Discrete Fourier Transform (DFT)

→ We begin/start with a brief review of the continuous-time Fourier transform, a frequency domain representation of a continuous-time signal, and its ~~operation~~ properties, as it will provide a better understanding of the frequency-domain representation of the discrete-time signals and systems.

→ The frequency-domain representation of a continuous-time signal $x_a(t)$ is given by the continuous-time Fourier transform (CTFT)

$$\boxed{X_a(j\Omega) = \int_{-\infty}^{\infty} x_a(t)\, e^{-j\Omega t}\, dt} \quad \text{—} (1)$$

The CTFT often is referred to as the Fourier spectrum. The continuous-time signal $x_a(t)$ can be recovered from its CTFT $X_a(j\omega)$ via the inverse Fourier integral

$$\boxed{x_a(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_a(j\Omega)\, e^{j\Omega t}\, d\Omega} \quad \text{—} (2)$$

where Ω denotes the continuous-time angular frequency variable in radians per sec.

It can be seen from the definition given by eq.(1) that, in general, the CTFT is a complex function of Ω in the range $-\infty < \Omega < \infty$. It can be expressed in polar form as

$$X_a(j\Omega) = |X_a(j\Omega)|\, e^{j\theta_a(\Omega)}$$

where $\theta_a = \arg\{X_a(j\Omega)\}$

The quantity $|X_a(j\Omega)|$ is called the magnitude spectrum, and the quantity $\theta_a(\Omega)$ is called the phase spectrum, with both spectra being real functions of Ω.

The Discrete-Time Fourier Transform → The frequency-domain

representation of a discrete-time sequence is given by the discrete-time Fourier transform (DTFT), which express the sequence as a weighted combination of the complex exponential sequence $\{e^{jwn}\}$, where w is the real normalized frequency variable. The discrete-time Fourier transform $X(e^{jw})$ of a sequence $x(n)$ is defined by

$$X(w) \text{ or } X(e^{jw}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-jwn} \quad \text{—(1)}$$

IDTFT is

$$\boxed{x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{jw})\, e^{jwn}\, dw} \quad \text{—(2)}$$

→ one basic property of the Fourier transform, namely the periodicity property of the transform.
In general, the F.T $X(e^{jw})$ is complex function of the real variable w and can be written in rectangular form as

$$\boxed{X(e^{jw}) = X_{re}(e^{jw}) + j\,X_{im}(e^{jw})} \quad \text{—(3)}$$

where $X_{re}(e^{jw})$ and $X_{im}(e^{jw})$ are, respectively, the real & imaginary parts of $X(e^{jw})$ and are real functions of w.
From eq (3), it follows that

$$X_{re}(e^{jw}) = \frac{1}{2}\{X(e^{jw}) + X^*(e^{jw})\} \quad \text{—3(a)}$$

$$X_{im}(e^{jw}) = \frac{1}{2j}\{X(e^{jw}) - X^*(e^{jw})\} \quad \text{—3(b)}$$

where $X^*(e^{jw})$ denotes the complex conjugate of $X(e^{jw})$.

Note that 'w' is the continuous function of angular frequency range from $-\pi$ to π. This means eventhough $x(n)$ is discrete, it spectrum $X(w)$ is continuous. Such continuous function cannot be evaluated on digital processor. To overcome the problem of digital signal processor by using DFT.

The Discrete Fourier Transform (DFT) : DFT is used for transforming a discrete-time sequence, $x(n)$ of a finite length of N points into a discrete-frequency sequence of the same finite length as $x(n)$. it is defined as

$$X(K) = \sum_{n=0}^{N-1} x(n)\, e^{-j2\pi kn/N} , \quad 0 \le k \le N-1 \quad \text{—(1)}$$

where N is number of point in signal
$k = 0, 1, --- N-1$.

and inverse DFT (IDFT) is defined as

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(K)\, e^{j2\pi kn/N} , \quad n = 0, 1 --- N-1 \quad \text{—(2)}$$

Response of DFT → $X(K)$ is the DFT of the signal $x(n)$ and can be written in rectangular form as

$$X(K) = X_R(K) + jX_I(K)$$

where $X_R(K)$ & $X_I(K)$ are respectively the real and imaginary part of $X(K)$.

The magnitude or amplitude of $X(K)$ is given by

$$|X(K)| = \sqrt{X_R^2(K) + X_I^2(K)}$$

and the phase of $X(K)$ is given by

$$\angle X(K) = \tan^{-1}\left[\frac{X_I(K)}{X_R(K)}\right]$$

The formula for DFT & IDFT may be expressed as

$$X(K) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn} , \quad k = 0, 1 --- N-1.$$

$$\& \quad x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(K)\, W_N^{-kn} , \quad n = 0, 1 --- N-1$$

where $W_N = e^{-j2\pi/N}$ – twiddle factor ≡ phase rotation factor

Ex 1 Find the 8-point DFT of the signal

$x(n) = \{1, 1, 1, 1, 1, 1, 1, 0, 0\}$

Also sketch its magnitude and phase response.

Soln → DFT of $x(n)$ is

$$X(K) = \sum_{n=0}^{N-1} x(n) W_N^{nk} = \sum_{n=0}^{7} x(n) W_N^{nk}$$

$$\text{where } W_N = e^{-j\frac{2\pi}{8}} = e^{-j\pi/4}$$

$$= 1 + W_N^{K} + W_N^{2K} + W_N^{3k} + W_N^{4k} + W_N^{5k}$$

$$X(k) = 1 + e^{-j\pi k/4} + e^{-j2\pi k/4} + e^{-j3\pi k/4} + e^{-j4\pi k/4} + e^{-j5\pi k/4}$$

Putting $K = 0, 1, 2, 3, 4, 5, 6, 7$ in above expression

$K=0$, $X(0) = 6$

$K=1$, $X(1) = -0.7071 - j1.7071$

$K=2$, $X(2) = 1 - j$

$K=3$, $X(3) = 0.7071 + j0.2929$

$K=4$, $X(4) = 0$

$K=5$, $X(5) = 0.7071 - j0.2929$

$K=6$, $X(6) = 1 + j$

$K=7$, $X(7) = -0.7071 + j1.7071$

DFT of $x(n)$ is given as

$$X(K) = \{6, -0.7071 - j1.7071, 1 - j, 0.7071 + j0.2929, 0, 0.7071 - j.2929, 1 + j, -0.7071 + j1.7071\}$$

Amplitude response of $X(K)$ is given as

$$|X(K)| = \{6, 1.8478, 1.4142, 0.7654, 0, 0.7654, 1.4142, 1.8478\}$$

Phase respons of $X(k)$ is given as

$$\angle X(K) = \{0, -1.9635, -0.785, 0.3927, 0, -0.3927, 0.785, 1.9635\}$$

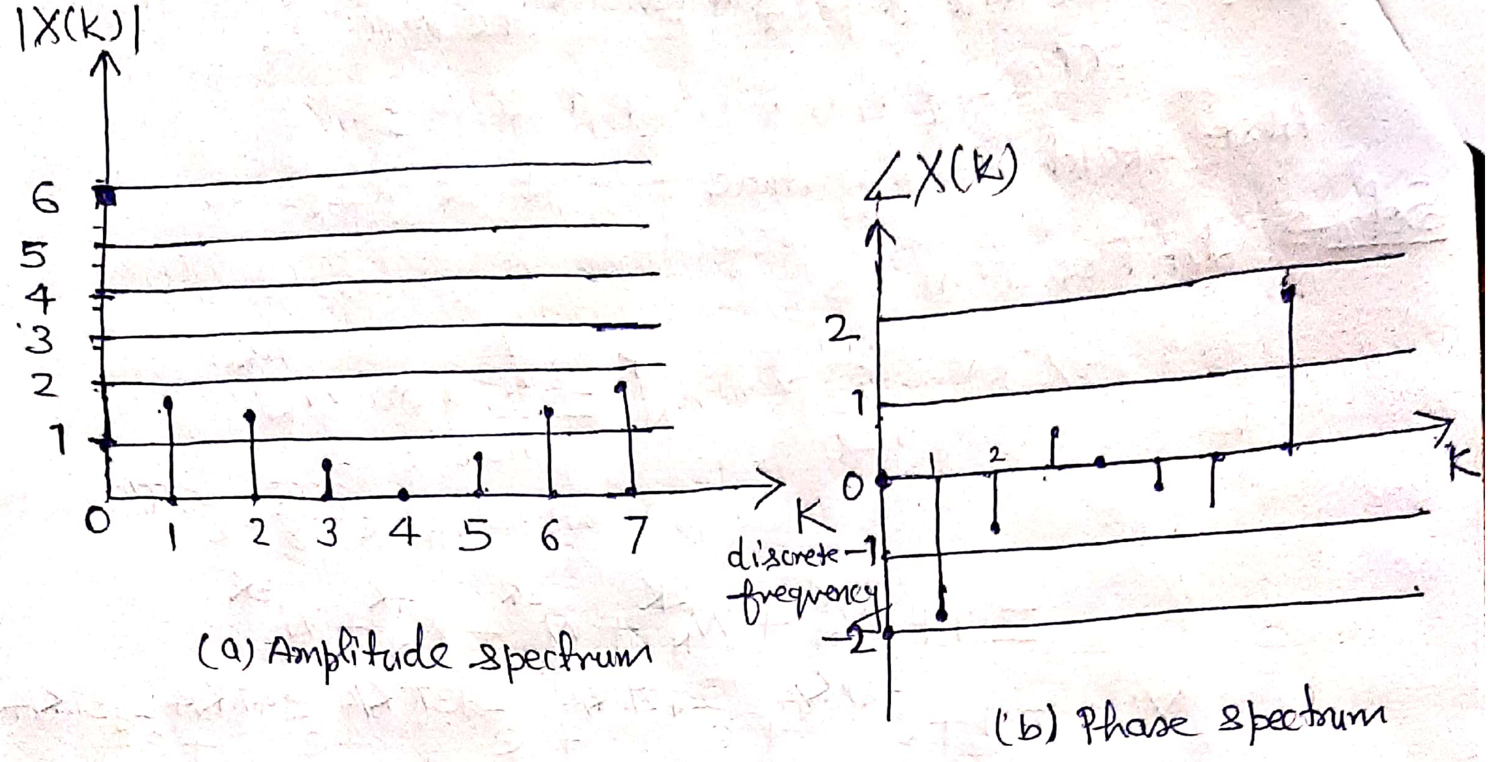

Fig 1: Amplitue and Phase spectrum

Ex 2. Find the DFT of the sequence

$$x(n) = \begin{cases} 1, & 2 \le n \le 6 \\ 0 & \text{for } n = 0, 1, 7, 8, 9 \end{cases}$$

Given that $N = 10$.

Soln → DFT of $x(n)$ is

$$X(k) = \sum_{n=0}^{N-1} x(n) \cdot W^{-nk}$$

$$= \sum_{n=2}^{6} (1) \cdot W^{-nk}$$

$$= \sum_{n=0}^{6} W^{-nk} - \sum_{n=0}^{1} W^{-nk}$$

$$= \frac{1-(W^{-k})^{6+1}}{1-W^{-k}} - \frac{1-(W^{-k})^{1+1}}{1-W^{-k}}$$

$$= \frac{1-W^{-7k}}{1-W^{-k}} - \frac{1-W^{2k}}{1-W^{-k}}$$

$$X(K) = \frac{1 - W^{-7K} - 1 + W^{-2K}}{1 - W^{-K}}$$

$$X(K) = \frac{W^{-2K} - W^{-7K}}{1 - W^{-K}}$$

Putting $W = e^{-j\frac{2\pi}{10}} = e^{-j\pi/5}$

$$X(K) = \frac{e^{-j\frac{2\pi K}{5}} - e^{-j\frac{7\pi K}{5}}}{1 - e^{-j\frac{\pi K}{5}}}$$

DFT as a Linear Transformation → The DFT as a linear transformation is explained and studied by using matrix form.

Computation of N-point DFT requires $N \times N$ complex multiplications and $N(N-1)$ complex additions.

If N point vector x_N of signal sequence $x(n)$, $n = 0, 1, \ldots N-1$

$$x_N = \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix} \quad \text{--- (1)}$$

& N-point vector X_N of the N DFT samples is given by

$$X_N = \begin{bmatrix} X(0) \\ X(1) \\ \vdots \\ X(N-1) \end{bmatrix} \quad \text{--- (2)}$$

W_N is the $N \times N$ DFT matrix given by

$$W_N = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N^1 & W_N^2 & \cdots & W_N^{N-1} \\ 1 & W_N^2 & W_N^4 & \cdots & W_N^{2(N-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & W_N^{N-1} & W_N^{2(N-1)} & \cdots & W_N^{(N-1)(N-1)} \end{bmatrix} \quad \text{--- (3)}$$

So, N point DFT in matrix form is defined as

$$\boxed{X_N = W_N \cdot x_N} \quad \text{—— (4)}$$

Where W_N is the matrix of the linear transform and is $N \times N$ symmetric matrix.

Similarly ~~IDFT~~ N-point IDFT in matrix form is

$$x_N = W_N^{-1} \cdot X_N$$

$$\boxed{x_N = \frac{1}{N} W_N^{*} \cdot X_N} \quad \text{—— (5)}$$

where $W_N^{-1} = \frac{1}{N} W_N^{*}$

$W_N = e^{-j2\pi/N}$ = twiddle factor or phase rotation factor