Magnetic dipole moment of Deuteron

The magnetic moments of the neutron[6] (μ_n), the proton[7] (μ_p), and the deuteron[6,28] (μ_d) have been very accurately measured. The difference, $(\mu_p + \mu_n) - \mu_d = 0.0222$ nm, is small, but definitely not zero. This suggests that the deuteron is mostly in a $^3S_1$ state, but spends a small amount of time in some other state.

We wish to find which mixture of states can explain the magnetic moment of the deuteron. Introducing the spin wave function for the neutron and proton as

$$\chi_1^1 = \alpha(p)\alpha(n)$$

$$\chi_1^0 = \frac{1}{\sqrt{2}}[\alpha(p)\beta(n) + \beta(p)\alpha(n)]$$

$$\chi_1^{-1} = \beta(p)\beta(n)$$

and

$$\chi_0^0 = \frac{1}{\sqrt{2}}[\alpha(p)\beta(n) - \beta(p)\alpha(n)],$$

we can write the wave function $\phi_{lSJ}(\mathbf{r})$ for the four cases discussed, in terms of three radial wave functions $u_l(r)$, $v_l(r)$ and $w_l(r)$, where the subscript l is the orbital angular momentum quantum number.

$$^3S_1:\quad \phi_{011}^{\,1}(\mathbf{r}) = \frac{u_0(r)}{r}\mathcal{Y}_{011}^{\,1} = \frac{u_0(r)}{r}Y_0^0\chi_1^1$$

$$^3D_1:\quad \phi_{211}^{\,1}(\mathbf{r}) = \frac{u_2(r)}{r}\mathcal{Y}_{211}^{\,1}$$

$$= \frac{u_2(r)}{r}\left[\sqrt{\frac{3}{5}}\,Y_2^2\chi_1^{-1} - \sqrt{\frac{3}{10}}\,Y_2^1\chi_1^0 + \sqrt{\frac{1}{10}}\,Y_2^0\chi_1^1\right]$$

$$^1P_1:\quad \phi_{101}^{\,1}(\mathbf{r}) = \frac{v_1(r)}{r}\mathcal{Y}_{101}^{\,1} = \frac{v_1(r)}{r}Y_1^1\chi_0^0$$

$$^3P_1:\quad \phi_{111}^{\,1}(\mathbf{r}) = \frac{w_1(r)}{r}\mathcal{Y}_{111}^{\,1}$$

$$= \frac{w_1(r)}{r}\left[\frac{1}{\sqrt{2}}\,Y_1^1\chi_1^0 - \frac{1}{\sqrt{2}}\,Y_1^0\chi_1^1\right]$$

Taking the magnetic moment projection along the z-axis and noting that in the center-of-mass system the orbital angular momentum associated with the proton is half of the relative orbital angular momentum ($\mathbf{l}_p = \frac{1}{2}\mathbf{l}$), the z-component of the deuteron magnetic moment is

$$\mu_z = \tfrac{1}{2}l_z + g_p s_{pz} + g_n s_{nz}$$

where g_p and g_n are the gyromagnetic ratios for the proton and neutron, and s_{pz} and s_{nz} are the z-components of their respective spins. Quantum mechanically, μ_z is an operator and the magnetic moment in the state ϕ_{lSJ}^M is found by taking the expectation value

$$\langle \mu_z \rangle = \int d^3r \phi_{lSJ}^{*M} \mu_z \phi_{lSJ}^M$$

This may be written in Dirac notation as

$$\langle \phi_{lSJ}^M | \mu_z | \phi_{lSJ}^M \rangle = \left\langle \frac{u_l}{r} \mathscr{Y}_{lSJ}^M \middle| \mu_z \middle| \frac{u_l}{r} \mathscr{Y}_{lSJ}^M \right\rangle$$

We shall compute these expectation values for the states $^3S_1$, $^3D_1$, $^1P_1$, $^3P_1$. We note that†

$$l_z Y_l^m = m Y_l^m; \qquad g_{p,n} = 2\mu_{p,n}; \qquad (\mathbf{s}_{p,n})_z = \tfrac{1}{2}(\boldsymbol{\sigma}_{p,n})_z$$

and that

$$\begin{aligned} s_{pz}\chi_1^1 &= \tfrac{1}{2}\chi_1^1 & s_{nz}\chi_1^1 &= \tfrac{1}{2}\chi_1^1 \\ s_{pz}\chi_1^0 &= \tfrac{1}{2}\chi_0^0 & s_{nz}\chi_1^0 &= -\tfrac{1}{2}\chi_0^0 \\ s_{pz}\chi_1^{-1} &= -\tfrac{1}{2}\chi_1^{-1} & s_{nz}\chi_1^{-1} &= -\tfrac{1}{2}\chi_1^{-1} \\ s_{pz}\chi_0^0 &= \tfrac{1}{2}\chi_1^0 & s_{nz}\chi_0^0 &= -\tfrac{1}{2}\chi_1^0 \end{aligned}$$

By making use of the relations we have the following expressions:

$$\mu_z \mathscr{Y}_{011}^1 = \tfrac{1}{2}(g_p + g_n)\mathscr{Y}_{011}^1 = \frac{g_p + g_n}{2} Y_0^0 \chi_1^1$$

$$\mu_z \mathscr{Y}_{211}^1 = \sqrt{\frac{3}{5}}\left(1 - \frac{g_p + g_n}{2}\right) Y_2^2 \chi_1^{-1} - \frac{1}{2}\sqrt{\frac{3}{10}}\, Y_2^1[\chi_1^0 + (g_p - g_n)\chi_0^0] + \sqrt{\frac{1}{10}}\,\frac{g_p + g_n}{2}\, Y_2^0 \chi_1^1$$

$$\mu_z \mathscr{Y}_{101}^1 = \tfrac{1}{2} Y_1^1[\chi_0^0 + (g_p - g_n)\chi_1^0]$$

$$\mu_z \mathscr{Y}_{111}^1 = \frac{1}{2\sqrt{2}}\left[Y_1^1\chi_1^0 + g_p(Y_1^1\chi_0^0 - Y_1^0\chi_1^1) - g_n(Y_1^1\chi_0^0 + Y_1^0\chi_1^1)\right]$$

† We shall use the values[7,8] $\mu_p = 2.7928$ nm, $\mu_n = -1.9131$ nm.

By using the orthonormality of the $Y_l^{m_l}$'s when integrated over the solid angle and that of the $\chi_S^{m_s}$, we obtain the expectation values:

$$^3S_1: \quad \langle \phi_{011}^{1} | \mu_z | \phi_{011}^{1} \rangle = \frac{g_p + g_n}{2} \int_0^\infty u_0^2(r)\, dr = \frac{g_p + g_n}{2} = 0.8797 \text{ nm}$$

$$^3D_1: \quad \langle \phi_{211}^{1} | \mu_z | \phi_{211}^{1} \rangle = \left[\frac{3}{4} - \frac{1}{4}(g_p + g_n) \right] \int_0^\infty u_2^2(r)\, dr$$

$$= [\tfrac{3}{4} - \tfrac{1}{4}(g_p + g_n)] = 0.3101 \text{ nm}$$

$$^1P_1: \quad \langle \phi_{101}^{1} | \mu_z | \phi_{101}^{1} \rangle = \frac{1}{2} \int_0^\infty v_1^2(r)\, dr = 0.5000 \text{ nm}$$

$$^3P_1: \quad \langle \phi_{111}^{1} | \mu_z | \phi_{111}^{1} \rangle = [\tfrac{1}{4} + \tfrac{1}{4}(g_p + g_n)] \int_0^\infty w_1^2(r)\, dr$$

$$= [\tfrac{1}{4} + \tfrac{1}{4}(g_p + g_n)] = 0.6899 \text{ nm}$$

we discussed the spin and orbital contributions to the magnetic dipole moment. If the $\ell = 0$ assumption is correct, there should be no orbital contribution to the magnetic moment, and we can assume the total magnetic moment to be simply the combination of the neutron and proton magnetic moments:

$$\mu = \mu_n + \mu_p$$
$$= \frac{g_{sn}\mu_N}{\hbar} s_n + \frac{g_{sp}\mu_N}{\hbar} s_p$$

where $g_{sn} = -3.826084$ and $g_{sp} = 5.585691$. As we did in Section 3.5, we take the observed magnetic moment to be the z component of μ when the spins have their maximum value $(+\frac{1}{2}\hbar)$:

$$\mu = \tfrac{1}{2}\mu_N(g_{sn} + g_{sp})$$
$$= 0.879804\ \mu_N$$

The observed value is $0.8574376 \pm 0.0000004\ \mu_N$, in good but not quite exact agreement with the calculated value. The small discrepancy can be ascribed to any of a number of factors, such as contributions from the mesons exchanged between the neutron and proton; in the context of the present discussion, we can assume the discrepancy to arise from a small mixture of d state ($\ell = 2$) in the deuteron wave function:

$$\psi = a_s \psi(\ell = 0) + a_d \psi(\ell = 2)$$

Calculating the magnetic moment from this wave function gives

$$\mu = a_s^2 \mu(\ell = 0) + a_d^2 \mu(\ell = 2)$$

where $\mu(\ell = 0)$ is the value calculated in Equation 4.8 and $\mu(\ell = 2) = \frac{1}{4}(3 - g_{sp} - g_{sn})\mu_N$ is the value calculated for a d state. The observed value is consistent with $a_s^2 = 0.96$, $a_d^2 = 0.04$; that is, the deuteron is 96% $\ell = 0$ and only 4% $\ell = 2$. The assumption of the pure $\ell = 0$ state, which we made in calculating the well depth, is thus pretty good but not quite exact.

Reference Books

Introductory Nuclear Physics by K. S. Krane
Nuclear Physics Theory and Experiments by Roy and Nigam