The Uncertainty Principle

100 Years 1920-2020

University of Lucknow | Centenary Year

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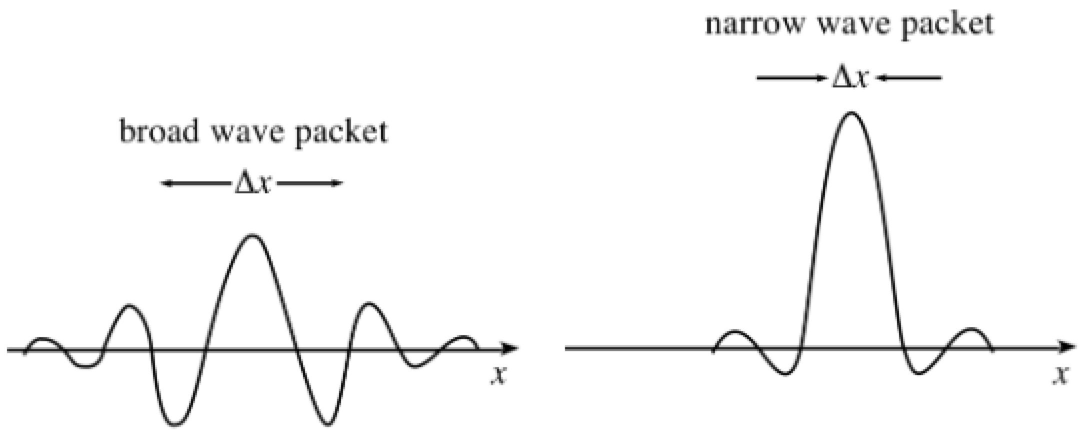
De Broglie proposed that a moving body must be regarded as a de Broglie wave group and not as a localized entity. This suggests that there is a fundamental limit to the accuracy with which we can measure its particle properties.

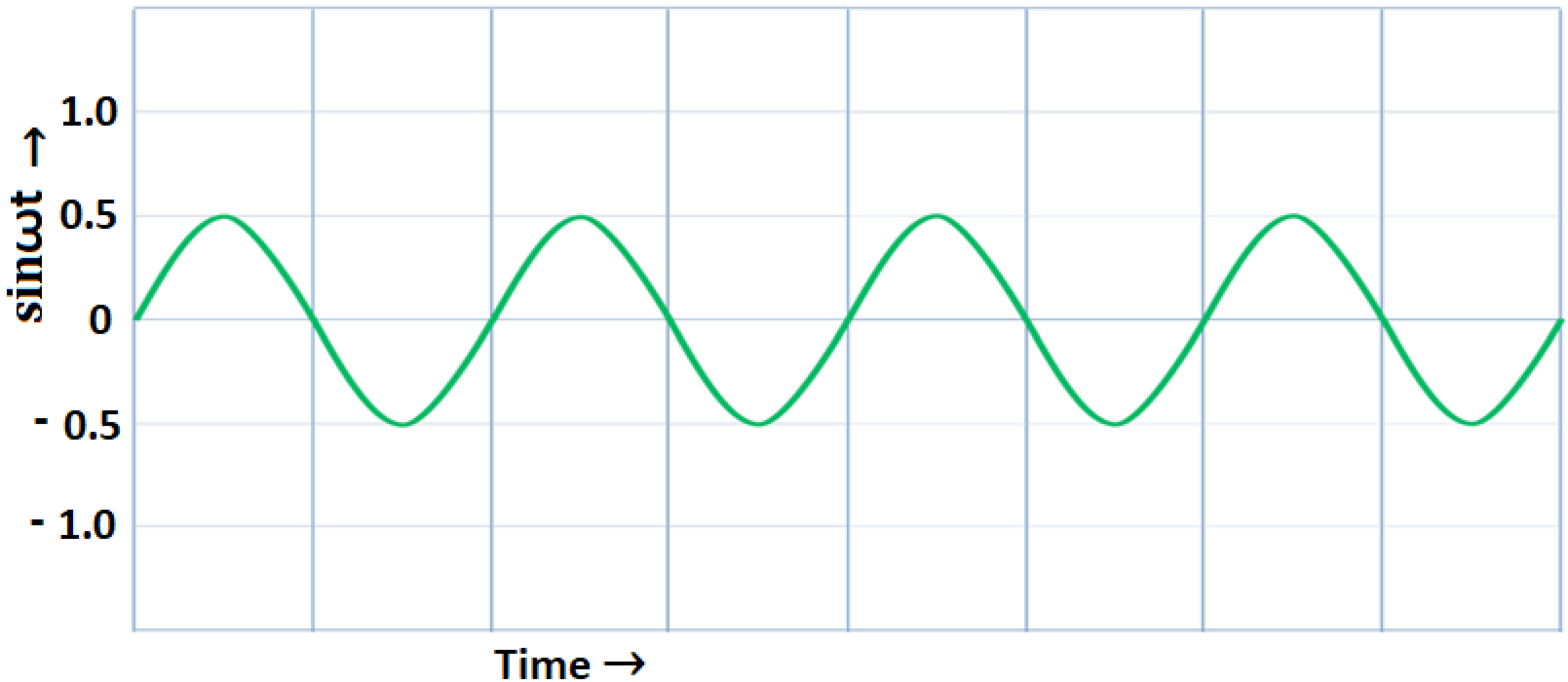
Figure shows three such de Broglie wave groups. The location of the particle may be anywhere within the group. If the group is very wide it allows better wavelength estimate but the location of the particle becomes uncertain, if the group is narrow the position of the particle is readily found but the wavelength is difficult to measure.



Any measurement of the momentum of the particle can be done by letting it interact with an external particle or radiatio i.e. by letting it interact with some external agent. In making the measurement of position we are going to disturb the position of the particle.

The very nature of the wave group allows us to relate the inherent uncertainty Δx in the measurement of a particle position with the inherent uncertainty Δp in the simultaneous measurement of its momentum.

How can we represent a wave group? Can a sinusoidal wave represent a wave group?



How can we represent a wave group?

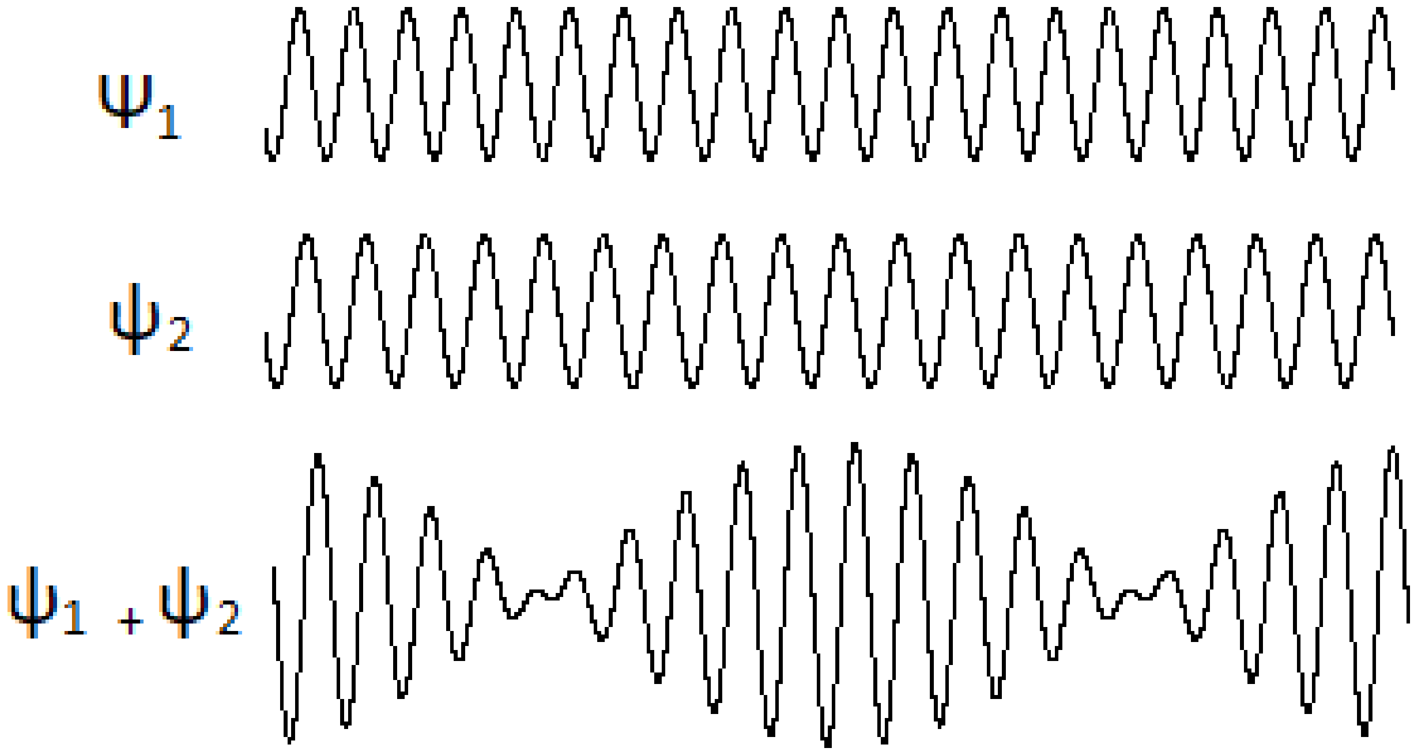
The simplest example of the wave group formation is by the superposition of two wave trains slightly different in angular frequency ω and propagation constant k. They superimpose to give a series of groups.

$$y_1 = A\, cos(\omega t - kx)$$

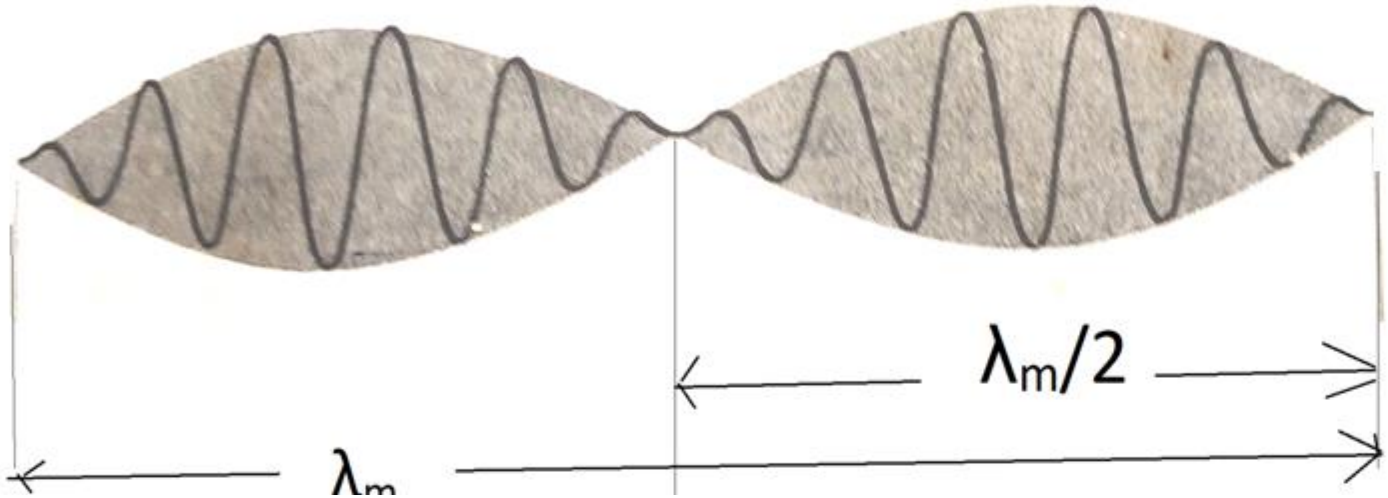
$$y_2 = A cos(\omega + \Delta\omega)t - (k + \Delta k)x$$

$$\psi = \psi_1 + \psi_2$$

$$\approx 2A\, cos(\omega t - kx)\, cos\left(\frac{\Delta\omega}{2}t - \frac{\Delta k}{2}x\right)$$



ψ1
ψ2
ψ1 + ψ2



$$\Delta x = \frac{\lambda_m}{2}$$

The width of each group is equal to half of the wavelength λ_m of the modulation. It is reasonable to assume that this width is the same order of inherent uncertainty Δx in the position of the group

The modulation wavelength is related to its propagation constant

$$\lambda_m = \frac{2\pi}{k_m}$$

The propagation constant of the modulation is

$$k_m = \frac{1}{2}\Delta k \qquad\qquad \lambda_m = \frac{2\pi}{{}^1/_2\,\Delta k}$$

$$\Delta x = \frac{2\pi}{\Delta k}$$

Since the waves that constitute the group are a result of superposition of waves of propagation constants k and k+Δk, even the most accurate measurement of k will have an uncertainty of Δk

$$\Delta k = \frac{2\pi}{\Delta x}$$

The de Broglie wavelength of the particle of momentum p is

$$\lambda = \frac{h}{p}$$

The propagation constant corresponding to this wavelength is

$$k = \frac{2\pi}{\lambda} = \frac{2\pi p}{h}$$

Thus the uncertainty of Δk in the propagation constant k results in uncertainty in momentum by Δp

$$\Delta p = \frac{h\Delta k}{2\pi} \qquad \Delta p = \frac{h}{\Delta x} \qquad \Delta p \Delta x = h$$

Which on a more accurate calculation will give → $\Delta p \Delta x \geq \hbar$

Energy-Time Uncertainty

Let us measure the energy emitted sometime during the time interval Δt in an atomic process.

The limited time available restricts the accuracy with which we can determine the frequency ν of the electromagnetic waves.

Assume that the minimum uncertainty in the number of waves we can count in a wave group is one wave.

Since the frequency of the wave is equal to the number of them we count divided by the time interval, the uncertainty $\Delta\nu$ in the frequency measurement is $\Delta\nu = \frac{1}{\Delta t}$

$$\Delta E = h\Delta\nu$$

so

The corresponding energy uncertainty is

$$\Delta E = \frac{h}{\Delta t}$$

Thus

$$\Delta E \, \Delta t = h$$

A more realistic calculation will show

$$\Delta E \, \Delta t \geq \hbar$$

Thus, the uncertainty statement for momentum-position is given as

$$\Delta p \Delta x \geq \hbar$$

Similar uncertainty statements for angular momentum-angular displacement and energy-time are give

$$\Delta L \Delta \theta \geq \hbar$$

$$\Delta E \Delta t \geq \hbar$$

Does an electron reside inside a nucleus?

The typical radius of a nucleus may be taken as 10^{-14} m. The uncertainty in the position of the electron may be assumed to be of the same order viz. 10^{-14} m. The corresponding uncertainty in the value of momentum should be

$$\Delta p \geq \frac{\hbar}{\Delta x} \quad \geq \frac{1.054 \times 10^{-34} j - sec}{10^{-14}\, m} \quad \geq 1.1 \times 10^{-20}\, kg - m/sec$$

The momentum should be of the order of the uncertainty in the momentum. An electron with momentum $1.1 \times 10^{-20}\, kg - m/sec$

has kinetic energy many times greater than its rest mass energy. Hence use the relativistic formula

$$T = pc = 3.3 \times 10^{-12}\, joule \approx 20\, MeV$$

Electrons observed have energy having couple of electron volts. No electron has been observed to have this kinetic energy of 20 MeV. Hence electron doesn't belong to the nucleus.

If electron belongs to an atom typical radius or uncertainty in the position should be $5 \times 10^{-11} m$

The corresponding value of uncertainty in momentum should be

$$\Delta p = 2.2 \times 10^{-24}\ kg - m/sec$$

An electron with this momentum is non relativistic. Hence the kinetic energy of electron should be

$$T = \frac{p^2}{2m} = 2.7 \times 10^{-18} joule \approx 17eV$$

The kinetic energy of the elections observed generally is of this order. Hence electron may reside in the atom.

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