Notes on Matrix

Definition:

Let A be an $m \times n$ matrix. ***The rank of A*** is the maximal number of linearly independent row vectors.

Definition : Let A be an $m \times n$ matrix and B be its ***row-echelon form***. The rank of A is the number of pivots of B.

Determining Linear independence

Using Matrices how to find if m vectors are linearly independent:

1. Make the vectors the rows of a $m \times n$ matrix (where the vectors are of size n)

2 Find the rank of the matrix.

3 If the rank is m then the vectors are linearly independent. If the rank is less than m, then the vectors are linearly dependant

Let A be any $m \times n$ matrix. Then A consists of n column vectors $a_1, a_2, \ldots, a_n$, which are m-vectors.

Definition (in another form)

The rank of A is the maximal number of linearly independent column vectors in A, i.e. the maximal number of linearly independent vectors among $\{a_1, a_2, \ldots, a_n, \}$. If A = 0, ***then the rank of A is 0.***

Methods of Finding Rank of a Matrix:

(a) Minor Method

Let A be any $m \times n$ matrix. Then it has square sub matrices are called minors of **different** orders. The determinates of these square sub matrices are minors of A. To determine the rank of A we will follow these steps.

Step 1: Start with the highest order minor (or minors) of A. Let their order be **r .** If any one of them is non-zero then rank (A) = r.

Step 2: If all of them are zero, start with minors of next lower order zero minor. The order of that(r−1), and so on until we get a non–zero .The order of that minor will be the rank of the matrix A. ***Thus, the order of the highest non-zero minor is the rank of the matrix.***

A matrix A is said to be of rank r if zero minor of order r.if

(a) It has at least one non-zero minor of order r

(b) All the minors of order higher than r are zero.

(ii) If A is a null matrix, then rank (A) = 0

(iii) If A is non-zero $m \times n$ matrix, then $1 < rank(A) \leq minimum\ of\ m\ and\ n$

(iv) If A is a non-zero singular $n \times n$ matrix, then r***ank (A) = n***

(b) Echelon Form Method:

(c) Normal form Method:

Although above method already discussed in the regular classes.

Example: Test whether the vectors (1,-1,1), (2,1,1) and (3,0,2) are linearly dependent using rank method.

Solution: let $\boldsymbol{A} = \begin{bmatrix} \mathbf{1} & \mathbf{-1} & \mathbf{1} \\ \mathbf{2} & \mathbf{1} & \mathbf{1} \\ \mathbf{3} & \mathbf{0} & \mathbf{2} \end{bmatrix}$

$R_2 \to R_2 - 2R_1$ $\boldsymbol{A} = \begin{bmatrix} \mathbf{1} & \mathbf{-1} & \mathbf{1} \\ \mathbf{0} & \mathbf{3} & \mathbf{-1} \\ \mathbf{3} & \mathbf{0} & \mathbf{2} \end{bmatrix}$

$R_3 \to R_3 - 3R_1$ $\boldsymbol{A} = \begin{bmatrix} \mathbf{1} & \mathbf{-1} & \mathbf{1} \\ \mathbf{0} & \mathbf{3} & \mathbf{-1} \\ \mathbf{0} & \mathbf{3} & \mathbf{-1} \end{bmatrix}$

$R_3 \to R_3 - R_1$ $\boldsymbol{A} = \begin{bmatrix} \mathbf{1} & \mathbf{-1} & \mathbf{1} \\ \mathbf{0} & \mathbf{3} & \mathbf{-1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}$

Number of non zero rows is 2. So rank of the given matrix = 2.If number of non zero vectors = number of given vectors,then we can decide that the vectors are linearly independent. Otherwise we can say it is linearly dependent.Here rank of the given matrix is 2 which is less than the number of given vectors.So that we can decide the given vectors are ***linearly dependent***

Properties of Eigenvalues and Eigenvectors:

Let A be an $n \times n$ invertible matrix. The following are true:

1. If A is triangular, then the diagonal elements of A are the eigenvalues of A.

2. If λ is an eigenvalue of A with eigenvector $\vec{x}$, then $\frac{1}{\lambda}$is an eigenvalue of A^{-1} with eigenvector $\vec{x}$.

3. If λ is an eigenvalue of A then λ is an eigenvalue of A^T .

4. The sum of the eigenvalues of A is equal to tr(A), the trace of A.

5. The product of the eigenvalues of A is the equal to det(A), the determinant of A.

Invertible Matrix Theorem

Let A be an $n \times n$ matrix. The following statements are equivalent.

(a) A is invertible. (B) A does not have an eigenvalue of 0

Linear System of Equations

Suppose $a, b \in R$. Consider the system of equations $AX = B$.

•The system is consistent and has a unique solution for X

•The system is consistent and has an infinite number of solutions for X

• The system is inconsistent and has no solution for X

There are two basic cases to consider:

det A $\neq 0$ or A = 0

Case 1: det $\mathbf{A} \neq \mathbf{0}$

In this case A^{-1} exists and the **unique solution to** $AX = B$ is X = A^{-1} B

Case 2: **det A** = 0

In this case A^{-1} does not exist.

If $B \neq 0$ then the system $AX = B$. has **no solution**..

If $B = 0$ hen the system $AX = B$. has an **nfinite number of solutions**

We note that a homogeneous system AX = O has a unique solution X = O if **det A** $\neq \mathbf{0}$

(This is called the trivial solution) or an infinite number of solutions if **det A** = 0

<u>Key Idea</u>

Step1. First write the matrix equation of the given system of equation in the form $AX = B$.

Step2. Write argument matrix $[A; B]$

Step3. Apply E row transformation to reduce it in **Echelon form**

Step4. Determine rank of A and $[A; B]$. The different cases will arise depending upon the the number of equation, number of variables and relationship between ranks of A and $[A; B]$

Case1. When there are n equation in n variable, then coefficient matrix is a square matrix of order $n \times n$

(i) If $\rho(A) = \rho\,[A: B] = r = n$ (where n is the number of variables) then the system has a **unique solution. Where** $\rho(A)$ is the rank of matrix A

(ii) (ii) If $\rho(A) = \rho\,[A: B] = r < n$. then the system has **infinitely many solutions** in terms of remaining n – r unknowns which are arbitrary.

(iii) If $n - r = 1$ (then solution is one variable independent solution and let equal to K). n – r = 2 (then solution is two variable independent solution and let variables equal to K_1, and K_2) and so on.

Example 1 Solve the inhomogeneous system of equations

$x + y = 1$

$2x + y = 2$

Solution: Given system of equation can be expressed as $AX = B$ where

$A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix}, B = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Clearly det $\mathbf{A = 1 \neq 0}$

The system of equations has the unique solution:

$X = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ ans.

Another approach for finding consistency of system of equations. Conditions for Solution of Linear Equations:

In matrix notation these equations can be put in the form $AX = B$.

Where $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ b_{21} & b_{22} & b_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}$ $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ $B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ (1)

ie $[A{:}B] = \left[\begin{matrix} a_{11} & a_{12} & a_{13} \\ b_{21} & b_{22} & b_{23} \\ c_{31} & c_{32} & c_{33} \end{matrix} \vdots \begin{matrix} b_1 \\ b_2 \\ b_3 \end{matrix}\right]$ is argumented matrix

If the ranks of A and augmented matrix $[A{:}B]$ are equal, then the system is said to be **consistent** **otherwise inconsistent.**

There are following conditions for existing the solution of any system of linear equations:

(iv) If $\rho(A) = \rho\,[A{:}B] = r = n$ (where n is the number of variables) then the system has a **unique solution. Where** $\rho(A)$ is the rank of matrix A

(v) (ii) If $\rho(A) = \rho\,[A{:}B] = r < n$. then the system has **infinitely many solutions** in terms of remaining n – r unknowns which are arbitrary.

(vi) If $n - r = 1$ (then solution is one variable independent solution and let equal to K). n – r = 2 (then solution is two variable independent solution and let variables equal to K_1, and K_2) and so on.

Example1 $7x + 2y + 3x = 16$

$2x + 11y + 5z = 25$

$x + 3y + 4z = 13$

Solution: $A = \begin{bmatrix} 7 & 2 & 3 \\ 2 & 11 & 5 \\ 1 & 3 & 4 \end{bmatrix}$ $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ $B = \begin{bmatrix} 16 \\ 25 \\ 13 \end{bmatrix}$

The augmented matrix $[A{:}B] = \left[\begin{matrix} 7 & 2 & 3 \\ 2 & 11 & 5 \\ 1 & 3 & 4 \end{matrix} \vdots \begin{matrix} 16 \\ 25 \\ 13 \end{matrix}\right]$

Adopting similar procedure we can find $\rho(A) = \rho\,[A{:}B] = 3 \Rightarrow r = n = 3$

The system is consistent. The given system has a **unique solution.**

Theorem 1: *Let A be the m × n coefficient matrix corresponding to a homogeneous system of equations, and suppose A has rank r. Then, the solution to the corresponding system has n−r parameters.*

Example: Test the consistency of following system of linear equations and hence find the solution.

$4x - y = 12$

$-x + 5y - 2z = 0$

$-2y + 4z = -8$

Solution: The augmented matrix $[A{:}B] = \left[\begin{matrix} 4 & -1 & 0 \\ -1 & 5 & -2 \\ 0 & -2 & 4 \end{matrix} \vdots \begin{matrix} 12 \\ 0 \\ -8 \end{matrix}\right]$

Applying $R_1 \leftrightarrow R_2$, we get

$$[A:B] = \begin{bmatrix} 1 & 5 & -2 & \vdots & 0 \\ 4 & -1 & 0 & \vdots & 12 \\ 0 & -2 & 4 & \vdots & -8 \end{bmatrix}$$

$$R_2 \to R_2 + 4R_1 \quad \begin{bmatrix} 1 & 5 & -2 & \vdots & 0 \\ 0 & 19 & -8 & \vdots & 12 \\ 0 & -2 & 4 & \vdots & -8 \end{bmatrix}$$

$$R_1 \to - R_1, R_3 \to R_3 + R_2, \begin{bmatrix} 1 & 5 & -2 & \vdots & 0 \\ 0 & 19 & -8 & \vdots & 12 \\ 0 & 0 & \frac{60}{19} & \vdots & \frac{-128}{19} \end{bmatrix}$$

find $\rho(A) = \rho\,[A:B] = 3 \Rightarrow r = n = 3$ (The system is consistent).

Hence, there is a unique solution

$$\Rightarrow \begin{bmatrix} 1 & 5 & -2 \\ 0 & 19 & -8 \\ 0 & 0 & \frac{60}{19} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 12 \\ \frac{-128}{19} \end{bmatrix}$$

$\Rightarrow x + 5y - 2z = 0$ (i)

$19y - 8z = 12$ (ii)

$\frac{60}{19} z = \frac{-128}{19}$ (iii)

On solving $x = \frac{44}{15}$ $y = -\frac{4}{15}$ and $z = -\frac{32}{15}$ answer.

Example3 : *Investigate for what values of λ, μ the equations*

$x + y + z = 6,$

$x + 2y + 3z = 10,$

$x + 2y + \lambda z = \mu$

have (i)no solution

(ii) a unique solution

(iii) an infinity of solutions.

Solution: The augmented matrix

$$[A:B] = \begin{bmatrix} 1 & 1 & 1 & \vdots & 6 \\ 1 & 2 & 3 & \vdots & 10 \\ 1 & 2 & \lambda & \vdots & \mu \end{bmatrix}$$

$$R_2 \to R_2 - R_1, R_3 \to R_3 - R_1 \Rightarrow \begin{bmatrix} 1 & 1 & 1 & \vdots & 6 \\ 0 & 1 & 2 & \vdots & 4 \\ 0 & 1 & 1-\lambda & \vdots & \mu - 6 \end{bmatrix}$$

(i) For no solution $\rho(A) \neq \rho[A; B]$ it is only possible when $\lambda = 3$.

(ii) For unique solution $\rho(A) = \rho[A\ B]$ it is only possible when $\lambda - 3 \neq 0$ i.e., $\lambda \neq 3$ and $\mu \neq 10$

(iii) For infinite number of solutions ρ (A) = ρ [A : B] = r < n it is only possible when λ = 3 and μ = 10. Answer.

System Of Homogeneous Equations

If in the set of equations (1) , $b_1 = b_2 = b_3 = .. = 0$, the set of equation is said to be homogeneous. **Result 1**:If r = n, i.e., the rank of coefficient matrix is equal to the number of variables, then there is always a trivial solution (x= y = z... = = 0).

Result 2: If r < n, i.e., the rank of coefficient matrix is smaller than the number of variables, then there exist a non-trivial solution.

Result 3: For non-trivial solution always |A| = 0.

Example: *Solve the following system of homogeneous equations:*

x + 2y + 3z = 0

3x + 4y + 4z = 0

7x + 10y + 12z = 0

$$\text{Solution: Here, } A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 4 \\ 7 & 10 & 12 \end{bmatrix}$$

$$R_3 \to R_3 - 2R_2 \Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -8 \\ 0 & 4 & 9 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -5 \\ 0 & 0 & 1 \end{bmatrix}$$

This shows rank (A) = 3 = number of unknowns. Hence, the given system has a trivial solution i.e., x = y = z = 0.

Row echelon form:

Matrix is said to be in **row echelon form** if the following conditions hold:

1. The first non-zero element in each row, called the leading coefficient, is 1.
2. Each leading coefficient is in a column to the right of the previous row leading coefficient.
3. Rows with all zeros are below rows with at least one non-zero element.

Reduced row echelon form:

Matrix is said to be **row reduced echelon form** if the following conditions hold

- . All the conditions for **row echelon form**
- leading entry in each nonzero row is 1; and
- each column containing a leading 1 has zeros everywhere else

Method to get the row-reduced echelon form of a given matrix A

Step 1: Consider the first column of the matrix A. If all the entries in the first column are zero, move to the second column. Else, find a row, sayi^{th} row, which contains a non-zero entry in the first column. Now, interchange the first row with the i^{th}row. Suppose the non-zero entry in the (1, 1)- position is $\alpha \neq 0$. Divide the whole row by α so that the (1, 1) entry of the new matrix is 1. Now, use the 1 to make all the entries below this 1 equal to 0.

Step 2: If all entries in the first column after the first step are zero, consider the right m × (n − 1) sub matrix of the matrix obtained in step 1 and proceed as in step 1. Else, forget the first row and first column. Start with the lower (m − 1) × (n − 1) sub matrix of the matrix obtained in the first step and proceed as in step 1.

Step 3: Keep repeating this process till we reach a stage where all the entries below a particular row, say r, are zero. Suppose at this stage we have obtained a matrix C. Then C has the following form: 1. the first non-zero entry in each row of C is 1. These 1's are the leading terms of C and the columns containing these leading terms are the leading columns. 2. the entries of C below the leading term are all zero.

Step 4: Now use the leading term in the r th row to make all entries in the r th leading column equal to zero.

Step 5: Next, use the leading term in the $(r-1)^{th}$ row to make all entries in the $(r-1)^{th}$leading column equal to zero and continue till we come to the first leading term or column. The final matrix is the row-reduced echelon form of the matrix A.

Theorem 1 : *The row reduced echelon form of a matrix is unique.*

Theorem2: *Let A and B be two distinct augmented matrices for two homogeneous systems of m equations in n variables, such that A and B are each in reduced row-echelon form. Then, the two systems do not have exactly the same solution*

Theorem3: *If R is the reduced row-echelon form of a square matrix, then either R has a row of zeros or R is an identity matrix*

Proof of the theorem is given but result may help in solving numerical problems

Gaussian Elimination Method

Gaussian elimination is an efficient method for solving any linear system using systematic elimination of variables. It is an exact method which solves a given system of equations in **n** unknowns by transforming the coefficient matrix into an ***upper triangular matrix*** and then solve for the unknowns ***by back substitution***. This method can also be used to find *the* ***rank of the matrix***. It can be described in the following way:

The operations of the Gaussian elimination method are

- Write down the augmented matrix of the linear system.

•Use elementary row operations to reduce the matrix to an echelon form.

• Solve the linear system of the echelon form using back substitution.

• Interchange any two equations.

•. Replace an equation by a nonzero constant multiple of itself.

• Replace an equation by the sum of that equation and a constant multiple of any other equation

•. Continue until the final matrix is in row-reduced form

Gaussian elimination is the technique for finding the reduced row echelon form of a matrix using the above procedure. It can be abbreviated to:

1. Create a leading 1.

. Use this leading 1 to put zeros underneath it.

3. Repeat the above steps until all possible rows have leading 1s.

4. Put zeros above these leading 1.

Exapmle1. x + y+ z = 6

2x + 3y + 4z = 20

3x+ 4y + 2z = 17

Solution: The augmented matrix

$$[A:B] = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 2 & 3 & 4 \vdots & 20 \\ 3 & 4 & 2 & 17 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 3R_1$

$$[A:B] = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 \vdots & 8 \\ 0 & 1 & -1 & -1 \end{bmatrix}$$

$$R_3 \to R_3 - R_2, = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 \vdots & 8 \\ 0 & 0 & -3 & -9 \end{bmatrix}$$

Back Substitutions:$-3z = -9 \Rightarrow z = 3$, $y = 2, x = 1$ **Answer.**

Example2. Solve the system of equations

$x + 2y + z = 14$

$3x - y - 4z = 7$

$-x + y + 3z = 2$

Solution: Here argument matrix

$$[A:B] = \begin{bmatrix} 1 & 2 & 1 & 14 \\ 3 & -1 & -4 \vdots & 7 \\ -1 & 1 & 3 & 2 \end{bmatrix}$$

$R_2 \to R_2$ –3 R_1 , $R_3 \to R_3$ + R_1 $[A:B] = \begin{bmatrix} 1 & 2 & 1 & 14 \\ 0 & -7 & -7 \vdots & -35 \\ 0 & 3 & 4 & 16 \end{bmatrix}$

$R_3 \to R_3$ -3 R_2 , $R_2 \to R_2/7$ $[A:B] = \begin{bmatrix} 1 & 2 & 1 & 14 \\ 0 & 1 & 1 \vdots & 5 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

Clearly $z = 1$, $y + z = 5$ ie $y = 4$ and $x = 5$

Solve using Gauss Elimination method

$\boldsymbol{x + y + z = 6}$

$\boldsymbol{x - y + 2z = 5}$

$\boldsymbol{2x + 3z = 12}$

Solution: Here argument matrix $[A:B] = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & -1 & 2 \vdots & 5 \\ 2 & 0 & 3 & 12 \end{bmatrix}$

$R_2 \to R_2 - R_1$, $R_3 \to R_3$ -2 R_1 $[A:B] = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & -2 & 1 \vdots & -1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$

$R_3 \to R_3$ -2 R_2 $[A:B] = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & -2 & 1 \vdots & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Clearly $0x + 0y + 0z = 1$ this cannot satisfy any value of x, y, z .hence the given system has no solution .In other word rank of A is 2 and rank of AB is 3 so rank of A$< r < rank$ $[AB]$

To perform Gauss-Jordan Elimination:

Definition. A matrix in row-echelon form is said to be in Gauss-Jordan form, if all the entries above leading entries are zero.

In this method, the matrix of the coefficients in the equations, augmented by a column containing the corresponding constants, is reduced to an upper diagonal matrix using elementary row operations.

1. Swap the rows so that all rows with all zero entries are on the bottom
2. Swap the rows so that the row with the largest, leftmost nonzero entry is on top.
3. Multiply the top row by a scalar so that top row's leading entry becomes 1.
4. Add/subtract multiples of the top row to the other rows so that all other entries in the column containing the top row's leading entry are all zero.
5. Repeat steps 2-4 for the next leftmost nonzero entry until all the leading entries are 1.
6. Swap the rows so that the leading entry of each nonzero row is to the right of the leading entry of the row above it.

   Remark:

   Gauss-Jordan elimination is a variation of Gaussian elimination. The difference is that the elementary row operations in Gauss-Jordan elimination continue until we reach the reduced echelon form of the matrix. This means a few extra row operations, but easier calculations in the final step since back substitution is now longer needed. There are also some theoretical advantages of Gauss-Jordan elimination since the reduced echelon form is unique

Example *: Find the inverse of the matrix using Gauss-Jordan method*

$$\begin{bmatrix} \mathbf{2} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{2} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & \mathbf{2} \end{bmatrix}$$

Solution: Consider the matrix

$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \vdots \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ A sequence of steps in the Gauss-Jordan method

Our aim should be convert first matrix as unit matrix (through elementary row transformation only) and corresponding change in second matrix will be inverse.

Step1. $R_1 \to R_1/2$ $\begin{bmatrix} 1 & 1/2 & 1/2 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \vdots \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Step2 $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$ 2 $\begin{bmatrix} 1 & 1/2 & 1/2 \\ 0 & 3/2 & 1/2 \\ 0 & 1/2 & 3/2 \end{bmatrix} \vdots \begin{bmatrix} 1/2 & 0 & 0 \\ -1/2 & 1 & 0 \\ -1/2 & 0 & 1 \end{bmatrix}$

Step3. $R_2 \to 2R_2/3$ $\begin{bmatrix} 1 & 1/2 & 1/2 \\ 0 & 1 & 1/3 \\ 0 & 1/2 & 3/2 \end{bmatrix} \vdots \begin{bmatrix} 1/2 & 0 & 0 \\ -1/3 & 2/3 & 0 \\ -1/2 & 0 & 1 \end{bmatrix}$

Step4 $R_3 \to R_3 - R_2/2$ $\begin{bmatrix} 1 & 1/2 & 1/2 \\ 0 & 1 & 1/3 \\ 0 & 0 & 4/3 \end{bmatrix} \vdots \begin{bmatrix} 1/2 & 0 & 0 \\ -1/3 & 2/3 & 0 \\ -1/3 & -1/3 & 1 \end{bmatrix}$

Step5 $R_3 \to 3R_3/4$ $\begin{bmatrix} 1 & 1/2 & 1/2 \\ 0 & 1 & 1/3 \\ 0 & 0 & 1 \end{bmatrix} \vdots \begin{bmatrix} 1/2 & 0 & 0 \\ -1/3 & 2/3 & 0 \\ -1/4 & -1/4 & 3/4 \end{bmatrix}$

In next step we have

$$\begin{bmatrix} 1 & 1/2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \vdots \begin{bmatrix} 5/4 & 1/8 & -3/8 \\ -1/4 & 3/4 & -1/4 \\ -1/4 & -1/4 & 3/4 \end{bmatrix}$$

Lastly $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \vdots \begin{bmatrix} 3/4 & -1/4 & -1/4 \\ -1/4 & 3/4 & -1/4 \\ -1/4 & -1/4 & 3/4 \end{bmatrix}$

Hence final inverse $\begin{matrix} 3/4 & -1/4 & -1/4 \\ -1/4 & 3/4 & -1/4 \\ -1/4 & -1/4 & 3/4 \end{matrix}$

Some important Results:

1. *The inverse of a matrix is unique*

2. A square matrix A has an inverse if and only if $|A| \neq 0$ .It means only non singular matrix has inverse.

3. If A is non singular matrix and p is non zero positive integer, then $(A^p)^{-1}$=$(A^{-1})^p$

4. If A is non singular matrix of order n such that $AX = AY$ then $X = Y$

5. If A and B are any two $n \times n$ matrices such that $AB = O$ where O is null matrix, the at least one of them is singular

6. : If A is not invertible, then $A\vec{x} = \vec{0}$ could have no solutions.

7. Solving $A\vec{x} = \vec{b}$ using Gaussian elimination is faster than using the inverse of A

Example2. Solve by **Gauss – Jordan method**.

$6x - y + z = 13$

$x + y + z = 9$

$10x - y + z = 19$

Solution : First write second equation in place of first (because coefficient of x is unity)

$$[A:B] = \begin{bmatrix} 1 & 1 & 1 & 9 \\ 6 & -1 & 1 \vdots & 13 \\ 10 & -1 & 1 & 19 \end{bmatrix}$$

$$R_2 \to R_2 - 6R_1, R_3 \to R_3 - 10R_1 [A:B] = \begin{bmatrix} 1 & 1 & 1 & 9 \\ 0 & -7 & -5 \vdots & -41 \\ 0 & -9 & -11 & -71 \end{bmatrix}$$

$$R_3 \to R_3 - R_2 \, , R_3 \leftrightarrow R_2 \;\; [A:B] = \begin{bmatrix} 1 & 1 & 1 & 9 \\ 0 & -2 & -6 \vdots & -30 \\ 0 & -7 & -5 & -41 \end{bmatrix}$$

$$R_2 \to R_2/-2 \; [A:B] = \begin{bmatrix} 1 & 1 & 1 & 9 \\ 0 & 1 & 3 \vdots & 15 \\ 0 & -7 & -5 & -41 \end{bmatrix}$$

$$R_3 \to R_3 + 7R_2 \; [A:B] = \begin{bmatrix} 1 & 1 & 1 & 9 \\ 0 & 1 & 3 \vdots & 15 \\ 0 & 0 & 16 & 64 \end{bmatrix}$$

$$R_3 \to R_3/16 \; [A:B] = \begin{bmatrix} 1 & 1 & 1 & 9 \\ 0 & 1 & 3 \vdots & 15 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

$$R_2 \to R_2 - 3R_3, R_1 \to R_1 - R_3, [A:B] = \begin{bmatrix} 1 & 1 & 0 & 5 \\ 0 & 1 & 0 \vdots & 3 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

$$R_1 \to R_1 - R_2 \; [A:B] = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 \vdots & 3 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

$clearly \; x = 2 \, , y = 3 \; z = 4$ answer

Solve by Gauss – Jordan method:

1. $x - 3y - 8z = -10$

$3x + y = 4$

$2x + 5y + 6z = 13$

[**Ans. x = y = z = 1**]

2. $x + y + z = 6$

$2x + 3y - 2z = 2$

$5x + y + 2z = 13$

[**Ans. x = 1, y = 2, z = 3**]

3. $3x + y + 2z = 3$

$2x - 3y - z = -3$

$x + 2y + z = 4$

[**Ans. x = 1, y = 2, z = −1**]

4. $4x + 3y + 3z = -2$

$x + z = 0$

$4x + 4y + 3z = -3$

[**Ans. x = 1, y = 1, z = −1**]

It is a solution where all xi are zero i.e., $x_1 = x_2 \ldots = x_n = 0$.

(I) If $A \neq 0$ then the system has a **unique solution** $X = \frac{B}{A}$

(II)If $A = 0$ and i. $B \neq 0$ then the system has **no solution**..

(III) If $B = 0$ then the system has i**nfinite number of solutions**, namely all $x \in R$.

Definition: Rank of Matrices

Given the linear system $Ax = B$ and the augmented matrix $(A|B)$.

1 If $\text{rank}(A) = \text{rank}(A|B) =$ the number of rows in x, then the system has a **unique solution**.

2 If $\text{rank}(A) = \text{rank}(A|B) < \textit{the number of rows in } x,$

$\textit{then the system has } \infty - \textit{many solutions}.$

- If $\text{rank}(A) < rank(A|B)$, *then the system is* ***inconsistent***.

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