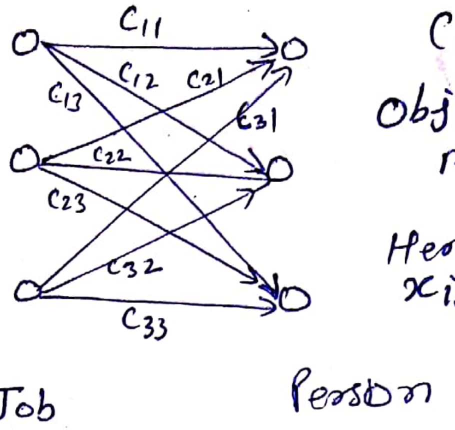
# Assignment Problem




$C_{ij}$ - cost of allocation

objective function ;

minimize $\sum_i \sum_j C_{ij} x_{ij}$

Here $x_{ij} = 1$ ; if job is assiged to person j.

$= 0$ ; otherwise.

· there are an equal no. of jobs and people.

At the end these jobs haveto be allocated such that each person gets only on job and each job goes to only one person. How to do that in minimum cost is called assignment problem.

★ In general if there are n jobs that have to assign to n peoeple, this can be done in n! ways. Among these n! ways which one has the least cost is the assignment problem.

★ It need not always job and people, It may be job & machines, machines & people, students and project etc. (i.e. equal no. of two different things).

## Example 1 :-

4 X 4 Assignment Problem

People

| Jobs | | | | |
|---|---|---|---|---|
| | 8 | 6 | 4 | 8 |
| | 6 | 5 | 5 | 8 |
| | 9 | 10 | 11 | 12 |
| | 7 | 6 | 8 | 10 |

→ cost coefficient matrix or $C_{ij}$ matrix

$C_{ij} \geq 0$

**Sol$^n$** Its square matrix hence balanced problem.

**Note:** * Matrix and its transpose will always give a same solution.

First we try to reduce this 4X4 matrix into another matrix which would have zeros and then make the assignments at the zero positions and try to get a feasible solution and such a feasible solution will be optimum because it will have zero costs.

Row & column subtraction :- Subtracting the same element from a row and subtracting a same element from a column does not effect the solution.

Now subtract row minimum from each row i.e. 4 from from 1st row, 5 from 2nd row, 9 from 3rd row and 6 from 4th row and we get new matrix i.e. ⟶

| | | | |
|---|---|---|---|
| 4 | 2 | 0 | 4 |
| 1 | 0 | 0 | 3 |
| 0 | 1 | 2 | 3 |
| 1 | 0 | 2 | 4 |

We can see that each row has at least on zero.

Now what is applicable to row also applicable to the column.

In the previous matrix column minimum in the first, second and third column is zero. Subtracting zero will give the same value. But 4th column has column minimum is 3 and hence

Subtract column minimum from each column we get

| | | | |
|---|---|---|---|
| 4 | 2 | 0 | 1 |
| 1 | 0 | 0 | 0 |
| 0 | 1 | 2 | 0 |
| 1 | 0 | 2 | 1 |

Now we can see that each row and each column has at least one zero.

Now start making Assignments.

First row assignment:

| | | | | |
|---|---|---|---|---|
| 4 | 2 | $\boxed{0}$ | 1 | → only one zero, so we can make the assignment |
| 1 | $\cancel{0}$ | $\cancel{0}$ | 0 | → two zeros temporarily cannot make assignment |
| 0 | 1 | 2 | 0 | → ~~Dont~~ 2-zeros Dont make assign. |
| 1 | $\boxed{0}$ | 2 | 1 | → one one zero |

Column assignment:

| | | | |
|---|---|---|---|
| 4 | 2 | $\boxed{0}$ | 1 |
| 1 | $\cancel{0}$ | $\cancel{0}$ | $\boxed{0}$ |
| $\boxed{0}$ | 1 | 2 | $\cancel{0}$ |
| 1 | $\boxed{0}$ | 2 | 1 |
| one zero | Already ~~Zero~~ Assigned | Already Assigned | Only one zero |

From this we can see each row and each column has one assignment.

$x_{13} = 1$

$x_{24} = 1$

$x_{31} = 1$

$x_{42} = 1$

The previous matrix gives the optimal solution to the reduced matrix and therefore it is optimum to the original matrix.

Hence assignment to the orignal matrix are

| | | | |
|---|---|---|---|
| 8 | 6 | $\boxed{4}$ | 8 |
| 6 | 5 | 5 | $\boxed{8}$ |
| $\boxed{9}$ | 10 | 11 | 12 |
| 7 | $\boxed{6}$ | 8 | 12 |

Total cost of assignment = 4+8+9+6
=27

Problem 2 :-

| | | | |
|---|---|---|---|
| 8 | 6 | 4 | 9 |
| 7 | 5 | 6 | 9 |
| 9 | 10 | 11 | 12 |
| 9 | 6 | 8 | 11 |

subtract row minimum from each row

| | | | |
|---|---|---|---|
| 4 | 2 | 0 | 5 |
| 2 | 0 | 1 | 4 |
| 0 | 1 | 2 | 3 |
| 3 | 0 | 2 | 5 |

subtract column minimum from each column

Then start assignment row assignment

| | | | | |
|---|---|---|---|---|
| 4 | 2 | $\boxed{0}$ | 2 | |
| 2 | $\boxed{0}$ | 1 | 1 | |
| 0 | 1 | 2 | 0 | X |
| 3 | $\cancel{0}$ | 2 | 2 | X |

Column assignment

| | | | | |
|---|---|---|---|---|
| 4 | 2 | $\boxed{0}$ | 2 | |
| 2 | $\boxed{0}$ | 1 | 1 | |
| $\boxed{0}$ | 1 | 2 | $\cancel{0}$ | × |
| 3 | $\cancel{0}$ | 2 | 2 | × |
| | Alread | Alread | | |

Now we realize that we can not make more assignment because all the zeros in this matrix either being boxed means they have been assigned or they are crossed which means they are not fit for assignment. In this we observe that we have made only three assignments as against four assignment.

Now we make some changes to this matrix

We makes some trick:

Thick an unassign row. If there is a zero in that row tick the corresponding column. If there is an assignment in that column then tick that row. Again if there is a zero then tick that column, so there is a zero but column is already ticked, so no more ticking is posible.

| | | | | |
|---|---|---|---|---|
| 4 | 2 | $\boxed{0}$ | 2 | |
| 2 | $\boxed{0}$ | 1 | 1 | ✓ |
| $\boxed{0}$ | 1 | 2 | $\cancel{0}$ | |
| 3 | $\cancel{0}$ | 2 | 2 | ✓ |
| | ✓ | | | |

Draw lines through unticked rows and ticked column. In the above procedure we draw three lines and these three lines covers all the zeros.

Now ~~#~~ take the minimum of the numbers where the no line is passing through.

min is = 1

this min is called $\theta$ ⇒ $\boxed{\theta = 1}$

- Subtract $\theta$ from all uncovered nos.
- Add $\theta$ to the nos. having two line passing through them.
- Numbers with one link passing through them remain same.

Now new matrix is

| | | | |
|---|---|---|---|
| 4 | 3 | 0 | 2 |
| 1 | 0 | 0 | 0 |
| 0 | 2 | 2 | 0 |
| 2 | 0 | 1 | 1 |

this new matrix will give the same solution. Now starts assignment

| | | | |
|---|---|---|---|
| 4 | 3 | $\boxed{0}$ | 2 |
| 1 | ~~0~~ | ~~0~~ | $\boxed{0}$ |
| $\boxed{0}$ | 2 | 2 | ~~0~~ |
| 2 | $\boxed{0}$ | 1 | 1 |

Now there is total 4-assignments.

Hence the solution is

| | | | |
|---|---|---|---|
| 8 | 6 | $\boxed{4}$ | 9 |
| 7 | 5 | 6 | $\boxed{9}$ |
| $\boxed{9}$ | 10 | 11 | 12 |
| 9 | $\boxed{6}$ | 8 | 11 |

Total cost $= 4+9+9+6$
$= 28$

**Problem 3 :-** Find the least cost of assignment for the given cost matrix of Job and peoples.

| | | | |
|---|---|---|---|
| 8 | 6 | 4 | 7 |
| 6 | 5 | 5 | 8 |
| 9 | 10 | 11 | 12 |
| 7 | 6 | 8 | 10 |

**Solution:**

row subtraction

| | | | |
|---|---|---|---|
| 4 | 2 | 0 | 3 |
| 1 | 0 | 0 | 3 |
| 0 | 1 | 2 | 3 |
| 1 | 0 | 2 | 4 |

Column subtraction

| | | | |
|---|---|---|---|
| 4 | 2 | 0 | 0 |
| 1 | 0 | 0 | 0 |
| 0 | 1 | 2 | 0 |
| 1 | 0 | 2 | 1 |

Start assignment

| | | | | |
|---|---|---|---|---|
| 4 | 2 | 0 | 0 | × |
| 1 | ~~0~~ | 0 | 0 | × |
| $\boxed{0}$ | 1 | 2 | ~~0~~ | × |
| 1 | $\boxed{0}$ | 2 | 1 | |
| | × | | × | |

Now we can see that all the zeros are not boxed or crossed. So we can not start our ticking and line drawing procedure.

In these situation we make some arbitrary assignment there is a tie in row 1 & row 2. Break it arbitrarily.

| | | | |
|---|---|---|---|
| 4 | 2 | $\boxed{0}$ | ~~0~~ |
| 1 | ~~0~~ | ~~0~~ | $\boxed{0}$ |
| $\boxed{0}$ | 1 | 2 | ~~0~~ |
| 1 | $\boxed{0}$ | 2 | 1 |

Alternately we can choose $x_{14}$.

Hence Alternate optima

$x_{13} = x_{24} = x_{31} = x_{42} = 1$

$x_{14} = x_{23} = x_{31} = x_{42}$

| | | | |
|---|---|---|---|
| 8 | 6 | 4 | 7 |
| 6 | 5 | 5 | 8 |
| $\boxed{9}$ | 10 | 11 | 12 |
| 7 | $\boxed{6}$ | 8 | 10 |

In both the cases

$z = 27$