Hypergeometric Function

Pochhammer Symbol: For a positive integer n, the Pochhammer Symbol is denoted and defined by

$$(a)_n = a(a+1)(a+2)(a+3)\ldots\ldots\ldots\{a+(n-1)\}$$

and $(a)_0 = 1$.

(i) $(a)_n = \frac{\Gamma(a+n)}{\Gamma a}$

(ii) $(a)_{n+1} = a(a+1)_n$

(iii) $(a+n)(a)_n = (a)_{n+1}$

General Hypergeometric Function: The general hypergeometric function is denoted and defined by

$${}_mF_n\ (a_1, a_2, \ldots.a_m; b, b_2, \ldots.b_n; x) = \sum \frac{(a_1)_r \ldots\ldots (a_m)_r}{(b_1)_r \quad (b_n)_r} \frac{x^r}{r!}$$

It is also denoted by

$${}_mF_n \begin{bmatrix} a_1, a_2, \ldots.a_m; & \\ b, b_2, \ldots.b_n\ ; & x \end{bmatrix}$$

Where
m − refers to number of parameters in numerator
n − refers to number of parameters in denominator.

Kummer's function: For $m = n = 1$, the function is known as Kummer's function or confluent hypergeometric function, we have

$${}_1F_1\ (a; b; x) = F(a; b; x) = \sum \frac{(a)_n}{(b)_n} \frac{x^n}{n!}$$

$$F(a; b; x) = 1 + \frac{a}{b}\frac{x}{1!} + \frac{a(a+1)}{b(b+1)}\frac{x^2}{2!} + \frac{a(a+1)(a+2)}{b(b+1)(b+2)}\frac{x^3}{3!} + \cdots \ldots\ldots\ldots\ldots..$$

Hypergeometric function: For $m = 2, n = 1$, the function is known as hypergeometric function, we have

$${}_2F_1\ (a, b; c; x) = F(a, b; c; x) = \sum \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!}$$

$$F(a, b; c; x) = 1 + \frac{ab}{c}\frac{x}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)}\frac{x^2}{2!} + \cdots \ldots\ldots\ldots\ldots..$$

This is called as **hypergeometric series**.

$$F(a,b;c;x) = F(b,a;c;x)$$

This is **symmetric property** of hypergeometric function.

$x(1-x)\frac{d^2y}{dx^2} + [c-(a+b+1)x]\frac{dy}{dx} - aby = 0$ is known as **hypergeometric equation**.

The general solution of this hypergeometric differential equation is given by

$$\mathrm{y} = \mathrm{A}\ {}_2F_1\ (a,b;c;x) + Bx^{1-c}\ {}_2F_1\ (1-c+a, 1-c+b; 2-c; x)$$

where A, B are arbitrary constants.

Differentiation of hypergeometric functions:

$$\frac{\mathrm{d}}{\mathrm{dx}}\ [\mathrm{F}\ (a,b;c;x)] = \frac{ab}{c}\ F(a+1,b+1;c+1;x)$$

$$\frac{\mathrm{d}^2}{\mathrm{dx}^2}\ [\mathrm{F}\ (a,b;c;x)] = \frac{(a)_2(b)_2}{(c)_2}\ F(a+2,b+2;c+2;x)$$

……………………………………………………………………………

$$\frac{\mathrm{d}^\mathrm{n}}{\mathrm{dx}^\mathrm{n}}\ [\mathrm{F}\ (a,b;c;x)] = \frac{(a)_n(b)_n}{(c)_n}\ F(a+n,b+n;c+n;x)$$

Also $\left[\frac{\mathrm{d}^\mathrm{n}}{\mathrm{dx}^\mathrm{n}}\ [\mathrm{F}\ (a,b;c;x)]\right]_{\mathrm{x}=0} = \frac{(a)_n(b)_n}{(c)_n}$

Integral representation of hypergeometric function:

$$\mathrm{F}\ (a,b;c;x) = \frac{\Gamma c}{\Gamma b\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-xt)^{-a}dt$$

Or

$$\mathrm{F}\ (a,b;c;x) = \frac{1}{\mathrm{B(b,c-b)}} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-xt)^{-a}dt \quad if\ c > b > 0$$

Recurrence Relations:

$$(\mathrm{a}-\mathrm{b})\mathrm{F}\ (a,b;c;x) = a\ \mathrm{F}\ (a+1,b;c;x) - b\mathrm{F}\ (a,b+1;c;x)$$

Question 1: Prove that $P_n(x) = {}_2F_1\left(-n, n+1; 1; \frac{1-x}{2}\right)$

Solution: Using Rodrigue's formula we have

$$P_n(x) = \frac{1}{n!2^n}\frac{d^n}{dx^n}(x^2-1)^n$$

$$= \frac{(-1)^n}{n!2^n}\frac{d^n}{dx^n}(1-x^2)^n$$

$$= \frac{(-1)^n}{n!}\frac{d^n}{dx^n}\left[(1-x)^n\frac{(1+x)^n}{2^n}\right]$$

$$= \frac{(-1)^n}{n!}\frac{d^n}{dx^n}\left[(1-x)^n\left\{1-\left(\frac{1-x}{2}\right)\right\}^n\right]$$

$$= \frac{(-1)^n}{n!}\frac{d^n}{dx^n}\left[(1-x)^n\left\{1-n\left(\frac{1-x}{2}\right)+\frac{n(n-1)}{2!}\left(\frac{1-x}{2}\right)^2-\cdots\ldots\ldots.\right\}\right]$$

$$= \frac{(-1)^n}{n!}\frac{d^n}{dx^n}\left[(1-x)^n-\frac{n}{2}(1-x)^{n+1}+\frac{n(n-1)}{2!2^2}(1-x)^{n+2}-\right]$$

$$= \frac{(-1)^n}{n!}\left[(-1)^n\ n!-(-1)^n\frac{(n+1)!}{1!}\frac{n}{2}(1-x)+(-1)^n\frac{(n+2)!}{2!}\frac{n(n-1)}{2!2^2}(1-x)^2-\cdots\ldots.\right]$$

$$\because \boldsymbol{\frac{d^n}{dx^n}(a-bx)^m = (-1)^n b^n \frac{m}{(m-n)!}(a-bx)^{m-n}}$$

$$= \frac{(-1)^{2n}}{n!}\left[n!-\frac{n!}{1!}\frac{n(n+1)}{2}(1-x)+\frac{n!}{2!}\frac{n(n-1)(n+2)(n+1)}{2!2^2}(1-x)^2-\cdots\ldots.\right]$$

$$= \left[1+\frac{-n(n+1)}{1.1!}\frac{(1-x)}{2}+\frac{-n(-n+1)(n+1)(n+2)}{1.2.2!}\left(\frac{1-x}{2}\right)^2-\cdots\ldots.\right]$$

$$= {}_2F_1\left(-n, n+1; 1; \frac{1-x}{2}\right)$$

Hence $\qquad P_n(x) = {}_2F_1\left(-n, n+1; 1; \frac{1-x}{2}\right).$

Question 2: Prove that

(i) $F(1, b; c; x) = (1-x)^{-1}$

(ii) $\log_e(1+x) = x\ {}_2F_1(1,1;2;-x)$

Solution:

(i) By the definition we know that

$$ {}_2F_1\ (a,b;c;x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!} $$

$$ {}_2F_1\ (a,b;c;x) = 1 + \frac{ab}{c}\frac{x}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)}\frac{x^2}{2!} + \cdots \ldots \ldots $$

Taking $a = 1, c = b$, we have

$$ F(1,b;b;x) = 1 + \frac{1.b}{b}\frac{x}{1!} + \frac{1(1+1)b(b+1)}{b(b+1)}\frac{x^2}{2!} + \cdots \ldots \ldots $$

$$ = 1 + \frac{(-1)}{1!}(-x) + \frac{-1.-2}{2!}(-x)^2 + \cdots \ldots \ldots $$

$$ = (1-x)^{-1} $$

Hence $\quad F\ (1,b;c;x) = (1-x)^{-1}$

(ii) We know that the hypergeometric series will be

$$ {}_2F_1\ (a,b;c;x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!} $$

$$ {}_2F_1\ (a,b;c;x) = 1 + \frac{ab}{c}\frac{x}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)}\frac{x^2}{2!} + \cdots \ldots \ldots $$

Taking $a = 1, b = 1, c = 2, and\ x = -x$, we have

$$ {}_2F_1(1,1;2;-x) = 1 + \frac{1.1}{1}\frac{(-x)}{1!} + \frac{1(1+1).1.(1+1)}{2(2+1)}\frac{(-x)^2}{2!} + \cdots \ldots \ldots $$

Multiply by x on both side

$$ x\,{}_2F_1(1,1;2;-x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots \ldots \ldots $$

$$ = \log_e(1+x) $$

Hence $\quad x\ {}_2F_1\ (1,1;2;-x) = \log_e(1+x)$

Question 3: Prove that $P_n(\cos\Theta) = \cos^n\Theta\,{}_2F_1\ \left(-\frac{n}{2}, -\frac{n-1}{2}; 1; -\tan^2\Theta\right)$

Solution: Using Laplace first integral, we have

$$ P_n(x) = \frac{1}{\pi}\int_0^{\pi} \left[x \pm \sqrt{x^2-1}\ \cos\phi\right]^n d\phi $$

Taking $x = \cos\Theta \Rightarrow \quad \sqrt{x^2-1} = i\sin\Theta$ Also taking positive sign in the above integral. We get

$$P_n(x) = \frac{1}{\pi}\int_0^{\pi} [\cos\theta + i\sin\theta\cos\phi]^{\,n}\, d\phi$$

$$= \frac{\cos^{n}\theta}{\pi}\int_0^{\pi} [1 + i\tan\theta\cos\phi]^{\,n}\, d\phi$$

Using Binomial expansion

$$P_n(x) = \frac{\cos^{n}\theta}{\pi}\int_0^{\pi} \left[1 + n\, i\tan\theta\cos\phi + \frac{n(n-1)}{2!}\ (i\tan\theta\cos\phi)^2 \ldots\ldots\ldots\right] d\phi$$

$$= \frac{\cos^{n}\theta}{\pi}\left[\int_0^{\pi} 1\, d\phi + n\, i\tan\theta\int_0^{\pi} \cos\phi\, d\phi + \frac{n(n-1)}{2!}\ (i\tan\theta)^2\int_0^{\pi} (\cos\phi)^2 d\phi \ldots\ldots\right]$$

$$= \frac{\cos^{n}\theta}{\pi}\left[\pi + n\, i\tan\theta(0) - \frac{n(n-1)}{2!}\ (\tan\theta)^2\left(\frac{\pi}{2}\right) + \cdots\ldots\right]$$

$$= \cos^{n}\theta\left[1 + \frac{\left(-\frac{n}{2}\right)\left(-\frac{n-1}{2}\right)}{1}\left(-\frac{\tan^2\theta}{1!}\right) + \cdots\ldots\right]$$

$$= \cos^{n}\theta\ {}_2F_1\left(-\frac{n}{2}, -\frac{n-1}{2}; 1; -\tan^2\theta\right)$$

Hence $\qquad P_n(\cos\theta) = \cos^{n}\theta\ {}_2F_1\left(-\frac{n}{2}, -\frac{n-1}{2}; 1; -\tan^2\theta\right)$

Question 4: If $|x| < 1\ \ and\ \ \left|\frac{x}{1-x}\right| < 1$, then show that

$${}_2F_1\,(a,b;c;x) = (1-\mathrm{x})^{-\mathrm{a}}\ {}_2F_1\left(a, c-b; c; \frac{-x}{1-x}\right)$$

Solution: The Integral representation of hypergeometric function will be

$${}_2F_1(a,b;c;x) = \frac{\Gamma c}{\Gamma b\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-xt)^{-a}dt \qquad (1)$$

Put $1-t = z\ \Rightarrow\ \ dt = -dz$

$${}_2F_1(a,b;c;x) = \frac{\Gamma c}{\Gamma b\Gamma(c-b)}\int_1^0 (1-z)^{b-1}z^{c-b-1}(1-x+xz)^{-a}(-dz)$$

$$= \frac{\Gamma c(1-x)^{-a}}{\Gamma b\Gamma(c-b)}\int_0^1 (1-z)^{b-1}z^{c-b-1}\left(1+\frac{xz}{1-x}\right)^{-a} dz \qquad (2)$$

Also if we take $b = c-b\ and\ x = \frac{-x}{1-x}$ in Eq.(1), we have

$${}_2F_1\left(a, c-b; c; \frac{-x}{1-x}\right) = \frac{\Gamma c}{\Gamma(c-b)\Gamma b}\int_0^1 (1-t)^{b-1}t^{c-b-1}\left(1+\frac{xt}{1-x}\right)^{-a} dt$$

$$ {}_2F_1\left(a, c-b; c; \frac{-x}{1-x}\right) = \frac{\Gamma c}{\Gamma(c-b)\Gamma b} \int_0^1 (1-z)^{b-1} z^{c-b-1} \left(1+\frac{xz}{1-x}\right)^{-a} dz \qquad (3) $$

From Eq.(2) and (3), we have

$$ {}_2F_1(a, b; c; x) = (1-x)^{-a}\, {}_2F_1\left(a, c-b; c; \frac{-x}{1-x}\right) $$

Hence

$$ F(a, b; c; x) = (1-x)^{-a} F\left(a, c-b; c; \frac{-x}{1-x}\right) $$

Question 5: Show that

(i) $\quad {}_2F_1\left(\frac{1}{2}, 1; \frac{3}{2}; \mathrm{x}^2\right) = \frac{1}{2x} log\left(\frac{1+x}{1-x}\right)$ or $\frac{1}{2}\tan^{-1} x \qquad for\ |x| < 1$

(ii) $\quad \lim_{b\to\infty} F\left(1, b; 1; \frac{x}{b}\right) = \mathrm{e}^{\mathrm{x}}$

Solution:

(i) We know that the hypergeometric series will be

$$ {}_2F_1\ (a, b; c; x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!} $$

Replace $a = \frac{1}{2}, b = 1, c = \frac{3}{2}$, $x = \mathrm{x}^2$ in above expression

$$ {}_2F_1\ \left(\frac{1}{2}, 1; \frac{3}{2}; \mathrm{x}^2\right) = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n (1)_n}{\left(\frac{3}{2}\right)_n} \frac{x^{2n}}{(1)_n} \qquad \text{as } (1)_n = n! $$

$$ = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n}{\left(\frac{3}{2}\right)_n} x^{2n} $$

$$ = \sum_{n=0}^{\infty} \frac{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}\ldots..(n-\frac{1}{2})}{\frac{3}{2}\cdot\frac{5}{2}\ldots..(n-\frac{1}{2})(n+\frac{1}{2})} x^{2n} $$

$$ = \sum_{n=0}^{\infty} \frac{x^{2n}}{2n+1} $$

$$ = \frac{1}{2x}\left[\sum_{n=0}^{\infty} (-1)^{n+1} \frac{x^n}{n} - \sum_{n=1}^{\infty} \frac{x^n}{n}\right] $$

$$ = \frac{1}{2x}[\log(1+x) - \log(1-x)], \qquad for\ |x| < 1 $$

$$ = \frac{1}{2x} log\left(\frac{1+x}{1-x}\right) $$

Hence $\quad {}_2F_1\left(\frac{1}{2},1;\frac{3}{2};\mathrm{x}^2\right)=\frac{1}{2x}log\left(\frac{1+x}{1-x}\right) \qquad for\ |x|<1$

(ii) We know that the hypergeometric series will be

$${}_2F_1\ (a,b;c;x)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{x^n}{n!}$$

Replace $a=1,b=b,c=1\ ,\ x=\frac{x}{b}$ in above expression

$${}_2F_1\ \left(1,b;1;\frac{x}{b}\right)=\sum_{n=0}^{\infty}\frac{(1)_n(b)_n}{(1)_n}\frac{1}{n!}\left(\frac{x}{b}\right)^n$$

$$=\sum_{n=0}^{\infty}\frac{(b)_n}{n!}\left(\frac{x}{b}\right)^n$$

$$=1+\sum_{n=1}^{\infty}\frac{(b)_n}{n!}\left(\frac{x}{b}\right)^n$$

$$=1+\sum_{n=1}^{\infty}\frac{b(b+1)(b+2)\ldots\ldots(b+n-1)}{n!}\frac{(x)^n}{(b)^n}$$

$$F\left(1,b;1;\frac{x}{b}\right)=1+\sum_{n=1}^{\infty}1.\left(1+\frac{1}{b}\right)\left(1+\frac{1}{b}\right)\left(1+\frac{1}{b}\right)\ldots..\left(1+\frac{1}{b}\right)\frac{x^n}{n!}$$

$$\lim_{b\to\infty}F\left(1,b;1;\frac{x}{b}\right)=\lim_{b\to\infty}\left[1+\sum_{n=1}^{\infty}1.\left(1+\frac{1}{b}\right)\left(1+\frac{1}{b}\right)\left(1+\frac{1}{b}\right)\ldots..\left(1+\frac{1}{b}\right)\frac{x^n}{n!}\right]$$

$$\lim_{b\to\infty}F\left(1,b;1;\frac{x}{b}\right)=1+\sum_{n=1}^{\infty}\frac{x^n}{n!}$$

$$\lim_{b\to\infty}F\left(1,b;1;\frac{x}{b}\right)=\sum_{n=0}^{\infty}\frac{x^n}{n!}=\mathrm{e}^{\mathrm{x}}$$

Hence $\quad \lim_{b\to\infty}F\left(1,b;1;\frac{x}{b}\right)=\mathrm{e}^{\mathrm{x}}$

Theorem 1: Differentiation of hypergeometric functions:

$$\frac{\mathrm{d}^{\mathrm{n}}}{\mathrm{dx}^{\mathrm{n}}}\ [\mathrm{F}\ (a,b;c;x)] = \frac{(a)_n(b)_n}{(c)_n}\ F(a+n,b+n;c+n;x)$$

$$\text{Also } \left[\frac{\mathrm{d}^{\mathrm{n}}}{\mathrm{dx}^{\mathrm{n}}}\ [\mathrm{F}\ (a,b;c;x)]\right]_{\mathrm{x}=0} = \frac{(a)_n(b)_n}{(c)_n}$$

Proof: We know that the hypergeometric series will be

$$_2F_1\ (a,b;c;x) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n}\ \frac{x^n}{n!} \tag{1}$$

Differentiate w. r. t. x on both side, we get

$$\frac{\mathrm{d}}{\mathrm{dx}}\ [\mathrm{F}\ (a,b;c;x)] = \sum_{n=1}^{\infty} \frac{(a)_n(b)_n}{(c)_n}\ \frac{n\,x^{n-1}}{n!}$$

$$= \sum_{n=1}^{\infty} \frac{(a)_n(b)_n}{(c)_n}\ \frac{x^{n-1}}{(n-1)!}$$

Taking $m = n-1$

$$= \sum_{m=0}^{\infty} \frac{(a)_{m+1}(b)_{m+1}}{(c)_{m+1}}\ \frac{x^m}{m!}$$

Using $(a)_{n+1} = a(a+1)_n$ etc

$$= \sum_{m=0}^{\infty} \frac{a(a+1)_m b(b+1)_m}{c(c+1)_m}\ \frac{x^m}{m!}$$

$$= \frac{ab}{c}\sum_{m=0}^{\infty} \frac{(a+1)_m\ (b+1)_m}{(c+1)_m}\ \frac{x^m}{m!}$$

$$\frac{d}{dx}\ [\mathrm{F}\ (a,b;c;x)] = \frac{ab}{c}\mathrm{F}\ (a+1,b+1;c+1;x) \tag{2}$$

which shows that the results is true for $n = 1$.

Let us suppose that the results is true for a particular value $n = m$ then

$$\frac{d^m}{dx^m}\ [F\ (a,b;c;x)] = \frac{(a)_m(b)_m}{(c)_m}\ F(a+m,b+m;c+m;x) \tag{3}$$

Differentiate w. r. t. x on both side, we get

$$\frac{d^{m+1}}{dx^{m+1}}\ [F\ (a,b;c;x)] = \frac{(a)_m(b)_m}{(c)_m}\ \frac{d}{dx}F(a+m,b+m;c+m;x)$$

$$= \frac{(a)_m(b)_m}{(c)_m}\frac{(a+m)(b+m)}{(c+m)}\ F(a+m+1,b+m+1;c+m+1;x) \qquad \text{from (2)}$$

$$\frac{d^{m+1}}{dx^{m+1}}\ [F\ (a,b;c;x)] = \frac{(a)_{m+1}(b)_{m+1}}{(c)_{m+1}}\ F(a+m+1,b+m+1;c+m+1;x)$$

Which shows that the result is true for $n = m+1$ also.

So by mathematical induction, we can say that it is true for every positive integral value of n.

Hence $\frac{d^n}{dx^n}\,[F\,(a,b;c;x)] = \frac{(a)_n(b)_n}{(c)_n}\,F(a+n,b+n;c+n;x)$

Put $x = 0$ in above result, we get

$\left[\frac{d^n}{dx^n}\,[F\,(a,b;c;x)]\right]_{x=0} = \frac{(a)_n(b)_n}{(c)_n}$

Theorem 2: Integral representation of hypergeometric function

$$F\,(a,b;c;x) = \frac{\Gamma c}{\Gamma b\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-xt)^{-a}dt$$

Proof: We know that the hypergeometric series will be

$$F(a,b;c;x) = \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\,\frac{x^n}{n!}$$

$$F(a,b;c;x) = \sum_{n=0}^{\infty}(a)_n\frac{\Gamma(b+n)}{\Gamma b}\frac{\Gamma c}{\Gamma(c+n)}\,\frac{x^n}{n!}$$

$$= \frac{\Gamma c}{\Gamma b\Gamma(c-b)}\sum_{n=0}^{\infty}(a)_n\frac{\Gamma(b+n)\Gamma(c-b)}{\Gamma[b+(c+n)-b]}\,\frac{x^n}{n!}$$

$$= \frac{\Gamma c}{\Gamma b\Gamma(c-b)}\sum_{n=0}^{\infty}(a)_n\frac{\Gamma(b+n)\Gamma(c-b)}{\Gamma[(b+n)+(c-b]}\,\frac{x^n}{n!}$$

$$= \frac{\Gamma c}{\Gamma b\Gamma(c-b)}\sum_{n=0}^{\infty}(a)_n B\{(b+n),(c-b)\}\,\frac{x^n}{n!}$$

$$= \frac{\Gamma c}{\Gamma b\Gamma(c-b)}\sum_{n=0}^{\infty}(a)_n\left(\int_0^1 t^{b+n-1}(1-t)^{c-b-1}dt\right)\frac{x^n}{n!}$$

$$= \frac{\Gamma c}{\Gamma b\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}\left(\sum_{n=0}^{\infty}(a)_n\,\frac{(xt)^n}{n!}\right)dt$$

Since $\sum_{n=0}^{\infty}(a)_n\,\frac{(xt)^n}{n!} = (1-xt)^{-a}$, we get

$$= \frac{\Gamma c}{\Gamma b\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-xt)^{-a}dt$$

So we have

$$F\,(a,b;c;x) = \frac{\Gamma c}{\Gamma b\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-xt)^{-a}dt$$

Theorem 3: Kummer's relation

$$\mathrm{F}\,(a;b;x)=e^{x}\mathrm{F}\,(b-a;b;-x)$$

Proof: We know that

$$\mathrm{F}\,(a;b;x)=\frac{\Gamma b}{\Gamma b\Gamma(b-a)}\int_0^1(1-t)^{b-a-1}t^{a-1}e^{xt}\,dt$$

Put $a=b-a\quad and\ x=-x$ in above relation, we get

$$\mathrm{F}\,(b-a;b;-x)=\frac{\Gamma b}{\Gamma(b-a)\Gamma a}\int_0^1(1-t)^{a-1}t^{b-a-1}e^{-xt}\,dt$$

Taking $1-t=z,\quad dt=-dz$, we have

$$\mathrm{F}\,(b-a;b;-x)=\frac{\Gamma b}{\Gamma(b-a)\Gamma a}\int_1^0 z^{a-1}(1-z)^{b-a-1}e^{-x(1-z)}(-dz)$$

$$=\frac{\Gamma b}{\Gamma(b-a)\Gamma a}\int_0^1 z^{a-1}(1-z)^{b-a-1}e^{-x(1-z)}dz$$

$$=\frac{\Gamma b e^{-x}}{\Gamma(b-a)\Gamma a}\int_0^1 z^{a-1}(1-z)^{b-a-1}e^{xz}\,dz$$

$$=e^{-x}.\frac{\Gamma b e^{-x}}{\Gamma(b-a)\Gamma a}\int_0^1 z^{a-1}(1-z)^{b-a-1}e^{xz}\,dz$$

$$\mathrm{F}\,(b-a;b;-x)=e^{-x}.\,\mathrm{F}\,(a;b;x)$$

Hence

$$\mathrm{F}\,(a;b;x)=e^{x}\mathrm{F}\,(b-a;b;-x)$$

Theorem 4: Contiguity relationship

$$(\mathrm{a}-\mathrm{b})\mathrm{F}\,(a,b;c;x)=a\,\mathrm{F}\,(a+1,b;c;x)-b\mathrm{F}\,(a,b+1;c;x)$$

Proof: The given expression in R.H.S is

$$a\,\mathrm{F}\,(a+1,b;c;x)-b\mathrm{F}\,(a,b+1;c;x)$$

$$=a\sum_{n=0}^{\infty}\frac{(a+1)_n(b)_n}{(c)_n}\,\frac{x^n}{n!}-\sum_{n=0}^{\infty}\frac{(a)_n(b+1)_n}{(c)_n}\,\frac{x^n}{n!}$$

$$=\sum_{n=0}^{\infty}\frac{a(a+1)_n(b)_n}{(c)_n}\,\frac{x^n}{n!}-\sum_{n=0}^{\infty}\frac{(a)_n b(b+1)_n}{(c)_n}\,\frac{x^n}{n!}$$

But $\quad a(a+1)_n=(a+n)(a)_n\quad and\quad b(b+1)_n=(b+n)(b)_n$, then we have

$$=\sum_{n=0}^{\infty}\frac{(a+n)(a)_n(b)_n}{(c)_n}\,\frac{x^n}{n!}-\sum_{n=0}^{\infty}\frac{(a)_n(b+n)(b)_n}{(c)_n}\,\frac{x^n}{n!}$$

$$= \sum_{n=0}^{\infty} [(a+n) - (b+n)] \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!}$$

$$= (a-b) \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!}$$

$$a\ \mathrm{F}\ (a+1, b; c; x) - b\mathrm{F}\ (a, b+1; c; x) = (\mathrm{a}-\mathrm{b})\mathrm{F}\ (a, b; c; x)$$

Hence

$$(\mathrm{a}-\mathrm{b})\mathrm{F}\ (a, b; c; x) = a\ \mathrm{F}\ (a+1, b; c; x) - b\mathrm{F}\ (a, b+1; c; x)$$