Coherent Scattering of neutrons
by
ortho and para hydrogen

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This is the coherent scattering of neutrons by protons in hydrogen molecule.

The sign of the scattering length cannot be determined by the scattering by a single nucleus since the cross section depends only on the square of the scattering length.

The scattering of very low energy neutrons of energy lying between 0.0008 eV and 0.0025 eV, known as thermal neutrons, from hydrogen gas (H_2 gas) can be considered as coherent scattering by two scattering centres if the wavelength of neutrons is large compared to the inter-nuclear distance which is approximately 0.78 Å(*i.e.*, 0.78×10^{-10} m). In this case, the scattering amplitudes add up and consequently the molecular scattering cross section σ_{H_2} will then be four times the free–proton cross section. For higher energy neutrons for which the de Broglie wave length is shorter than the inter-nuclear distance, the scattering can be viewed as incoherent and consequently only the cross sections add up and not the amplitudes. So, the molecular hydrogen cross section σ_{H_2} is just twice the proton cross section.

$$\sigma_{H_2} = \begin{cases} 4\sigma_p & (\text{ for neutron energies} < 0.0025 \text{ eV}) \\ 2\sigma_p & (\text{ for neutron energies} > 0.0025 \text{ eV}) \end{cases}$$

The above estimate does not include the reduced mass correction and this has to be considered. In the case of low energy neutron scattering, the hydrogen molecule recoils as a whole and hence a correction for the reduced mass is to be included when comparing with free proton cross section.

$$\frac{1}{\mu_{H_2}} = \frac{1}{M} + \frac{1}{2M}; \qquad \mu_{H_2} = \frac{2M}{3};$$
$$\frac{1}{\mu_H} = \frac{1}{M} + \frac{1}{M}; \qquad \mu_H = \frac{1}{2}M.$$

It follows that

$$\frac{\mu_{H_2}}{\mu_H} = \frac{4}{3}.$$

the scattering cross section increases steeply at the onset of coherent scattering by thermal neutrons. It is known that the n-p scattering amplitude is spin-dependent. Including the spin dependence, the n-p scattering amplitude can be written as

$$a = a_t P_t + a_s P_s,$$

where P_t and P_s are the projection operators for the triplet and singlet spin states of the n-p system.

$$P_t = \frac{1}{4}(3 + \sigma_1 \cdot \sigma_2); \qquad P_s = \frac{1}{4}(1 - \sigma_1 \cdot \sigma_2).$$

Substituting the expressions for the projection operators we get,

$$a = \frac{1}{4}(3a_t + a_s) + \frac{1}{4}(a_t - a_s)\sigma_n \cdot \sigma_p,$$

where we have used the indices n and p for the nucleons 1 and 2.

For scattering of neutrons of very long wavelength by H_2, the new scattering amplitudes corrected for recoil effects are

$$a_t' = \frac{4}{3}a_t; \qquad a_s' = \frac{4}{3}a_s.$$

The molecular scattering amplitude is

$$\begin{aligned} a_M &= a'(1) + a'(2) \\ &= \frac{1}{2}(3a_t' + a_s') + \frac{1}{4}(a_t' - a_s')\sigma_n \cdot (\sigma_{p1} + \sigma_{p2}) \\ &= \frac{1}{2}(3a_t' + a_s') + \frac{1}{2}(a_t' - a_s')(\sigma_n \cdot S_M), \end{aligned}$$

where

$$S_M = \frac{1}{2}(\sigma_{p1} + \sigma_{p2}),$$

is the resultant spin of the hydrogen molecule. Squaring, we obtain

$$|a_M|^2 = \frac{1}{4}(3a_t' + a_s')^2 + \frac{1}{2}(3a_t' + a_s')(a_t' - a_s')\boldsymbol{\sigma}_n \cdot \boldsymbol{S}_M + \frac{1}{4}(a_t' - a_s')^2(\boldsymbol{\sigma}_n \cdot \boldsymbol{S}_M)^2.$$

In the above expression the expectation values of the operators $\boldsymbol{\sigma}_n \cdot \boldsymbol{S}_M$ and $(\boldsymbol{\sigma}_n \cdot \boldsymbol{S}_M)^2$ are to be inserted.

$$\begin{aligned}
\langle \boldsymbol{\sigma}_n \cdot \boldsymbol{S}_M \rangle &= 0. \\
\left\langle (\boldsymbol{\sigma}_n \cdot \boldsymbol{S}_M)^2 \right\rangle &= \langle \boldsymbol{S}_M \cdot \boldsymbol{S}_M \rangle + i\langle \boldsymbol{\sigma}_n \cdot \boldsymbol{S}_M \times \boldsymbol{S}_M \rangle \\
&= \langle \boldsymbol{S}_M \cdot \boldsymbol{S}_M \rangle - \langle \boldsymbol{\sigma}_n \cdot \boldsymbol{S}_M \rangle, \quad \text{since } \boldsymbol{S}_M \times \boldsymbol{S}_M = i\boldsymbol{S}_M \\
&= S_M(S_M + 1), \quad \text{since } \langle \boldsymbol{\sigma}_n \cdot \boldsymbol{S}_M \rangle = 0.
\end{aligned}$$

Substituting these expectation values then we get the differential cross section for the scattering of thermal neutrons by hydrogen molecule.

$$\frac{d\sigma}{d\Omega} = |a_M|^2 = \frac{1}{4}(3a_t' + a_s')^2 + \frac{1}{4}(a_t' - a_s')^2 S_M(S_M + 1).$$

Performing the angular integration, we get the total cross section

$$\sigma = 4\pi |a_M|^2.$$

The hydrogen molecule can exist either in spin triplet state ($S_M = 1$) state or in spin singlet state ($S_M = 0$). The hydrogen molecule in spin triplet state is known as ortho-hydrogen and the hydrogen molecule in spin singlet state is known as para-hydrogen. Substituting $S_M = 1$ we obtain the neutron scattering cross section for ortho-hydrogen,

$$\sigma_{\text{ortho}} = \pi \left\{ (3a_t' + a_s')^2 + 2(a_t' - a_s')^2 \right\}.$$

Similarly, substituting $S_M = 0$ we obtain the neutron scattering cross section for para-hydrogen.

$$\sigma_{\text{para}} = \pi (3a_t' + a_s')^2.$$

The ratio of the two cross sections yield valuable information on the ratio of the singlet to the triplet scattering amplitude.

$$\begin{aligned}
\frac{\sigma_{\text{ortho}}}{\sigma_{\text{para}}} &= 1 + 2\frac{(a_t' - a_s')^2}{(3a_t' + a_s')^2} \\
&= 1 + 2\frac{\left(1 - \frac{a_s'}{a_t'}\right)^2}{\left(3 + \frac{a_s'}{a_t'}\right)^2} \\
&= 1 + 2\frac{\left(1 - \frac{a_s}{a_t}\right)^2}{\left(3 + \frac{a_s}{a_t}\right)^2},
\end{aligned}$$

If the n-p force is spin-independent, then

$$a_t = a_s.$$

hence we write

$$\frac{\sigma_{\text{ortho}}}{\sigma_{\text{para}}} = 1.$$

The experimental values for the effective cross sections at 0.002 eV are

$$\sigma_{\text{ortho}} \approx 120\,\text{barns}; \qquad \sigma_{\text{para}} \approx 4\,\text{barns}.$$

Thus the ratio

$$\frac{\sigma_{\text{ortho}}}{\sigma_{\text{para}}} \approx 30,$$

which definitely rules out the spin-independence of n-p force.

Reference Books

- Introductory Nuclear Physics by K. S. Krane
- Nuclear Physics by Devnathan