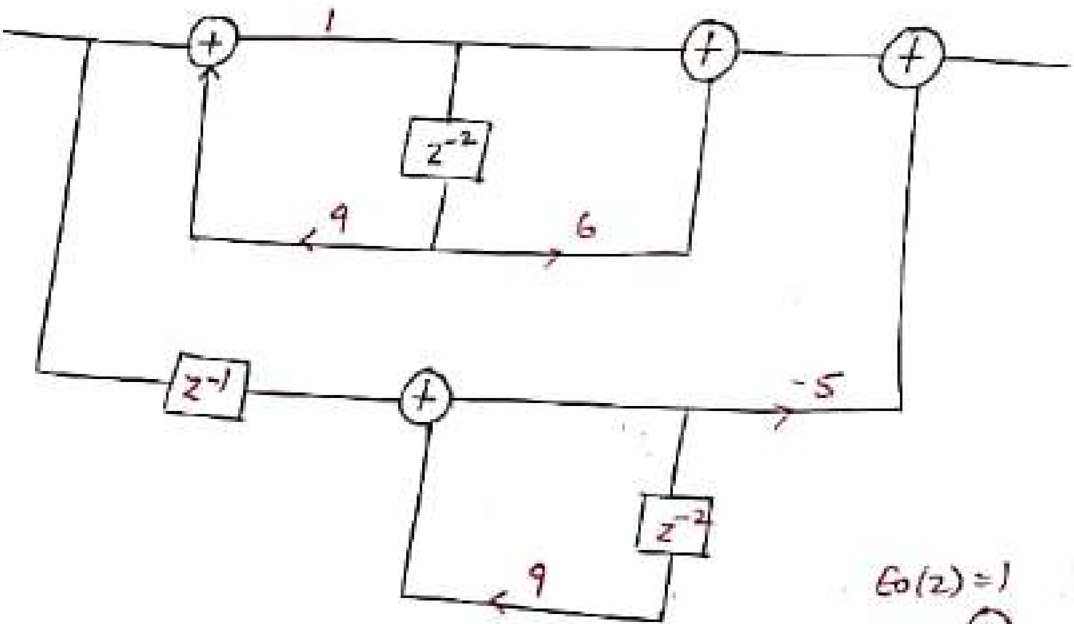
$$= \frac{(1-2z^{-1})\cdot(1-3z^{-1}) - (1+2z^{-1})\cdot(1+3z^{-1})}{2\left[(1)^2 - (3z^{-1})^2\right]}$$

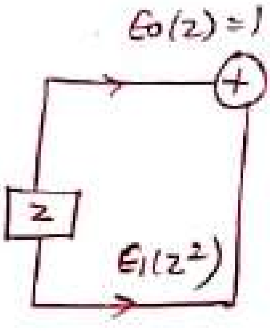
$$= \frac{1-3z^{-1}-2z^{-1}+6z^{-2}-1-2z^{-1}-3z^{-1}-6z^{-2}}{2\left[1-9z^{-2}\right]}$$

$$z^{-1}E_1(z^2) = \frac{-5z^{-1}}{1-9z^{-2}}$$

$$\therefore \quad H(z) = E_0(z^2) + z^{-1}E_1(z^2)$$

$$= \frac{1+6z^{-2}}{1-9z^{-2}} + \left(\frac{-5z^{-1}}{1-9z^{-2}}\right)$$



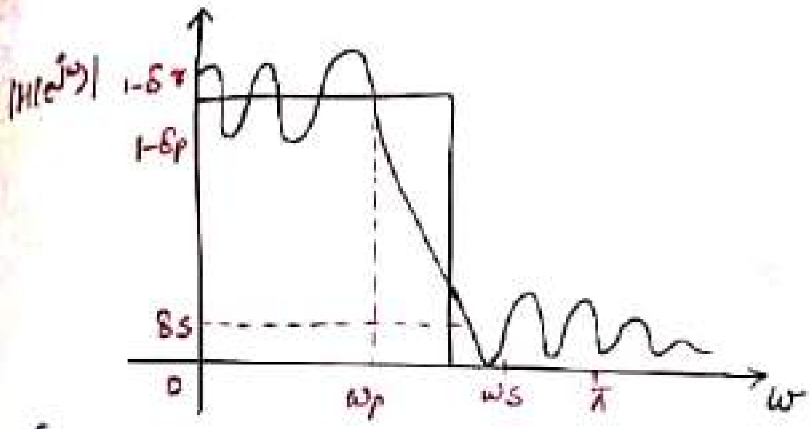


8th April, 19

Filter Specifications -

(A) Absolute Specification -

(for FIR)



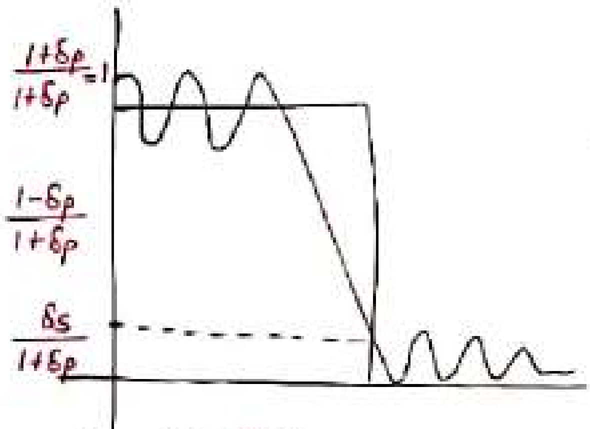
δ_p

δ_s $\quad \Delta\omega = \omega_s - \omega_p$

$$1-\delta_p \le |H(e^{j\omega})| \le 1+\delta_p \qquad 0 \le \omega \le \omega_p$$

$$|H(e^{j\omega})| \le \delta_s \qquad \omega_s \le \omega \le \omega_p$$

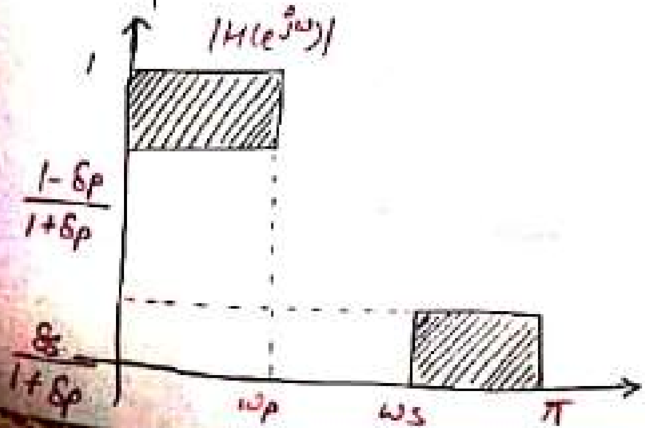
(B) Relative Specification -



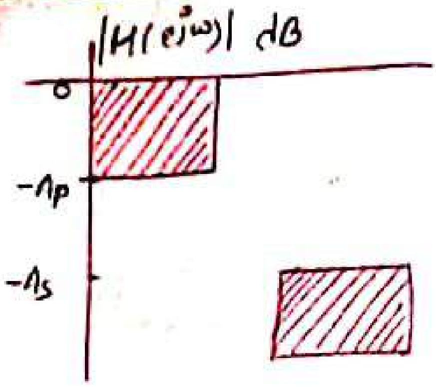
$$\frac{1-\delta_p}{1+\delta_p} \le |H(e^{j\omega})| \le 1$$

$$0 \le \omega \le \omega_p$$

$$|H(e^{j\omega})| \le \frac{\delta_s}{1+\delta_p}$$



$$\omega_s \le \omega \le \pi$$



$$A_p = 20 \log \left(\frac{1+\delta_p}{1-\delta_p}\right)$$

$$A_s = 20 \log \left(\frac{1+\delta_p}{\delta_s}\right) \approx -20 \log \delta_s$$

$$\delta_p = \frac{10^{A_p/20} - 1}{10^{A_p/20} - 1}$$ and, $$\delta_s = 10^{-A_s/20}$$

- ## Analog Filter Specification-

(for IR)

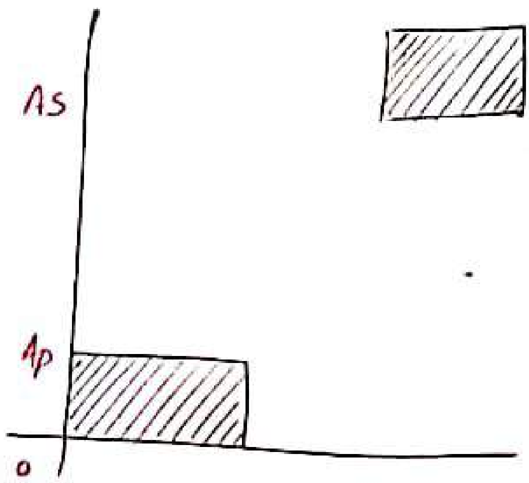
* Attenuation in pass band is minimum.

* Attenuation in stop band is maximum.

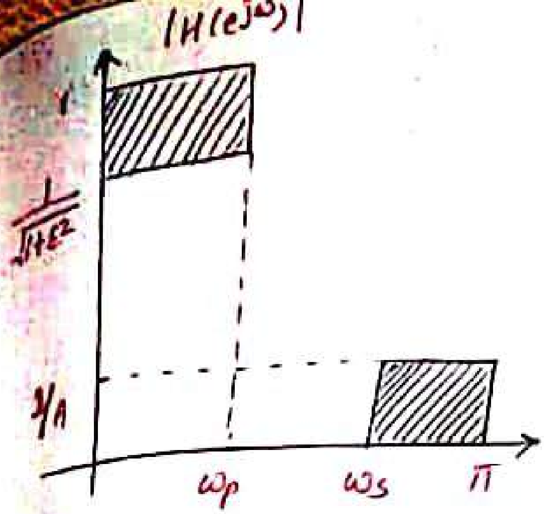
$\epsilon \rightarrow$ Pass band ripple factor

$\delta_s \rightarrow$ Stop band

$\delta_p \rightarrow$ Pass band tolerance



$A \rightarrow$ Stop band ripple factor



Analog filter Specifications

$$\delta_1 = \frac{1-\delta_p}{1+\delta_p} = \frac{1}{\sqrt{1+\varepsilon^2}} = \quad \varepsilon = \sqrt{\frac{1}{\delta_1^2} - 1}$$

$$\frac{1}{A} = \frac{\delta_s}{1+\delta_p} \text{ is } \delta_2$$

$$\left\{ \begin{array}{l} \delta_1 = \dfrac{1-\delta_p}{1+\delta_p} \\ \delta_2 = \dfrac{\delta_s}{1+\delta_p} \end{array} \right.$$

$$A_p = 20 \log \left(\sqrt{1+\varepsilon^2}\right)$$

$$A_s = 20 \log (A)$$

$$\varepsilon = \sqrt{10^{A_p/10} - 1}$$

$$A = 10^{A_s/20}$$

- **FIR Filter Designing -**

(1) **Windowing Method -**

Infinite sequence are converted into finite sequence.

(a) **Rectangular windowing method -**

Signal is multiplied by the rectangular window. It increases oscillations and ripples in pass and stop band. It is called Gibb's effect.

Hd(n) is the input of infinite range.

$Hd(n).\omega(n)$

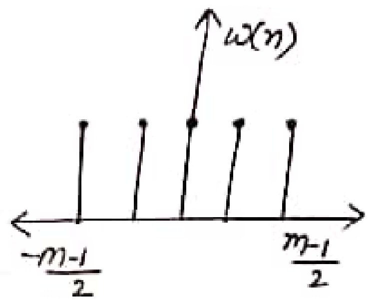
$$Hm(n) = Hd(n).\,\omega(n)$$

Hm(n) → range is finite

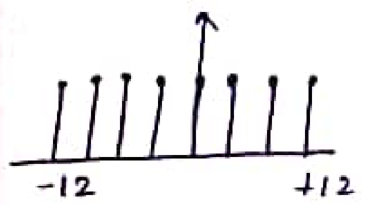
In terms of frequency,

$$Hm(e^{j\omega}) = Hd(e^{j\omega}) \circledast \omega(e^{j\omega})$$ Periodic convolution

$$= \frac{1}{2\pi}\int Hd(e^{j\omega}).\,\omega(e^{j(\omega-\theta)})\,d\theta$$



Assume the length of the filter is 25.



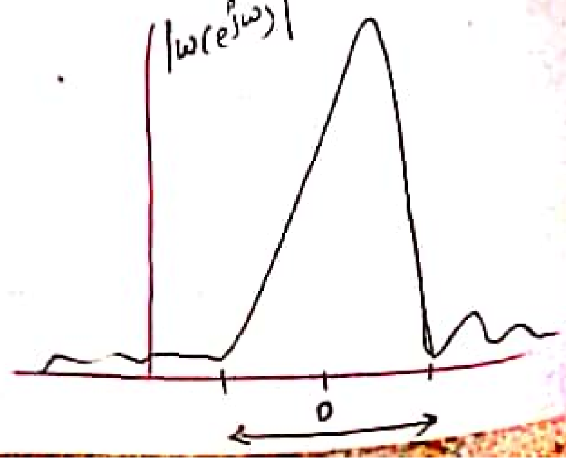
$$\omega(e^{j\omega}) = \sum_{n=-\left(\frac{m-1}{2}\right)}^{\frac{m-1}{2}} 1.\ e^{-j\omega n}$$

$$\Rightarrow k = n + \left(\frac{m-1}{2}\right)$$

$$\therefore \omega(e^{j\omega}) = \sum_{k=0}^{m-1} e^{-j\omega\left[k - \frac{m-1}{2}\right]}$$

$$= e^{+j\omega\left(\frac{m-1}{2}\right)} \sum_{k=0}^{m-1} e^{-j\omega k}$$

$$\omega(e^{j\omega})^{*} = \frac{\sin\frac{\omega m}{2}}{\sin\frac{\omega}{2}}$$



To find the transition width –

$$\sin \frac{\omega m}{2} = 0 = \sin n\pi$$

$n = \pm 1, \pm 2, \pm 3, \ldots$

$n = 1$

$$\frac{\omega m}{2} = n\pi$$

$$\boxed{\omega = \frac{2\pi}{m}}$$

Transition width, $D = 2\omega = \frac{4\pi}{m}$, where m is the length of the filter

$$\sin \frac{\omega m}{2} = 1 = \sin \frac{n\pi}{2}$$

$n = \pm 3, \pm 5, \pm 7, \ldots$

For $n = 3$,

$$\frac{\omega m}{2} = \frac{3\pi}{2}$$

$$\boxed{\omega = \frac{3\pi}{m}}$$; for maximum side lobe

$$|w(e^{j\omega})|_{\omega = \frac{3\pi}{m}} = \frac{\sin \frac{3\pi}{m} \cdot \frac{m}{2}}{\sin \frac{3\pi}{m2}} = \frac{\sin \frac{3\pi}{2}}{\sin \frac{3\pi}{m2}}$$

where, m is large

$$= \frac{1}{(3\pi/m2)}$$

$$\boxed{|w(e^{j\omega})|_{\omega = \frac{3\pi}{m}} = \frac{2m}{3\pi}}$$

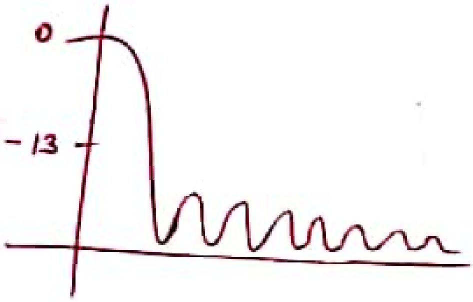
Magnitude of first side lobe.

Magnitude of the main lobe = m.

Decrement in lobe in respect to the main lobe

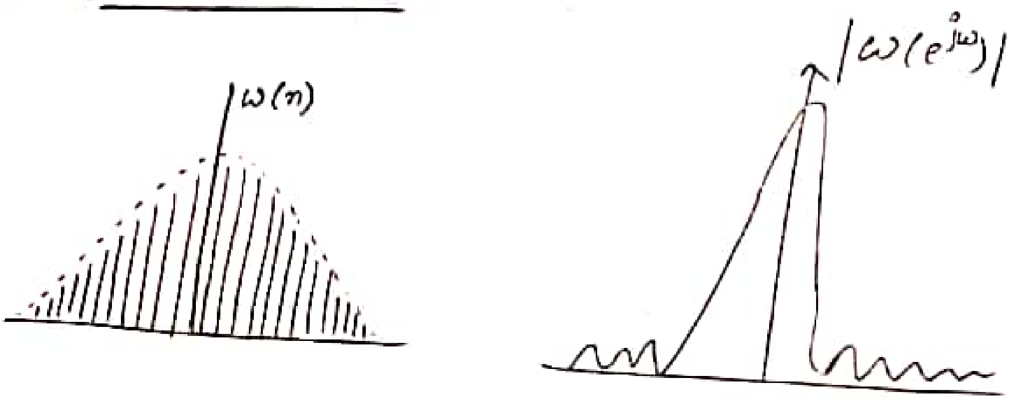
$$= \frac{\text{side lobe}}{\text{main lobe}} \times 100$$

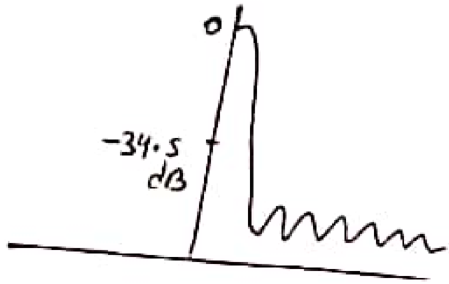
$$= \frac{\left(\frac{2m}{3\pi}\right)}{m} \times 100 = \frac{2}{3\pi} \times 100 = -13\,dB$$



On increasing the values, fluctuations will be lesser.

(b) Kaiser Window -





$$w(n) = \begin{cases} \dfrac{I_0\left[\beta\sqrt{1-\left(1-\frac{2n}{m-1}\right)^2}\right]}{I_0(\beta)} & ; \ 0 \le n \le m-1 \ \text{(for causal)} \\ 0 & ; \ \text{otherwise} \end{cases}$$

$$w(n) = \begin{cases} \dfrac{I_0\left[\beta\sqrt{1-\left(\dfrac{2n}{m-1}\right)^2}\right]}{I_0(\beta)} & -\dfrac{m-1}{2} \le n \le \dfrac{m-1}{2} \end{cases}$$

$$I_0(\beta) = 1 + \sum_{\beta=1}^{\infty}\left[\frac{1}{\beta!}\left(\frac{x}{2}\right)^k\right]^2$$

Filter designing using Keiser Window -

(1) Find δ_p and δ_s

$$\delta_p = \frac{10^{A_p/20} - 1}{10^{A_p/20} + 1} \quad \text{and} \quad \delta_s = 10^{-A_s/20}$$

$$\delta = \min(\delta_p, \delta_s)$$

here in, δ_s is always less.

(2) Find attenuation

$$\boxed{A = -20 \log_{10} \delta}$$

(3) Find transition width

$$\Delta\omega = \omega_s - \omega_p$$

$$\Delta f = \frac{\omega_s - \omega_p}{2\pi}$$

(4) Find m.

$$m \geq \begin{cases} \dfrac{A - 7.95}{14.36\,\Delta f} + 1 & A \geq 21dB \\ \dfrac{0.922}{\Delta f} + 1 & A < 21dB \end{cases}$$

(5) To find β

$$\beta = \begin{cases} 0 & A \le 21 \\ 0.5842(A-21)^{0.4} + 0.07886(A-21) & 21 < A < 50 \\ 0.1102(A-8.7) & A > 50 \end{cases}$$

(6) Compute $w(n)$

(7) Determine the desired impulse response $Hd(n)$ using inverse DTFT

(8) $Hm(n) = H_d(n) \cdot w(n)$

$h(n) = h(m-1-n)$

9th April, 19

Ques. Design a low pass filter, with the following specifications - $0.99 < |H(e^{j\omega})| < 1.01$ $\quad |\omega| \le 0.4\pi$

and the magnitude of

$|H(e^{j\omega})| \le 0.01$ $\quad 0.6\pi < |\omega| \le \pi$

Using Keiser window.

Soln.

$1 - \delta_p = 0.99$ $\qquad 1 + \delta_p = 1.01$

$\delta_p = 1 - 0.99$ $\qquad \delta_p = 1.01 - 1$

$= 0.01$ $\qquad = 0.01$

$$\boxed{\delta_p = 0.01}$$

$|H(e^{j\omega})| \le \delta_s$

$\therefore$ $\boxed{\delta_s = 0.01}$

from $0 \le |\omega| \le 0.4\pi$

$\omega_p = 0.4\pi$

from $0.6\pi \le |\omega| < \pi$

$\omega_s = 0.6\pi$

$\delta = \min(\delta_p, \delta_s)$

$\boxed{\delta = 0.01}$

$A = -20 \log(0.01) = 40dB$

$\Delta\omega = \omega_s - \omega_p$

$\boxed{\Delta\omega = 0.2\pi}$

$$\Delta f = \frac{\omega_s - \omega_p}{2\pi} = \frac{0.2\pi}{2\pi} = 0.1$$

$$m \geq \frac{A - 7.95}{14.36\,\Delta f} + 1 = \frac{40 - 7.95}{14.36 \times 0.1} + 1$$

$$= \frac{32.05}{1.436} + 1 = 23.3189$$

$\boxed{m = 25}$

for causal, we can take odd and even both but for anti-causal only odd will be taken.
Generally, causal is taken.

$$\beta = 0.5842\,(40-21)^{0.4} + 0.07866\,(40-21)$$

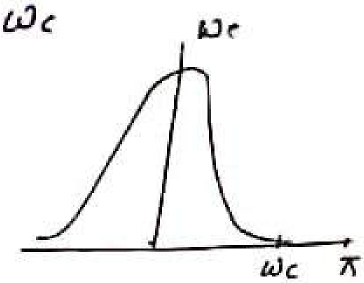
$25 < A \le 50$

$$= 0.5842\,(19)^{0.4} + 0.07866\,(19)$$

$$= 3.3953$$

$$\omega(n) = \frac{J_0\left(\beta\sqrt{1-\left(1-\frac{2n}{m-1}\right)^2}\right)}{J_0(\beta)} \quad 0 \le n \le m-1$$

$$H_m(e^{j\omega}) = \begin{cases} e^{-j\tau\omega} ; & \text{when } \omega \le \omega_c \\ 0 ; & \text{Otherwise} \end{cases}$$



$$\tau = \frac{m-1}{2}$$

By symmetry property, we will calculate from 0 to 12.

$h(0) = h(m-1-n)$

$h(0) = h(m-1-0)$

$h(0) = h(24)$

$h(1) = h(23)$

$$h_m(n) = \frac{1}{2\pi}\int_{-\omega_c}^{+\omega_c} e^{-j\tau\omega}\cdot e^{j\omega n}\cdot d\omega$$

$$= \frac{1}{2\pi}\int_{-\omega_c}^{+\omega_c} e^{j\omega(-\tau+n)}\cdot d\omega$$

$$= \frac{\sin 0.5\pi(n-12)}{\pi(n-12)}$$

$$\omega_c = \frac{1}{2}(\omega_p + \omega_s)$$

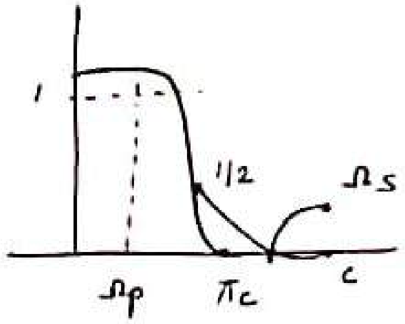
$$= \frac{1}{2}(0.4\pi + 0.6\pi)$$

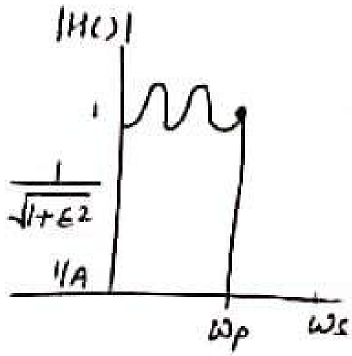
$$= 0.5\pi$$

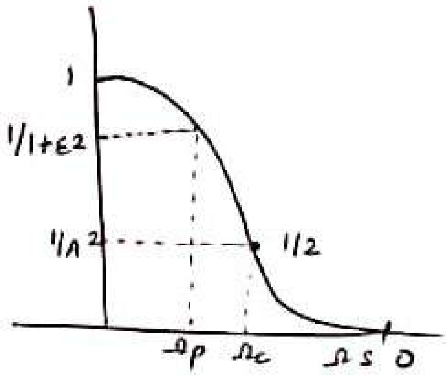
Analog Butterworth Filter (IIR filter) - 123

$$|H(j\Omega)|^2 = \frac{1}{1+\left(\frac{\Omega}{\Omega_c}\right)^{2N}}$$

zero oscillations







Ques1 Determination of filter parameters -

(a) N

(b) cut-off frequency, ω_c.

Case I : When specifications ϵ, A, ω_p, ω_s are given

At Ω_p,

$$|H(j\Omega_p)|^2 = \frac{1}{1+\left(\frac{\Omega_p}{\Omega_c}\right)^{2N}} = \frac{1}{1+\epsilon^2}$$

Similarly at Ω_s,

$$|H(e^{j\Omega_s})|^2 = \frac{1}{1+\left(\frac{\Omega_s}{\Omega_c}\right)^{2N}} = \frac{1}{A^2} \quad \text{(from graph)}$$

$$\varepsilon^2 = \left(\frac{\Omega_p}{\Omega_c}\right)^{2N}$$

$$A^2 - 1 = \left(\frac{\Omega_s}{\Omega_c}\right)^{2N}$$

$$\therefore \quad \frac{\varepsilon^2}{A^2-1} = \frac{\left(\frac{\Omega_p}{\Omega_c}\right)^{2N}}{\left(\frac{\Omega_s}{\Omega_c}\right)^{2N}}$$

$$\Rightarrow \frac{\varepsilon^2}{A^2-1} = \left(\frac{\Omega_p}{\Omega_s}\right)^{2N}$$

Taking log on both sides,

$$2N \log\left(\frac{\Omega_p}{\Omega_s}\right) = \log \frac{\varepsilon^2}{A^2-1}$$

$$N = \frac{1}{2} \log\left[\frac{\varepsilon^2}{A^2-1} - \frac{\Omega_p}{\Omega_s}\right]$$

Case-II : When specifications A_p, A_s, Ω_p and Ω_s are given

$$-A_p \le |H(j\Omega)|_{dB} \le 0$$

$$|H(j\Omega)|_{dB} \le -A_s$$

At Ω_p,

$$-A_p = 20 \log |H(j\Omega_p)|$$

At Ω_s,

$$-A_s = 20 \log |H(j\Omega_s)|$$

$$\Rightarrow -A_p = 10 \log |H(j\Omega_p)|^2$$

$$= 10 \log \left[\frac{1}{1+\left(\frac{\Omega_p}{\Omega_c}\right)^{2N}} \right)$$

$$= 10 \log \left(\frac{1}{1+\epsilon^2} \right)$$

$$-A_s = 10 \log |H(j\Omega_s)|^2$$

$$= 10 \log \left[\frac{1}{1+\left(\frac{\Omega_s}{\Omega_c}\right)^{2N}} \right]$$

$$= 10 \log \left(\frac{1}{A^2} \right)$$

$$\left(\frac{\Omega_p}{\Omega_c}\right)^{2N} = 10^{A_p/10} - 1 \quad \text{——— (1.)}$$

$$\left(\frac{\Omega_s}{\Omega_c}\right)^{2N} = 10^{A_s/10} - 1 \quad \text{——— (2.)}$$

$$\left(\frac{\Omega_p}{\Omega_s}\right)^{2N} = \frac{10^{A_p/10} - 1}{10^{A_s/10} - 1}$$

$$2N \cdot \log \frac{\Omega_p}{\Omega_s} = \log \left[\frac{10^{A_p/10} - 1}{10^{A_s/10} - 1} \right]$$

find Ω_c and N.

Ques. Determine the system function H(s) for first order Butterworth filter.

$$|H(j\omega)|^2 = H(j\omega) \cdot H(-j\omega) = \frac{1}{1+\left(\frac{\Omega}{\Omega_c}\right)^{2N}}$$

$$H(s) \cdot H(-s) = \frac{1}{1+\left(\frac{-s^2}{\Omega_c^2}\right)^N}$$

$$-s^{2N} + \Omega_c^{2N} = 0$$

$$-s^{2N} = -\Omega_c^{2N}$$

When N is even,

$$(-1)^N \cdot s^{2N} = e^{j(2k+1)\pi} \cdot \Omega_c^{2N}$$

$k = 0 \ldots\ldots (2N-1)$

$$s^{2N} = e^{j(2k+1)\pi} \cdot \Omega_c^{2N}$$

$k = 0 \ldots\ldots (2N-1)$

$$P_k = \Omega_c \cdot e^{j\left(\frac{2k+1}{2N}\right)\pi}$$

$$-s^2 = -e^{j\frac{(2k+1)}{2N}\pi} \cdot \Omega_c^{\frac{2N}{2N}} = \Omega_c \cdot e^{j\frac{k\pi}{N}} \text{ (for N is odd.)}$$

$$H(s) = \frac{\Omega_c^N}{\pi (s - P_k)}$$

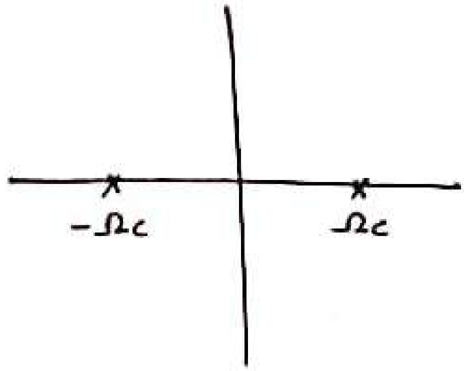
↳ only left sided poles.

for odd N, in this question;

$$P_k = \Omega_c \cdot e^{j\frac{k\pi}{N}}$$

$P_1 = \Omega_c$

$P_2 = \Omega_c\, e^{j\pi} = -\Omega_c$



The values of both the poles are real.

$$H(s) = \frac{\Omega_c^1}{[s-(-\Omega_c)]} = \frac{\Omega_c}{s+\Omega_c}$$

only the left pole

Ques: Determine the system function H(s) for second order Butterworth filter.

order = 2 i.e. N → Even.

$$s^{2N} = e^{j(2k+1)\pi} \cdot \Omega_c^{2N}$$

$$\Rightarrow P_k = \Omega_c \cdot e^{j\left(\frac{2k+1}{2N}\right)\cdot\pi}$$

$$= \Omega_c \cdot e^{j\left(\frac{2k\pi}{2N} + \frac{\pi}{2N}\right)}$$

$$= \Omega_c \cdot e^{j\frac{k\pi}{N}} \cdot e^{\frac{\pi j}{2N}}$$

$$= \Omega_c \cdot e^{j\frac{k\pi}{2}}\, e^{\frac{\pi j}{4}} = \Omega_c\, e^{j\left[\left(\frac{2k+1}{4}\right)\cdot\pi\right]}$$

$P_1 = \Omega c\, e^{j3\pi/4} = (-0.707 + j0.707)\Omega c$

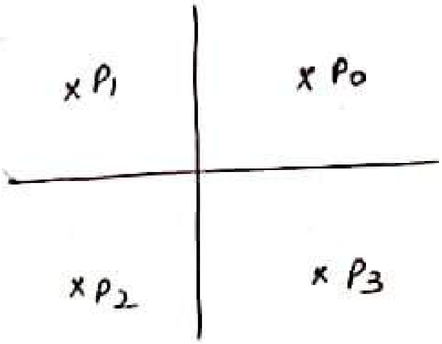
$P_2 = \Omega c\, e^{j\frac{5\pi}{4}} = (-0.707 - j0.707)\Omega c$

$P_3 = \Omega c\, e^{j\frac{7\pi}{4}} = (0.707 - 0.707j)\Omega c$

$P_0 = \Omega c\, e^{j\pi/4} = (0.707 + j0.707)\Omega c$

$$\therefore \quad H(s) = \frac{\Omega c^{2N}}{\prod (S - P_k)} = \frac{\Omega c^{2N}}{(S-P_1)(S-P_2)}$$

$$H(s) = \frac{\Omega c^{2(2)}}{(s - \Omega c\, e^{j3\pi/4}).(s - \Omega c . e^{j(-3\pi/4)})}$$



P_1 and P_2 are the conjugates of each other.

Ques. Compute the poles of an analog Butterworth filter that satisfies the conditions, ~~0.707~~

$$0.707 \le |H(j\Omega)| \le 1 \qquad 0 \le \Omega \le 2$$

$$|H(j\Omega)| \le 0.1 \qquad \Omega \geq 4$$

Determine H(s) and hence obtain H(z) using Bilinear transformation assuming T= 1s.

Impulse Invariant

$$\Omega p = \frac{\omega p}{T}$$

$$\Omega s = \frac{\omega s}{T} \qquad s = \frac{1}{T} \ln(z).$$

If we use Bilinear transformation -

$$\Omega_p = \frac{2}{T} \tan\left(\frac{\omega_p}{2}\right)$$

$$\Omega_s = \frac{2}{T} \tan\left(\frac{\omega_s}{2}\right)$$

$$S = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$$

$$\frac{1}{\sqrt{1+\varepsilon^2}} \leq H(j\Omega) \leq 1$$

$$|H(j\Omega)| \leq \frac{1}{A}$$

i.e. $\Omega_p = 2$

$\Omega_s = 4$

$$\frac{1}{\sqrt{1+\varepsilon^2}} = \frac{1}{\sqrt{2}}$$

$$\Rightarrow 1+\varepsilon^2 = 2$$

$$\Rightarrow \varepsilon^2 = 1$$

$$\begin{cases} \varepsilon = 1 \\ A = 10 \\ \Omega_p = 2 \\ \Omega_s = 4 \end{cases}$$

Case-I :-

~~$N = \frac{1}{2} \ln$~~

$$N = \frac{\log\left(\varepsilon^2 / A^2-1\right)}{2\log\left(\Omega_p / \Omega_s\right)}$$

$$= \log\left[\frac{\varepsilon^2}{(A^2-1)} - \left(\frac{\Omega_p}{\Omega_s}\right)^2\right]$$

$$= \log\left[\frac{1}{(100-1)} - \left(\frac{2}{4}\right)^2\right]$$

$$= \log\left[\frac{1}{99} - \frac{1}{4}\right] \doteq 3.31 \simeq 4 \text{ (Even)}$$

Ques 12.35

Tarun

~~$\omega_c =$~~

~~$\Omega_c =$~~

$$A^2 - 1 = \left(\frac{\Omega_s}{\Omega_c}\right)^{2N}$$

$$\Rightarrow 100 - 1 = \left(\frac{4}{\Omega_c}\right)^8$$

$$\Rightarrow 99 \cdot (\Omega_c)^8 = (4)^8$$

$$\Rightarrow \Omega_c = 2$$

k lies in between 0 to 7

$$P_k = \Omega_c \cdot e^{j\left(\frac{2k+1}{2N}\right)\cdot\pi}$$

only poles on left hand side are considered.

$P_0 = 2 \cdot e^{j\frac{\pi}{8}} =$ right ×

$P_1 = 2 \cdot e^{j\frac{3\pi}{8}} =$ right ×

$P_2 = 2 \cdot e^{j\frac{5\pi}{8}} =$ left ✓

$P_3 = 2 \cdot e^{j\frac{7\pi}{8}} =$ left ✓

$P_4 = 2 \cdot e^{j\frac{9\pi}{8}} =$ left ✓ $= P_3^* = 2e^{-j7\pi/8}$

$P_5 = 2 \cdot e^{j\frac{11\pi}{8}} = 2(0.383 - 0.942j)$ left ✓ $= P_2^* = 2e^{-j\frac{5\pi}{8}}$

$P_6 = 2 \cdot e^{j\frac{13\pi}{8}} =$ right ×

$P_7 = 2 \cdot e^{j\frac{15\pi}{8}} =$ right ×

P_2 and P_5 are conjugates of each other.

and P_3 and P_4 are the conjugates of each other.

$$H(S) = \frac{\Omega_c^N}{(S-P_2)\cdot(S-P_3)\cdot(S-P_4)\cdot(S-P_5)}$$

$H(z) =$

put, $S = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$

12th April, 19

- **Cheby Shev :-**

Cheby Shev is all pole filter and Types two is Poles and zero filter.

In type 1, magnitude response is equiripple in pass band and monotonic in stop band.

In type 2, it is monotonic in pass band and equiripple in stop band.

Type-2 is also called Inverse Cheby Shev filter.

Advantage -

↳ By choosing a filter that has an equiripple rather than a monotonic behaviour (ButterWorth), we can obtain a lower order filter.

Type-1 :-

Magnitude filter response,

$$|H(j\omega)|^2 = \frac{1}{1+\epsilon^2 T_N^2\left(\frac{\Omega}{\Omega_c}\right)}$$

where, $T_N(z)$ is the n^{th} order Chebyshev polynomial.

$$T_N(z) = \begin{cases} \cos(N\cos^{-1}z) & ; |z| \leq 1 \quad \text{Pass Band} \\ \cos(N\cosh^{-1}z) & ; |z| > 1 \end{cases}$$

(1) $z=0$

$$|H(j0)|^2 = \frac{1}{1+\epsilon^2 T_N^2(0)}$$

$N = 1, 2$

$T_1(0) = \cos(\pi/2) = 0$

$T_2(0) = \cos(2 \cdot \pi/2) = 1$

N even

$$|H(j0)^2| = \frac{1}{1+\epsilon^2}$$

N odd

$$|H(j0)^2| = 1$$

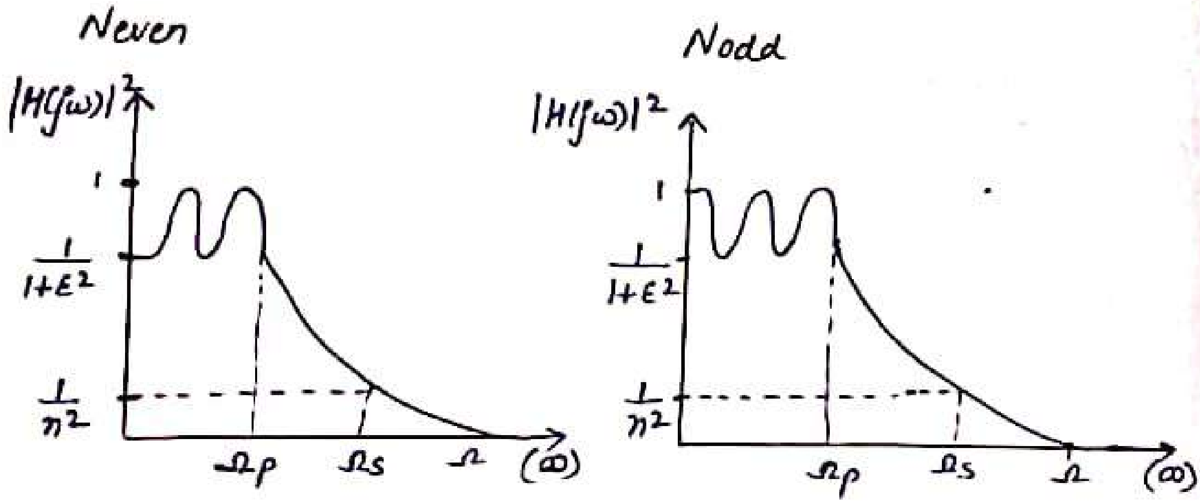
(2) $z=1$

$$|H(j\Omega_c)^2| = \frac{1}{1+\epsilon^2}$$

(3) $z>1$ $\quad (\Omega > \Omega_c)$

$$T_N(\infty) = \infty$$

$$|H(j\infty)|^2 = \frac{1}{1+\infty} = 0$$



- **Determination of filter parameters (N, Ω_c) of Cheby Shev filter -**

Case I :- When specification ϵ, A, Ω_p, Ω_s are known -

$$|H(j\omega)^2| = \frac{1}{1+\epsilon^2 T_N^2\left(\frac{\Omega}{\Omega_c}\right)}$$

General Specifications,

$$\frac{1}{1+\epsilon^2} \le |H(j\omega)^2| \le 1 \qquad |\Omega| \le \Omega_c$$

$$|H(j\omega)^2| \le \frac{1}{A^2}$$

(1.) at $\Omega = \Omega_p$

$$|H(j\omega)|^2 = \frac{1}{1+\epsilon^2} = \frac{1}{1+\epsilon^2 T_N^2\left(\frac{\Omega_p}{\Omega_c}\right)}$$

$T_N\left(\frac{\Omega_p}{\Omega_c}\right) = 1$

$\hookrightarrow T_N(1) = 1$

$$\Rightarrow \left(\frac{\Omega_p}{\Omega_c}\right) = 1$$

(2) at $\Omega = \Omega_s$

$$|H(j\omega)|^2 = \frac{1}{A^2} = \frac{1}{1+\varepsilon^2 T_N^2 \cdot \left(\frac{\Omega_s}{\Omega_c}\right)}$$

$$\varepsilon^2 \cdot T_N^2\left(\frac{\Omega_s}{\Omega_c}\right) = A^2 - 1$$

$$T_N^2 \cdot \left(\frac{\Omega_s}{\Omega_c}\right) = \frac{\sqrt{A^2-1}}{\varepsilon}$$

$$\cosh\left(N \cdot \cosh^{-1} \frac{\Omega_s}{\Omega_c}\right) = \frac{\sqrt{A^2-1}}{\varepsilon}$$

$$N = \frac{\cosh^{-1}\left(\sqrt{A^2-1}/\varepsilon\right)}{\cosh^{-1}\left(\frac{\Omega_s}{\Omega_c}\right)}$$

<u>Case-II</u> :- When specifications A_p, A_s and Ω_p are known -

(1) When $\Omega = \Omega_p$

$$-A_p = 20 \log |H(j\omega)| = 10 \log |H(j\omega)|^2$$

$$= 10 \log \left(\frac{1}{1+\epsilon^2} \right)$$

$$-A_p = 10 \log \frac{1}{1+\epsilon^2 T_N^2 \left(\frac{\Omega_p}{\Omega_c} \right)}$$

$$\boxed{\Omega_p = \Omega_c}$$

(2) At $\Omega = \Omega_s$

$$-A_s = 10 \log |H(j\Omega)|^2 = 10 \log \left(\frac{1}{A^2} \right)$$

$$-A_s = 10 \log \frac{1}{1+\epsilon^2 T_N^2 \left(\frac{\Omega_s}{\Omega_c} \right)} = 10 \log \frac{1}{A^2}$$

$$10^{-A_s/10} = \frac{1}{1+\epsilon^2 T_N^2 \left(\frac{\Omega_s}{\Omega_c} \right)}$$

$$\Rightarrow \quad 1+\epsilon^2 T_N^2 \left(\frac{\Omega_s}{\Omega_c} \right) = 10^{A_s/10}$$

$$T_N^2 \left(\frac{\Omega_s}{\Omega_c} \right) = \frac{10^{A_s/10} - 1}{\epsilon^2} = \frac{10^{A_s/10} - 1}{10^{A_p/10} - 1}$$

$$T_N \left(\frac{\Omega_s}{\Omega_p} \right) = \frac{\sqrt{10^{A_s/10} - 1}}{\sqrt{10^{A_p/10} - 1}}$$

$$\cosh \left(N \cosh^{-1} \left(\frac{\Omega_s}{\Omega_c} \right) \right) = \sqrt{\frac{10^{A_s/10} - 1}{10^{A_p/10} - 1}}$$

$$N \cosh^{-1}\left(\frac{\Omega_s}{\Omega_c}\right) = \frac{\cosh^{-1} \sqrt{(10^{A_s/10}-1)/(10^{A_p/10}-1)}}{\cosh^{-1}(\Omega_s/\Omega_c)}$$

Ques: Following are the specifications for a low pass filter, $\Omega_p = 1$ $\quad A_p = 0.5\, dB$

$\Omega_s = 2.33 \quad A_s = 22\, dB$

Compute the filter order for Che By Shev Butterworth.

$$N = \frac{\cosh^{-1}\left(\sqrt{\frac{10^{22/10}-1}{10^{.5/10}-1}}\right)}{\cosh^{-1}(2.33)} = \frac{}{1.489}$$

$N = 2.87$

$N \approx 3$

also;

$$N = \frac{\log\left(\epsilon^2/A^2-1\right)}{2\log\left(\frac{\Omega_p}{\Omega_s}\right)} = \frac{-3.11}{-0.735}$$

$N = 4.23 \approx 5$

Type - 2 :- Che By Shev filter -

$$|H(j\omega)|^2 = \frac{1}{1+\epsilon^2\left[\frac{T_N^2(\Omega_s/\Omega_p)}{T_N^2\left(\frac{\Omega_s}{\Omega}\right)}\right]} = \frac{T_N^2(\Omega_s/\Omega)}{T_N^2\left(\frac{\Omega_s}{\Omega}\right) + \epsilon^2 T_N^2\left(\frac{\Omega_s}{\Omega_p}\right)}$$

at $\Omega = 0$

$$|H(j\omega)|^2 = 1$$

at $\Omega = \Omega_p$

$$|H(j\omega)|^2 = \frac{1}{1+\epsilon^2}$$

at $\Omega = \Omega_s$

$$\frac{1}{1+\epsilon^2 T_N^2\left(\frac{\Omega_s}{\Omega_p}\right)} = \frac{1}{A^2}$$

$\therefore$

$$N = \frac{\cosh^{-1}\left(\sqrt{A^2-1}\,/\,\epsilon\right)}{\cosh^{-1}\left(\Omega_s/\Omega_p\right)}$$