B. Sc (Sem IV) (Mechanics)

Virtual Work

If we imagine that the forces in equilibrium undergo some small displacement, then the work done by the forces during that displacement is called virtual work and such a displacement is called virtual displacement.

The Principle of Virtual Work

The necessary and ~~sufficient~~ sufficient condition that a particle or a rigid body acted upon by a system of coplanar forces be in equilibrium is that the algebraic sum of the virtual works done by the forces during any small displacement consistent with the geometrical condition of the system is zero to the first degree of approximation.

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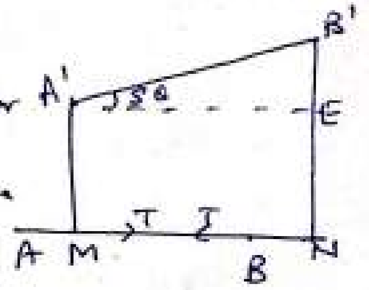
Forces which are omitted in forming the equation of virtual work

1) The work done by the tension of an inextensible string is zero during a small displacement.

Proof Let AB be an inextensible string of length l and T be the tension in the string AB. After a small displacement, so $A'B'$ be the position of string AB, so

$$AB = A'B' = l$$

$A'M$ and $B'N$ are perpendicular on AB.



Work done by the tension of the string during this displacement

$$T.AM - T.BN$$

(∵ displacement of B is in opposite dirn to T.)

$$= T.(AB - MB) - T.(MN - MB)$$

$$= T.(AB - MN)$$

$$= T.(AB - A'E)$$

$$= T.(AB - A'B'\cos\delta\theta)$$

$$= T.(l - l\cos\delta\theta)$$

$$= T.l(1 - \cos\delta\theta)$$

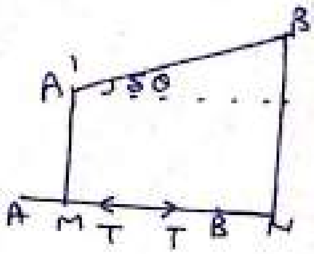
$$= T.l\left(1 - \left\{1 - \frac{(\delta\theta)^2}{2!} + \cdots\right\}\right)$$

$$= T.l(1-1) \quad (\because \delta\theta \text{ is very small})$$

$$= 0$$

(2) The work done by the thrust of an inextensible rod is zero during a small displacement.

<u>Proof</u>- Let T be the thrust in an inextensible rod AB joining two points A and B of a rigid body. Hence work done by the thrust in the rod AB during a small displacement (As in part 1)



$$= -T.AM + T.BN = 0$$

(3) The reaction R of any smooth surface with which the body is in contact does no work.

<u>Proof</u> : If the surface is smooth, the reaction R on the point of contact A is along the normal to the surface.

If A moves to a neighbouring point B, then the displacement AB is at right angles to the direction of the force. So the work done by R is zero.

If the surface is rough, the work done by the frictional force $\bar{F}$ is $\bar{F}.(-AB)$

<u>4</u> If a body rolls without sliding on any fixed surface, the work done in a small displacement by the reaction of the surface on the rolling body is zero.

(5) The work done by the mutual reaction between two bodies of a system is zero in any virtual displacement of the system.

Proof: Since action and reaction are equal and opposite and so the work done by the action balances that by reaction.

6 If a body is constrained to turn about a fixed point or a fixed axis, the virtual work of the reaction at the point or on the axis is zero.

Work Done by the Tension and Thrust of an Extensible String During a Small Displacement

1) To show that the work done by the tension T of an extensible string of length l during a small displacement is $-T.\delta l$.

Proof- Let T be the tension in an extensible string AB of length l joining two points A and B of a rigid body. After a small displacement let $A'B'$ be the length of the string AB i.e $A'B' = l + \delta l$.

$\delta\theta$ be the small angle between AB and $A'B'$.
The work done by the tension T of the string AB during this displacement $= T.AM - T.BN$

$= T.(AB - MB) - T.(MN - MB)$

$= T.(AB - MN)$

$= T.(AB - A'E)$

$= T.(AB - A'B'\cos\delta\theta)$

$= T.(l - (l+\delta l)\cos\delta\theta)$

$= T.\left(l - (l+\delta l)\left\{1 - \frac{(\delta\theta)^2}{2!} + \dots\right\}\right)$

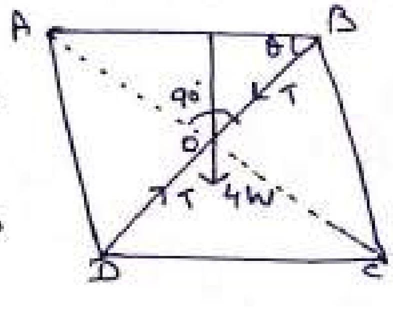
$= T.(l - (l+\delta l))$ $\quad$ ($\because$ $\delta\theta$ is very small, neglecting its higher power)

$= -T.\delta l.$

2) Similarly the work done by the thrust T of an extensible ~~b~~ rod of length l during a small displacement is $T.\delta l$.

Q(1) A string of length 'a', forms the shorter diagonal of a rhombus formed of four uniform rods, each of ~~length~~ length 'b' and weight W, which are hinged together. If one of the rods be supported in a horizontal position, prove that the tension of the string is

$$\frac{2W(2b^2-a^2)}{b\sqrt{(4b^2-a^2)}}.$$



Sol^n Let ABCD be a framework in the shape of rhombus formed of four equal uniform rods of length b and weight W. Let AB is fixed in horizontal position. and B and D are joined by a string of length a forming a shorter diagonal of the rhombus.

Let T be the tension in the string BD. The total weight 4W of the rods AB, BC, CD and DA acting at the point of intersection O of the diagonals AC and BD.

$\angle AOB = 90^\circ$ ($\because$ ABCD is rhombus)

Let $\angle ABO = \theta$. Draw $OM \perp AB$

Give a small displacement to the system, that θ changes to $\theta + \delta\theta$. AB remains fixed lengths of rods do not change, while the length of string BD changes.

$$\therefore \quad BD = 2BO = 2AB\cos\theta$$
$$= 2b\cos\theta.$$

The depth of O below the fixed line $AB = MO$

$$= BO\sin\theta$$
$$= (AB\cos\theta)\sin\theta$$
$$= b\sin\theta\cos\theta.$$

By the principle of virtual work,

$$-T\delta(BD) + 4W\delta(MO) = 0$$

$$\Rightarrow \quad -T\delta(2b\cos\theta) + 4W\delta(b\sin\theta\cos\theta) = 0$$

$$\Rightarrow \quad 2bT\sin\theta\,\delta\theta + 4Wb(\cos^2\theta - \sin^2\theta)\delta\theta = 0$$

or $\quad 2b\left[T\sin\theta + 2W(\cos^2\theta - \sin^2\theta)\right]\delta\theta = 0$

$\because \quad \delta\theta \neq 0$

$\therefore \quad 2b\left[T\sin\theta + 2W(\cos^2\theta - \sin^2\theta)\right] = 0$

or $\quad T\sin\theta + 2W(\cos^2\theta - \sin^2\theta) = 0$

or $$T = \frac{2W(\sin^2\theta - \cos^2\theta)}{\sin\theta}$$

$$= \frac{2W(1 - 2\cos^2\theta)}{\sqrt{1-\cos^2\theta}}$$

$\because \quad BD = a \quad$ so $\quad BO = \frac{a}{2}$

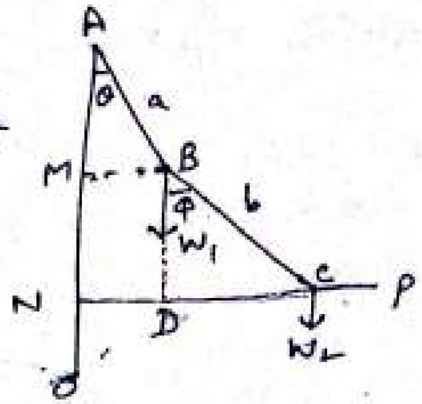
$\Rightarrow \quad \cos\theta = \frac{BO}{AB} = \frac{a/2}{b} = \frac{a}{2b}$

$$\therefore \quad T = \frac{2W\left\{1 - 2\left(\frac{a^2}{4b^2}\right)\right\}}{\sqrt{\{1 - a^2/4b^2\}}}$$

$$\boxed{T = \frac{2W\{2b^2 - a^2\}}{b\sqrt{4b^2 - a^2}}}$$

Q(2) Weights W_1, W_2 are fastened to a light inextensible string ABC at the points B, C, the end A being fixed. Prove that, if a horizontal force P is applied at C and in equilibrium AB, BC are inclined at angles θ, ϕ to the vertical, then

$$P = (W_1 + W_2)\tan\theta = W_2\tan\phi.$$



Solⁿ Let the length of the Portion AB of the string be 'a' and that of BC be 'b'.

Let A be fixed and AO is vertical.

The depth of B from A = AM
$= a\cos\theta$

The depth of C from A = AN
$= a\cos\theta + b\cos\phi$

The horizontal distance of C from AO
($\because$ AO is fixed line)

$= CN$
$= ND + DC$
$= MB + DC$
$= a\sin\theta + b\sin\phi$.

Give a small displacement to the system s.t. θ changes to $\theta+\delta\theta$ and ϕ changes to $\phi+\delta\phi$. (length of the string remains unchanged)

The by the principle of virtual work

$$W_1\delta(AM) + W_2\delta(AN) + P\delta(CN) = 0$$

or $$W_1\delta(a\cos\theta) + W_2\delta(a\cos\theta + b\cos\phi) + P\delta(a\sin\theta + b\sin\phi) = 0$$

or $$-aW_1\sin\theta\,\delta\theta - aW_2\sin\theta\,\delta\theta - bW_2\sin\phi\,\delta\phi + aP\cos\theta\,\delta\theta + bP\cos\phi\,\delta\phi = 0$$

$$a\left[P\cos\theta - (W_1+W_2)\sin\theta\right]\delta\theta = b\left[W_2\sin\phi - P\cos\phi\right]\delta\phi \quad \text{--- (1)}$$

where θ and ϕ are independent.

Consider the displacement, when θ changes and ϕ does not change, so $\delta\phi = 0$ & $\delta\theta \neq 0$

$\therefore$ from (1), we have

$$a\left[P\cos\theta - (W_1+W_2)\sin\theta\right]\delta\theta = 0$$

$\because \delta\theta \neq 0 \Rightarrow P\cos\theta - (W_1+W_2)\sin\theta = 0$, $\because a \neq 0$

$$\Rightarrow P = (W_1+W_2)\tan\theta.$$

Consider the displacement, when ϕ changes and θ does not change, so $\delta\theta = 0$, $\delta\phi \neq 0$

$\therefore$ from (1), we have

$$b\left[W_2\sin\phi - P\cos\phi\right]\delta\phi = 0$$

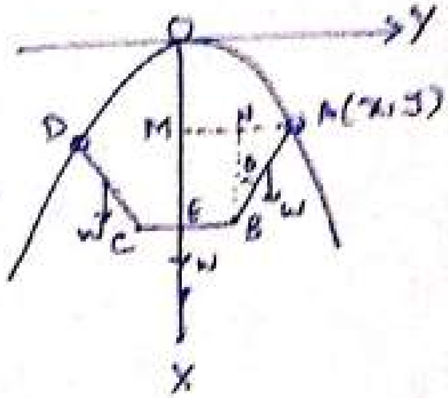
$\because \delta\phi \neq 0 \Rightarrow W_2\sin\phi - P\cos\phi = 0$, (as $b \neq 0$)

$$\Rightarrow P = W_2\tan\phi.$$

Hence $\boxed{P = (W_1+W_2)\tan\theta = W_2\tan\phi}$

Q(3) Three equal and similar rods AB, BC, CD freely jointed at B and C have small weightless rings attached to them at A and D. The rings slide on a smooth parabolic wire, whose axis is vertical and vertex upwards and whose latus rectum is half the sum of the lengths of the three rods. Prove that in the position of equlibrium, the ~~led~~ inclination θ of AB or CD to the vertical is given by

$$\cos\theta - \sin\theta + \sin 2\theta = 0.$$



Solⁿ Let $AB = BC = CD = 2a$.

$\therefore AB + BC + CD = 6a$

$\therefore$ Latus rectum of the parabola $= 3a$

(Since latus rectum is half the sum of length of rods)

Hence eqⁿ of parabola $y^2 = 3ax$

In the position of equlibrium, let θ be the inclination of AB or CD with vertical. Let W be the weights of rods AB, BC and CD. act at the middle points.

Let the co-ordinates of A be (x, y)

$\therefore OM = x$ and $MA = y$

$\therefore y = MN + NA$

$= EB + NA$

$= a + 2a\sin\theta$ (In fig.)

(x, y) lies on parabola $y^2 = 3ax$

$\therefore (a + 2a\sin\theta)^2 = 3ax$ ——— (1)

Differentiate it, we get

$$2(a + 2a\sin\theta).2a\cos\theta\,\delta\theta = 3a\,\delta x$$

or $\delta x = \frac{4}{3}a(1 + 2\sin\theta)\cos\theta\,\delta\theta$ —— (2)

Let OY is fixed line. The depth of the middle points of AB or CD below OY is $x + \frac{1}{2}ME$

$= x + \frac{1}{2}NB$

$= x + a\cos\theta.$

The depth of BC below OY $= OE$

$= x + ME$

$= x + NB$

$= x + 2a\cos\theta.$

Give a small displacement to the system so that θ changes to $\theta + \delta\theta$.

The eq^n of virtual work

$$2W\delta(x + a\cos\theta) + W\delta(x + 2a\cos\theta) = 0$$

$$\Rightarrow 2W[\delta x - a\sin\theta\,\delta\theta] + W[\delta x - 2a\sin\theta\,\delta\theta] = 0$$

or $3W\delta x - 4aW\sin\theta\,\delta\theta = 0$

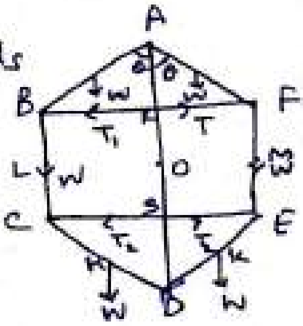
or $3W\cdot\frac{4}{3}a(1+2\sin\theta)\cos\theta\,\delta\theta - 4aW\sin\theta\,\delta\theta = 0$ (from (2))

or $4aW[\cos\theta + 2\sin\theta\cos\theta - \sin\theta]\delta\theta = 0$

$\because a \neq 0,\ W \neq 0,\ \delta\theta \neq 0$

$\Rightarrow \boxed{\cos\theta + \sin 2\theta - \sin\theta = 0}$

Q(4) Six equal heavy rods, freely hinged at the ends, form a regular hexagon ABCDEF, which when hung up by the point A is kept from altering its shape by two light rods BF and CE. Prove that the thrusts of these rods are $\frac{5\sqrt{3}}{2}W$ and $\frac{\sqrt{3}}{2}W$, where W is the weight of each rod.



Sol^n : Let the ~~leg~~ length of the rods AB, BC, CD, DE, EF and AF be $2a$ and θ be the angle which makes slant rods AB, AF, CD, DF make with vertical AD.

Let T_1 and T_2 be thrusts in rods BF and CE respectively.

Let A be fixed point.

The weight of each rods AB, BC, CD, DE, EF and AF act at their mid points.

Give a small displacement to the system, so that θ changes to $\theta + \delta\theta$ at the end A, while at the end D, θ does not change.

Let length BF changes, while length CE does not change, so that during this displacement the work done by the thrust T_2 of the rod CE is zero.

The centres of gravity of all six rods are slightly displaced

$\therefore$ $BF = 2.(2a\sin\theta) = 4a\sin\theta$

The depth of each of the points M and N below A is $a\cos\theta$, the depth of each of the points P and Q below A is

$$2a\cos\theta + a$$

$\because AO = AR + RO$
$= AR + LB$
$= 2a\cos\theta + a$
as L or M is mid point of rod of length $2a$.

The depth of each of the point H and K below A is

$$2a\cos\theta + 2a + \tfrac{1}{2}SD$$

(SD is fixed $\Rightarrow \delta(SD) = 0$

$SD =$

By the principle of virtual work,

$$T_1\,\delta(4a\sin\theta) + 2W\delta(a\cos\theta) + 2W\delta(2a\cos\theta + a) + 2W\delta\left(2a\cos\theta + 2a + \tfrac{1}{2}SD\right) = 0$$

$$(4aT_1\cos\theta - 2aW\sin\theta - 4aW\sin\theta - 4aW\sin\theta)\delta\theta = 0$$

$\because \delta\theta \neq 0$

$$\Rightarrow 4aT_1\cos\theta - 10aW\sin\theta = 0$$

or $T_1 = \dfrac{5}{2}W\tan\theta.$

But in the position of equilibrium, the hexagon is regular $\Rightarrow \theta = \pi/3$

$$\therefore T_1 = \frac{5}{2}W\tan\frac{\pi}{3} = \frac{5\sqrt{3}}{2}W$$

$$\boxed{T_1 = \frac{5\sqrt{3}}{2}W}$$

for thrust T_2,

Replace the rod BF by two equal and opposite forces T_1 and CE by two equal and opposite forces T_2. Give a small displacement to the system about line AD, so that θ changes to $\theta + \delta\theta$ at both ends A and D. In this case the total weight 6W of all ~~rods~~ six rods can be taken in the middle point O of AD.

$\therefore \quad BF = 4a\sin\theta, \quad CE = 4a\sin\theta$

$AO = 2a\sin\theta + a$

By the principle of virtual work, we have

$$T_1\,\delta(4a\sin\theta) + T_2\,\delta(4a\sin\theta) + 6W\,\delta(2a\cos\theta + a) = 0$$

$$\Rightarrow 4aT_1\cos\theta\,\delta\theta + 4aT_2\cos\theta\,\delta\theta - 12aW\sin\theta\,\delta\theta = 0$$

$$\Rightarrow 4a(T_1 + T_2)\cos\theta = 12aW\sin\theta \quad , (\because \delta\theta \neq 0)$$

$$\Rightarrow T_1 + T_2 = 3W\tan\theta.$$

But in the position of equilibrium $\theta = \pi/3$

$$\Rightarrow T_1 + T_2 = 3W\tan\frac{\pi}{3} = 3W.\sqrt{3}$$

$$= \cancel{3W.\frac{1}{\sqrt{3}}}$$

$$\Rightarrow T_1 + T_2 = \cancel{\sqrt{3}}\, 3\sqrt{3}W$$

$$\Rightarrow T_2 = 3\sqrt{3}W - T_1$$

$$= 3\sqrt{3}W - \frac{5\sqrt{3}}{2}W$$

$$= W\frac{\sqrt{3}}{2}$$

$$\boxed{T_2 = \frac{W\sqrt{3}}{2}}$$