Hypergeometric Functions:

Pochhammer Symbol- The pochhammer symbol is denoted and defined by

$$(a)_n = a(a+1)(a+2)............(a+n-1) \qquad (1)$$

where n is any positive integer and

$$(a)_0 = 1 \qquad (2)$$

Some Results:

1. $(a)_n = a(a+1)(a+2)............(a+n-1)$, on the RHS multiplying and dividing by $1.2.3................(a-1)$, so we get

$$(a)_n = \frac{1.2.3................(a-1)a(a+1)(a+2)..............(a+n-1)}{1.2.3..............(a-1)}$$

$$= \frac{\Gamma(a+n)}{\Gamma a}$$

2 $(a)_{n+1} = a(a+1)(a+2)...............[a+(n+1)-1]$

$$= a[(a+1)(a+2).............[(a+1)+n-1]$$

$$= a(a+1)_n$$

3 $(a+n)(a)_n = (a+n)a(a+1)(a+2).........(a+n-1)$

$$= a(a+1)(a+2)................(a+n-1)(a+n)$$

$$= a(a+1)(a+2)................(a+n-1)[a+(n+1)-1]$$

$$= (a)_{n+1}$$

General Hypergeometric Function: The general hypergeometric function is denoted and defined by,

$$ {}_mF_n(a_1, a_2.......a_m;\ b_1, b_2........b_n; x) = \sum_{r=0}^{\infty} \frac{(a_1)_r (a_2)_r...........(a_m)_r}{(b_1)_r (b_2)_r.......(b_n)_r} \left(\frac{x^r}{r!} \right) \qquad (3)$$

It can also be denoted by,

$$ {}_mF_n \left[\begin{matrix} a_1, a_2...........a_m; & \\ & x \\ b_1, b_2...........b_n; & \end{matrix} \right]$$

Note 1- when $m = n = 1$, then the general hypergeometric function,

$$ {}_1F_1(a;\ b;\ \ x) \quad or \quad F(a;\ b;\ \ x) = \sum_{r=0}^{\infty} \frac{(a)_r}{(b)_r} \left(\frac{x^r}{r!} \right)$$

$$= 1 + \frac{a}{1.b} x + \frac{a(a+1)}{1.2b(b+1)} x^2 + ...$$

is known as **Kummer's** function or **Confluent hypergeometric function**.

Note 2- when $m = 2,\ n = 1$, then the general hypergeometric function,

$$ {}_2F_1(a,b;\ c;\ x)\ \ or\ F(a,b;\ c;\ x) = \sum_{r=0}^{\infty} \frac{(a)_r (b)_r}{(c)_r} \left(\frac{x^r}{r!} \right) $$

$$ = 1 + \frac{a.b}{1.c} x + \frac{a(a+1)b(b+1)}{1.2.c(c+1)} x^2 + \ldots\ldots\ldots\ldots $$

then the general hypergeometric function is known as **hypergeometric series**.

Symmetric property: The hypergeometric function does not change if a & b are interchanged and c is fixed,

$$ F(a,\ b;\ \ c;\ \ x) = F(b,a;\ \ c;\ \ x)\ . $$

Gauss hypergeometric equation: The equation,

$x(1-x)\dfrac{d^2y}{dx^2} + [c-(a+b+1)]\dfrac{dy}{dx} - aby = 0$, is known as hypergeometric equation.

Examples: To show that,

1. ${}_2F_1(-n, 1;\ 1;\ \ -x) = (1+x)^n$ 2. $x\,.\,{}_2F_1(1, 1;\ 2;\ \ -x) = \log(1+x)$

3. $x.\,{}_2F_1\left(\frac{1}{2},\ \ \frac{1}{2};\ \frac{3}{2};\ \ \ x^2 \right) = \sin^{-1} x$

Solution: 1- Since, $(1+x)^n = 1 + nx + \dfrac{n(n-1)}{2!}x^2 + \dfrac{n(n-1)(n-2)}{3!}x^3 + \ldots\ldots\ldots\ldots$

$$ = 1 + \frac{(-n).1}{1!.1}(-x) + \frac{(-n)(-n+1).1.(1+1)}{2!.1.(1+1)}(-x)^2 $$

$$ + \frac{(-n)(-n+1)(-n+2).1.(1+1)(1+2)}{3!.1.(1+1)(1+2)}(-x)^3 + \ldots\ldots\ldots $$

$$ = {}_2F_1(-n,\ 1;\ 1\ ;\ \ -x)\ . $$

Solution: 2- Since, $\log(1+x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \ldots\ldots\ldots$

$$ = x\left[1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \ldots\ldots\ldots\ldots \right] $$

$$ = x\left[1 + \frac{1.1}{1!\ 2}(-x) + \frac{1.(1+1).1(1+1)}{2!\ 2(2+1)}(-x)^2 + \ldots\ldots\ldots\ldots \right] $$

$$ == {}_2F_1(1, 1\ ;\ \ 2\ ;\ \ -x)\ . $$

Solution: 3- Similarly, expanding the series of $\sin^{-1}(x)$ and on arranging the terms we get the required result.

Differentiation of hypergeometric function:

$$ \frac{d^n}{dx^n} F(a,\ b;\ \ c\ ;\ \ x) = \frac{(a)_n (b)_n}{(c)_n} F(a+n,\ b+n; c+n;\ \ x)\ . $$

Proof: We shall prove it by mathematical induction on n.

Since, $F(a, b; c; x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \left(\frac{x^n}{n!} \right)$ (1)

Differentiating with respect to x on both sides, we get

$$\frac{d}{dx} F(a, b; c; x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} (nx^{n-1}) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n(n-1)} (nx^{n-1})$$

$$= \sum_{n=1}^{\infty} \frac{(a)_n (b)_n}{(c)_n (n-1)!} x^{n-1}$$

Putting $n-1 = m$, we get

$$\frac{d}{dx} F(a, b; c; x) = \sum_{m=0}^{\infty} \frac{(a)_{m+1} (b)_{m+1}}{(c)_{m+1} m!} x^m$$

$$= \sum_{m=0}^{\infty} \frac{a(a+1)_m b(b+1)_m}{c(c+1)_m m!} . x^m$$

$$= \frac{ab}{c} \sum_{m=0}^{\infty} \frac{(a+1)_m (b+1)_m}{(c+1)_m m!} . x^m$$

$$= \frac{ab}{c} F(a+1, b+1; c+1; x) \tag{2}$$

so it is true for $n = 1$.

Let us assume, it is true for any particular value $n = m$, then

$$\frac{d^m}{dx^m} F(a, b; c; x) = \frac{(a)_m (b)_m}{(c)_m} F(a+m, b+m; c+m; x) \tag{3}$$

Differentiating both sides of equation (3) with respect to x, we get

$$\frac{d^{m+1}}{dx^{m+1}} F(a, b; c; x) = \frac{(a)_m (b)_m}{(c)_m} \frac{d}{dx} \left(F(a+m, b+m; c+m; x) \right)$$

$$= \frac{(a)_m (b)_m}{(c)_m} \left(\frac{(a+m)(b+m)}{(c+m)} \right) F(a+m+1, b+m+1; c+m+1; x)$$

$$= \frac{(a+m)(a)_m (b+m)(b)_m}{(c+m)(c)_m} F(a+m+1, b+m+1; c+m+1; x)$$

$$= \frac{(a)_{m+1} (b)_{m+1}}{(c)_{m+1}} F(a+m+1, b+m+1; c+m+1; x)$$

So it is true for any positive integer $n = m+1$. Hence it is true for all positive integer n.

Note: putting $x = 0$ in differentiation formula for hypergeometric function, we get

$$\left[\frac{d^n}{dx^n} F(a, b; c; x) \right]_{x=0} = \frac{(a)_n (b)_n}{(c)_n} F(a+n, b+n; c+n; 0)$$

$$=\frac{(a)_n(b)_n}{(c)_n}\left[\sum_{m=0}^{\infty}\frac{(a+n)_m(b+n)_m}{(c+n)_m}\left(\frac{x^m}{m!}\right)\right]_{x=0}$$

$$=\frac{(a)_n(b)_n}{(c)_n}[1+0+0........]$$

$$\left[\frac{d^n}{dx^n}F(a,b;c\,;\,x)\right]_{x=0}=\frac{(a)_n(b)_n}{(c)_n}$$

Integral representation formula for hypergeometric function:

$$F(a,b;\ c;\ x)=\frac{\Gamma c}{\Gamma b\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-xt)^{-a}\,dt,\qquad c>b>0$$

Or

$$F(a,b;\ c;\ x)=\frac{1}{B(b,\,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-xt)^{-a}\,dt,\qquad c>b>0 \tag{1}$$

Note: 1- putting $x=1$ in integral representation formula for hypergeometric function, we obtain the **Gauss theorem**,

$$F(a,b;\ c;\ x)=\frac{\Gamma c}{\Gamma b\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-t)^{-a}\,dt,\qquad c>b>0$$

$$=\frac{\Gamma c}{\Gamma b\,\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1+(-a)}\,dt$$

$$=\frac{\Gamma c}{\Gamma b\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-a-1}\,dt$$

$$=\frac{\Gamma c}{\Gamma b\Gamma(c-b)}B(b,\,c-b-a)$$

$$=\frac{\Gamma c}{\Gamma b\Gamma(c-b)}.\frac{\Gamma b\Gamma(c-b-a)}{\Gamma(b+c-b-a)}$$

$$F(a,b\,;c;1)=\frac{\Gamma c\Gamma(c-b-a)}{\Gamma(c-a)\Gamma(c-b)}\ .$$

Note: 2- putting $x=1$ & $a=-n$ in the integral representation formula for hypergeometric function, we obtain the **Vandermonde's theorem**

$$F(-n,\,b;\,c\,;1)=\frac{(c-b)_n}{(c)_n}.$$

Note: 3- putting $x=-1$ & $c=(b-a+1)$ in the integral representation formula for hypergeometric function, we obtain the **Kummer's theorem**

$$F(a, b; b-a+1; -1) = \frac{\Gamma(b-a+1)\Gamma\left(\frac{b}{2}+1\right)}{\Gamma(b+1)\Gamma\left(\frac{b}{2}-a+1\right)}.$$

Confluent hypergeometric equation or Kummer's equation: The equation,

$xy'' + (c-x)y' - ay = 0$ is called a confluent hypergeometric equation.

The function $F(a\,;\,c\,;\,x) = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n}\left(\frac{x^n}{n!}\right)$ is called a confluent hypergeometric function.

Differentiation of confluent hypergeometric function:

$\frac{d^n}{dx^n}F(a;\,c;\,x) = \frac{(a)_n}{(c)_n}F(a+n;\,c+n;\,x)$ and

$\left[\frac{d^n}{dx^n}F(a\,;c;\,x)\right]_{x=0} = \frac{(a)_n}{(c)_n}$

Integral representation formula for confluent hypergeometric function:

$$F(a\,;\,b\,;\,x) = \frac{\Gamma b}{\Gamma a\Gamma(b-a)}\int_0^1 t^{a-1}(1-t)^{b-a-1}e^{xt}\,dt$$

Contiguous hypergeometric function: The function $F(a', b'; c'; x)$ is said to be contiguous hypergeometric function to $F(a,b;c;\,x)$, if it is increased or decreased by 1 in any of the parameters a, b or c.

Thus we have six contiguous hypergeometric functions to $F(a,b;c;\,x)$ i.e

$F_{a^+} = F(a+1,b;\,c;\,x)$, $\quad F_{a^-} = F(a-1,\,b\,;\,c\,;\,x)$

$F_{b^+} = F(a,b+1;\,c;\,x)$, $\quad F_{b^-} = F(a,\,b-1\,;\,c\,;\,x)$

$F_{c^+} = F(a,b;\,c+1;\,x)$, $\quad F_{c^-} = F(a,\,b\,;\,c-1\,;\,x)$

Examples: To show that

a- $F(a\,;a\,;x) = e^x$.

b- $\lim_{a\to\infty} F(1,a;\,1\,;\frac{x}{a}) = e^x$.

c- $\frac{d^4}{dx^4}F(2,-2\,;\,5\,;\,x) = 0$.