SEQUENCE AND SERIES OF REAL NUMBERS

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What is a Sequence?

- In mathematics a sequence is an ordered list , like a set, it contains members called elements or terms of sequence. Most precisely, a sequence of real numbers is defined as a function S: N $\longrightarrow$ R, then for each n∈N, S(n) or S_n is a real number. The real numbers S_1 S_2 ,S_3,……. ,S_n are called terms of sequence. A sequence may be written as $\{S_1\ S_2\ S_3 \ldots\ldots. S_n\}$ or $\{S_n\}$.
- For example
- The n^{th}term of the sequence {-5n} is $S_n = -5n$ then the sequence becomes {-5,-10, -15, ……..-5n…….}.
- The n^{th} term of the sequence $\{S_n\}$ is $S_n = \frac{n}{n+1}$, the sequence is $\{1,\frac{2}{3},\frac{3}{4}, \ldots\}$.
- A sequence is also given by its recursion formula where $S_1 = 1$and $S_n = \sqrt{3S_n}$, then the sequence is $\{1,\sqrt{3},\sqrt{3\sqrt{3}}, \ldots \ldots\}$.

INFINITE SERIES

- A series is , roughly speaking , a description of the operation of adding infinitely many quantities one after the other, to a given starting quantity.
- An expression of the form $a_1 + a_2 + \cdots + a_n + \cdots$, where each a_n is real numbers , in which each term is followed by another term is known as infinite series of real numbers. It is denoted by $\sum_{n=1}^{\infty} a_n$ or $\sum a_n$, here a_n is n^{th}term of the series.
- The sum of n terms of series is denoted by S_n , thus
- $S_n = a_1 + a_2 + \cdots + a_n = \sum_{n=1}^{n} a_n.$

SEQUENCE OF PARTIAL SUM

- Suppose $\sum a_n$ is infinite series. We define a sequence $\{S_n\}$ as follows:
- $S_1 = a_1,$
- $S_2 = a_1 + a_2,$
- $S_3 = a_1 + a_2 + a_3,$
- -
- -
- $S_n = a_1 + a_2 + a_3 + \cdots + a_n$ and so on
- The sequence $\{S_n\}$ is called a sequence of partial sums of series $\sum a_n$.

CONVERGENCE, DIVERGENCE AND OSCILLATION OF A SERIES

1. **CONVERGENT :** A series $\sum a_n$ is said to be convergent if the sequence $\{S_n\}$ of partial sums of series converges to a real number S. i.e. $\lim_{n\to\infty} S_n = S$, where S is finite and unique.
2. **DIVERGENT:** A series is said to be divergent if the sequence $\{S_n\}$ of partial sum diverges to $+\infty$ or $-\infty$. i.e. i.e. $\lim_{n\to\infty} S_n = \pm\infty$.
3. **OSCILLATORY:** A series is oscillatory if sequence $\{S_n\}$ of partial sum of series oscillates i.e. $\lim_{n\to\infty} S_n$ does not tend to unique limit.

For example: consider the series

$$\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\cdots$$

Here $S_n = \frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\cdots+\frac{1}{n(n+1)}$

$$= \left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots+\left(\frac{1}{n}-\frac{1}{n+1}\right)$$

$S_n = \left(1-\frac{1}{n+1}\right)$ $\quad \lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(1-\frac{1}{n+1}\right) = (1-0) = 1(finite).$

Hence the series converges to 1.

- Consider the series

$$\sum 3^n = 3 + 3^2 + 3^3 + \cdots$$

Here $S_n = (3^n - 1)$, $\lim_{n\to\infty} S_n = \lim_{n\to\infty} 3^n - 1 = \infty$.

Hence the series is divergent.

<u>**Example:**</u> **Consider the series** $\sum(-\mathbf{1})^{\boldsymbol{n-1}}$.

- Here $a_n = (-1)^{n-1}$ $\quad now\ S_1 = a_1 = 1$
- $S_2 = a_1 + a_2 = 1 - 1 = 0,$
- $S_3 = a_1 + a_2 + a_3 = 1 - 1 + 1 = 1, \qquad S_4 = 0, \qquad S_5 = 1 \ldots.so\ on.$
- Therefore $\{S_n\} = \{1,0,1,0,\ldots\}$ Which Oscillates between 0 and 1.So the series is oscillatory.

<u>Elementary Properties of series</u>

The alteration of a finite number of terms of a series has no effect on convergence and divergence.

1. If a series converges or has an infinite sums, their sum is unique.
2. Multiplication of the terms of a series by a nonzero constant K doesnot effect the convergence or divergence of a series.

$$\sum_{n=1}^{\infty} Ka_n = k\sum_{n=1}^{\infty} a_n$$

Necessary Condition For Convergence:

Theorem: If a series converges, its general term tends toward zero as n becomes infinity i.e. if the series $\sum a_n$ converges, then $\lim_{n\to\infty} a_n = 0$. But the converse is not true.

Proof: let us consider the series $\sum a_n$. Consider $\{S_n\}$ be the sequence of partial sums of series $\sum a_n$.

$$S_n = a_1 + a_2 + \cdots + a_{n-1} + a_n$$
$$S_{n-1} = a_1 + a_2 + \cdots + a_{n-2} + a_{n-1}$$
$$S_n - S_{n-1} = a_n$$

Since the series $\sum a_n$ converges , so $\{S_n\}$ converges

Let $\lim_{n\to\infty} S_n = s \qquad then \lim_{n\to\infty} S_{n-1} = s$

Therefore $\lim_{n\to\infty} a_n = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} \qquad = s - s = 0$

Hence $\lim_{n\to\infty} a_n$=0

Thus the condition for convergence is necessary but not sufficient.

Consider the harmonic series

$$\sum \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots$$

Here $a_n = \frac{1}{n}$ $\qquad \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0$

But the series $\sum \frac{1}{n}$ is divergent . (by p-test)

RESULT FOR GEOMETRIC SERIES

The geometric series

$$\sum r^n = 1 + r + r^2 + r^3 + \cdots \quad (r > 0)$$

is convergent if $|r| < 1$ and divergent if $|r| \geq 1$.

Examples:

- The series $\sum \frac{1}{2^n} = \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^n} + \cdots$ is convergent.

Here the series is G.P. series where common ratio $r = 1/2 < 1$. so the above series converges.

- The series $\sum 3^n = 3 + 3^2 + 3^3 + \cdots$ is divergent, since $r = 3 > 1$.

POSITIVE TERM SERIES

An infinite series whose terms are positive or more generally, a non negative series (a series whose terms are nonnegative) are called positive term series.

<u>Theorem</u>: (A test for a positive series)

A positive term series $\sum a_n$ is convergent if and only if its sequence $\{S_n\}$ of partial sum bounded above.

Equivalently, a positive term series $\sum a_n$ converges iff $S_n < k \quad \forall n \in N$.

<u>Remark</u>: Since monotonic sequence either converge or diverge but never oscillate. Therefore positive term series are either converge or diverge.

Test for Divergence

Theorem: If $\sum a_n$ is a positive series such that $\lim_{n\to\infty} a_n \neq 0$, then the $\sum a_n$ diverges.

EXAMPLE(1): Test the convergence of series $\sum_{n=1}^{\infty} cos\frac{\pi}{2n}$.

Here $a_n = cos\frac{\pi}{2n}$, Then $\lim_{n\to\infty} a_n = \lim_{n\to\infty} cos\frac{\pi}{2n} = \cos 0 = 1 \neq 0.$

So by theorem $\lim_{n\to\infty} a_n \neq 0$ then the series is divergent.

Example (2): consider the series $\sqrt{\frac{1}{n}} + \sqrt{\frac{2}{6}} + \cdots + \sqrt{\frac{n}{2(n+1)}} + \cdots$

Here $a_n = \sqrt{\frac{n}{2(n+1)}}$ $\quad \lim_{n\to\infty} a_n = \lim_{n\to\infty} \left[\frac{n}{2n(1+1/n)}\right]^{1/2} = \lim_{n\to\infty} \left[\frac{1}{2(1+1/n)}\right]^{1/2}$

$$= \frac{1}{\sqrt{2}} \neq 0$$

Hence the series is divergent.

SOME COMPARISON TEST

- ## FIRST COMPARISON TEST:

Theorem: let $\sum a_n$ and $\sum b_n$ be two positive term series such that $a_n \leq k\ b_n \quad \forall n \geq m$

(k being a fixed positive number and m a fixed positive integer. Then

1. If $\sum b_n$ converges implies $\sum a_n$ converges.
2. If $\sum a_n$ diverges implies $\sum b_n$ diverges.

CONVERGENCE OF p-SERIES $\sum \frac{1}{n^p}$

The series $\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \cdots + \frac{1}{n^p} + \cdots$ (p > 0)

Converges if (p >1) and diverges if $(p \leq 1)$.

EXAMPLE consider the series $\sum e^{-n^2}$

$$e^x > x \qquad \forall x > 0\backslash$$

$$e^{n^2} > n^2 \qquad \forall n$$

$$\frac{1}{e^{n^2}} < \frac{1}{n^2} \qquad e^{-n^2} < \frac{1}{n^2} \qquad \forall n$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (by p-series test here p=2 >1)

Hence by first comparison test $\sum e^{-n^2}$ is convergent.

EXAMPLE: Consider series $\sum_{n=2}^{\infty} \frac{1}{n^2 \log n}$

Since we know that $\frac{1}{n^2 \log n} > \frac{1}{n^2} \forall n \geq 2$

Here series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent since $\mathrm{p} = 2 > 1$

So by first comparison test

$\sum_{n=2}^{\infty} \frac{1}{n^2 \log n}$ is convergent.

LIMIT FORM TEST

Theorem Let $\sum a_n$ and $\sum b_n$ be two positive term series such that

$$\lim_{n\to\infty} \frac{a_n}{b_n} = l \quad (l\ is\ nonzero\ and\ finite)$$

Then $\sum a_n$ and $\sum b_n$ converges and diverges together.

i.e. $\sum b_n$converges implies $\sum a_n$ converges.

$\sum b_n$ diverges implies $\sum a_n$ diverges.

REMARKS:

1. If $\mathrm{l} = 0$ $or\ l = \infty$ then above test may not hold good.
2. To apply limit test on the series $\sum a_n$, we have to select series $\sum b_n$ called auxillary series (which is usually p-series) in which the n^{th} term of b_n behaves as a_n, for large values of n written as $a_n \sim b_n$.

For large values of n we have

$$\frac{1}{n^2+1} \sim \frac{1}{n^2}$$

$$\frac{1}{\sqrt{n}+\sqrt{n+1}} \sim \frac{1}{\sqrt{n}+\sqrt{n}} = \frac{1}{2\sqrt{n}}.$$

We also take $\sum b_n$ as

$b_n = \frac{1}{n^{(a-b)}}$ where $a\ and\ b$ are higher indices of n in denominator and numerator.

For example $a_n = \frac{n}{n^3+\sqrt{n}}$ then $b_n = \frac{1}{n^{3-1}} = \frac{1}{n^2}$

And usually $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = 1.$

- The series $\sum \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \le 1.$

EXAMPLES

consider the series $\frac{1}{1.2.3} + \frac{3}{2.3.4} + \frac{5}{3.4.5} + \cdots$

The n^{th} term of the series is

$$a_n = \frac{2n-1}{n(n+1)(n+2)}$$

$$a_n = \frac{2-\frac{1}{n}}{n^2(1+\frac{1}{n})(1+\frac{2}{n})}$$

Let us consider the auxiliary series $b_n = \frac{1}{n^2}$

Then $$\frac{a_n}{b_n} = \frac{(2n-1)n^2}{n(n+1)(n+2)}$$

$$\frac{\boldsymbol{a_n}}{\boldsymbol{b_n}} = \frac{n(2n-1)}{(n+1)(n+2)}$$

$$\lim_{n\to\infty} \frac{\boldsymbol{a_n}}{\boldsymbol{b_n}} = \lim_{n\to\infty} \left(\frac{\boldsymbol{n}}{\boldsymbol{1+n}}\right)\left(\frac{\boldsymbol{2n-1}}{\boldsymbol{n+2}}\right)$$

$$\mathop{lim}_{n\to\infty} \frac{a_n}{b_n} = \mathop{lim}_{n\to\infty} \left(\frac{1}{1+{}^{1}/_{n}}\right)\left(\frac{2-1\backslash n}{1+2\backslash n}\right)$$

$$\lim_{n\to\infty} \frac{\boldsymbol{a_n}}{\boldsymbol{b_n}} = \left(\frac{\boldsymbol{1}}{\boldsymbol{1+0}}\right)\left(\frac{\boldsymbol{2-0}}{\boldsymbol{1+0}}\right)$$

$$= 2 \neq 0(finite)$$

So $\sum a_n$ and $\sum b_n$ converges or diverges together since the series $\sum b_n = \frac{1}{n^2}$ converges (because p=2>1). Hence $\sum a_n$ converges.

<u>Example:</u> Test the convergence of the series $\sum_{n=1}^{\infty}\left(\sqrt{n^4+1}-\sqrt{n^4-1}\right)$

Sol: Here n^{th} term of the series will be

$$a_n = \sqrt{n^4+1} - \sqrt{n^4-1} \times \frac{\sqrt{n^4+1}+\sqrt{n^4-1}}{\sqrt{n^4+1}+\sqrt{n^4-1}}$$

$$a_n = \frac{(n^4+1)-(n^4-1)}{\sqrt{n^4+1}+\sqrt{n^4-1}} \qquad \sim \frac{2}{\left\{n^2\sqrt{\left(1+\frac{1}{n^4}\right)}+n^2\sqrt{\left(1-\frac{1}{n^4}\right)}\right\}}$$

Consider $b_n = \frac{1}{n^2}$ then

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{2}{\left\{\sqrt{\left(1+\frac{1}{n^4}\right)}+\sqrt{\left(1-\frac{1}{n^4}\right)}\right\}}$$

$$= \frac{2}{2} = 1 \neq 0(finite)$$

So $\sum a_n$ and $\sum b_n$ converges or diverges together . Since the series $\sum b_n = \frac{1}{n^2}$ is convergent. Hence the given series is convergent.

Cauchy's n^{th} Root Test:

If $\sum a_n$ be a positive term series such that $\lim\limits_{n\to\infty}(a_n)^{1/n} = l$ then

a. **$\sum a_n$ is convergent , if $l < 1$**

b. **$\sum a_n$ is divergent, if $l > 1$**

c. **Test fail if $l = 1$.**

Example:

Test the convergence of the series

(i) $\sum_{n=1}^{\infty}(n^{1\backslash n}-1)^n$

Solution: The n^{th}term of the series be

$$a_n = (n^{1\backslash n}-1)^n$$

Now by Cauchy's n^{th} root test

$$(a_n)^{1\backslash n} = \left(n^{1\backslash n} - 1\right)$$

$$\lim_{n\to\infty}(a_n)^{1\backslash n} = \lim_{n\to\infty}\left(n^{1\backslash n} - 1\right) \;= 1 - 1 = 0 < 1$$

Hence by Cauchy's n^{th} root test given series is convergent.

Example: Test the convergence of the series

$$\frac{1}{2} + \frac{2}{3}x + \left(\frac{3}{4}\right)^2 x^2 + \left(\frac{4}{5}\right)^3 x^3 + \cdots \quad (x>0)$$

Solution: Here $a_n = \left(\frac{n+1}{n+2}\right)^n x^n$ (Neglecting the first term)

By Cauchy's n^{th} root test

$$\lim_{n\to\infty}(a_n)^{1\backslash n} = \lim_{n\to\infty}\left(\frac{n+1}{n+2}\right)\mathrm{x} \;= x$$

Hence by Cauchy's n^{th} root test $\sum a_n$ is convergent if x<1, $\sum a_n$ is divergent if x>1 and test fail if x=1.

Now when x=1

$$a_n = \left(\frac{n+1}{n+2}\right)^n 1^n$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\frac{1 + 1\backslash n}{1 + 2\backslash n}\right)^n = \frac{e}{e^2} = \frac{1}{e} \neq 0$$

So the series $\sum a_n$ is divergent at x=1.

Finally the series converges if x<1and diverges if $x \geq 1$.

Comparison of Ratio test Or Second Ratio Test:

If $\sum a_n$ and $\sum b_n$ are two series of positive terms such that $\frac{a_n}{a_{n+1}} \geq \frac{b_n}{b_{n+1}} \forall n \geq m$

Then (i) $\sum b_n$ converges implies that $\sum a_n$ converges.

(ii) $\sum a_n$ diverges implies that $\sum b_n$ diverges.

D' Alemberts Ratio Test:

Suppose $\sum a_n$ be series of positive term such that

$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = l$, then the series is

(i) Convergent if $l > 1$.

(ii) Divergent if $l < 1$.

(iii) The test fail to describe the nature of the series if $l = 1$.

Remarks:

(i) This test is applied when n^{th} term of the series involves factorials, product of several factors or combinations of powers and factorial.

(ii) The another equivalent form of ratio test is $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = l$, if $\sum a_n$ is series of positive term then

a. $\sum a_n$ converges if $l < 1$.

b. $\sum a_n$ diverges if $l > 1$.

c. Test fail if $l = 1$.

(iii) If $\lim\limits_{n\to\infty}\frac{a_n}{a_{n+1}}=\infty$, then $\sum a_n$ is convergent.

(iv) If $\lim\limits_{n\to\infty}\frac{a_{n+1}}{a_n}=0$, then $\sum a_n$ is convergent.

Example: Test the convergence of the given series

$$\frac{1}{3}+\frac{1.2}{3.5}+\frac{1.2.3}{3.5.7}+\frac{1.2.3.4}{3.5.7.9}+\cdots$$

Solution: Here $a_n=\frac{1.2.3.4\ldots n}{3.5.7.9\ldots(2n+1)}$ [since 3.5.7.9… are in A.P. n^{th} term is 3+2(n-1)=2n+1]

$$a_{n+1}=\frac{1.2.3.4\ldots n(n+1)}{3.5.7.9\ldots(2n+1)(2n+3)}$$

$$\lim_{n\to\infty}\frac{a_n}{a_{n+1}}=\lim_{n\to\infty}\frac{2n+3}{n+1}=\lim_{n\to\infty}\frac{2+3\backslash n}{1+1\backslash n}=2>1$$

Therefore by ratio test $\sum a_n$ is convergent.

Example: Test the convergence of the series $\sum\frac{x^n}{x+n}$

Solution: Here $a_n=\frac{x^n}{x+n}$ and $a_{n+1}=\frac{x^{n+1}}{x+(n+1)}$

Now $\lim\limits_{n\to\infty}\frac{a_n}{a_{n+1}}=\lim\limits_{n\to\infty}\frac{x+n+1}{x+n}\frac{x^n}{x^{n+1}}=\lim\limits_{n\to\infty}\frac{\{1+(1+x)\backslash n\}}{(1+x\backslash n)}\frac{1}{x}=\frac{(1+0)}{(1+0)}\frac{1}{x}=\frac{1}{x}$

By Ratio test $\sum a_n$ is convergent if $\frac{1}{x}>1$ $i.e.$ $x<1$and $\sum a_n$ is divergent if $\frac{1}{x}<1$ $i.e. x>1$. For x=1 test becomes fail and we have

$$a_n=\frac{1}{1+n}$$

$a_n = \frac{1}{n(1+1\backslash n)}$ choose $b_n = \frac{1}{n}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{(1+1\backslash n)} = 1 \neq 0(finite)$$

By Comparison test $\sum a_n$ and $\sum b_n$ are either both convergent or divergent.

Since auxillary series $\sum b_n = \frac{1}{n}$ is divergent so $\sum a_n$ is also divergent.

Hence the series $\sum a_n$ is convergent if $x < 1$ and divergent if $x \geq 1$.

Raabe's Test:

Suppose $\sum a_n$ be series of positive term such that

$\lim\limits_{n\to\infty} \left\{n\left(\frac{a_n}{a_{n+1}} - 1\right)\right\} = l$, then the series is

(i) **Convergent if $l > 1$.**

(ii) **Divergent if $l < 1$.**

(iii) **The test fail to describe the nature of the series if $l = 1$.**

Remarks:

(i) Raabe's Test is Stronger than D'Alemberts Ratio Test.

(ii) When Ratio test fail i.e. $\lim\limits_{n\to\infty} \frac{a_n}{a_{n+1}} = l$ then Rabbe's test may apply.

(iii) If $\lim\limits_{n\to\infty} \left\{n\left(\frac{a_n}{a_{n+1}} - 1\right)\right\} = +\infty$ then series $\sum a_n$ is convergent.

(iv) If $\lim\limits_{n\to\infty} \left\{n\left(\frac{a_n}{a_{n+1}} - 1\right)\right\} = -\infty$, then series $\sum a_n$ is divergent.

Example: Test the convergence of the series

$1 + a + \frac{a(a+1)}{2!} + \frac{a(a+1)(a+2)}{3!} + \cdots$

Solution: Here $\boldsymbol{a_n = \frac{a(a+1)(a+2)+\cdots+(a+n-1)}{n!}}$

$$\boldsymbol{a_{n+1} = \frac{a(a+1)(a+2)+\cdots+(a+n)}{n+1!}}$$

$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \lim_{n\to\infty} \frac{\boldsymbol{a(a+1)(a+2)\ldots(a+n-1)}}{\boldsymbol{n!}} \times \frac{\boldsymbol{(n+1)!}}{\boldsymbol{a(a+1)\ldots\ldots(a+n)}}$$

$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \lim_{n\to\infty} \frac{n+1}{a+n} = \lim_{n\to\infty} \frac{1+1\backslash n}{1+a\backslash n} = 1$$

Therefore D'Alembert's ratio test fail it is not able to describe the nature of series.

Thus we apply Rabbe's test

$$\lim_{n\to\infty} \left\{\boldsymbol{n}\left(\frac{\boldsymbol{a_n}}{\boldsymbol{a_{n+1}}} - \boldsymbol{1}\right)\right\} = \lim_{n\to\infty} \left\{\boldsymbol{n}\left(\frac{\boldsymbol{n+1}}{\boldsymbol{a+n}} - \boldsymbol{1}\right)\right\} = \lim_{n\to\infty} \left\{\frac{\boldsymbol{n}}{\boldsymbol{n}}\left(\frac{\boldsymbol{1-a}}{\boldsymbol{1+a}\backslash\mathrm{n}}\right)\right\}$$

$$\lim_{n\to\infty} \left\{\boldsymbol{n}\left(\frac{\boldsymbol{a_n}}{\boldsymbol{a_{n+1}}} - \boldsymbol{1}\right)\right\} = (\boldsymbol{1-a})$$

So the series is convergent if $\boldsymbol{1-a > 1 \; i.e. \; a < 0}$ and divergent if $\boldsymbol{1-a < 1 \; i.e. \; a > 0}$ and test fail if $\boldsymbol{1-a = 1 \; i.e. \; a = 0}$.

Now if $\boldsymbol{a = 0}$ then the series contains only first term and therefore the convergent.

Hence finally the series is convergent if $\boldsymbol{a \leq 0}$ and divergent if $\boldsymbol{a > 0}$.

Example: Test the convergence of series $\sum x^n (\log n)^p$

Solution: Here $a_n = x^n (\log x)^p$ $\qquad a_{n+1} = x^{n+1}(\log(n+1))^p$

$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \lim_{n\to\infty} \frac{x^n (\log x)^p}{x^{n+1}(\log(n+1))^p} = \lim_{n\to\infty} \frac{\mathbf{1}}{\boldsymbol{x}} \left[\frac{(\log(\boldsymbol{n}+\mathbf{1}))^{\boldsymbol{p}}}{(\log \boldsymbol{n})^{\boldsymbol{p}}}\right]^{-\mathbf{1}}$$

$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \lim_{n\to\infty} \frac{\mathbf{1}}{\boldsymbol{x}} \left[\frac{(\log \boldsymbol{n}(\mathbf{1}+\mathbf{1}\backslash \mathrm{n}))}{\log \boldsymbol{n}}\right]^{-\boldsymbol{p}} = \lim_{n\to\infty} \frac{\mathbf{1}}{\boldsymbol{x}} \left[\frac{(\log \boldsymbol{n} + \log(1+1\backslash \mathrm{n})}{\log \boldsymbol{n}}\right]^{-\boldsymbol{p}}$$

$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \lim_{n\to\infty} \frac{\mathbf{1}}{\boldsymbol{x}} \left[\frac{\log \boldsymbol{n} + \left(\frac{\mathbf{1}}{\boldsymbol{n}} - \frac{\mathbf{1}}{\mathbf{2}\boldsymbol{n}^{\mathbf{2}}} + \cdots\right)}{\log \boldsymbol{n}}\right]^{-\boldsymbol{p}}$$

$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \lim_{n\to\infty} \frac{\mathbf{1}}{\boldsymbol{x}} \left[\mathbf{1} + \frac{\mathbf{1}}{\mathrm{n}\log \boldsymbol{n}} - \frac{\mathbf{1}}{\mathbf{2}\boldsymbol{n}^{\mathbf{2}}\log \boldsymbol{n}} + \cdots\right]^{-\boldsymbol{p}}$$

$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \lim_{n\to\infty} \frac{\mathbf{1}}{\boldsymbol{x}} \left[\mathbf{1} - \frac{\boldsymbol{p}}{\mathrm{n}\log \boldsymbol{n}} + \frac{\boldsymbol{P}}{\mathbf{2}\boldsymbol{n}^{\mathbf{2}}\log \boldsymbol{n}} - \cdots\right] = \frac{\mathbf{1}}{\boldsymbol{x}}$$

Hence by ratio test the series is convergent if $\frac{\mathbf{1}}{\boldsymbol{x}} > \mathbf{1}$ $\;\boldsymbol{i.e.x} < \mathbf{1}$ and divergent if $\frac{\mathbf{1}}{\boldsymbol{x}} < \mathbf{1}\; \boldsymbol{i.e.x} > \mathbf{1}$. Test fail when x=1.

For x=1, Applying Rabbe’s test

$$\lim_{n\to\infty}\left\{\boldsymbol{n}\left(\frac{\boldsymbol{a}_{\mathbf{n}}}{\boldsymbol{a}_{\mathbf{n+1}}}-\mathbf{1}\right)\right\}=\lim_{n\to\infty}\boldsymbol{n}\left(\mathbf{1}-\frac{\boldsymbol{p}}{\mathrm{nlog}\,\boldsymbol{n}}+\frac{\boldsymbol{p}}{\mathbf{2}\boldsymbol{n}^{\mathbf{2}}\log\boldsymbol{n}}-\cdots-\mathbf{1}\right)$$

$$=\lim_{n\to\infty}\frac{-\boldsymbol{n}}{\boldsymbol{n}}\left(\frac{\boldsymbol{p}}{\log\boldsymbol{n}}-\frac{\boldsymbol{p}}{2\mathrm{nlog}\,\boldsymbol{n}}-\cdots\right)=0<1$$

So by Raabe's test the series diverges if x=1. Hence finally series converges if x<1 and diverges if $x \geq 1$.

Example: Test the convergence of the series

$$1+\frac{p}{q}+\frac{p(p+1)}{q(q+1)}+\frac{p(p+1)(p+2)}{q(q+1)(q+2)}+\cdots(P>0)(q>0)$$

Solution: Neglecting First term

$a_n=\frac{p(p+1)(p+2)\ldots(p+n-1)}{q(q+1)(q+2)\ldots(q+n-1)}$, $\quad a_{n+1}=\frac{p(p+1)(p+2)\ldots(p+n-1)(p+n)}{q(q+1)(q+2)\ldots(q+n-1)(q+n)}$

$$\lim_{n\to\infty}\frac{a_n}{a_{n+1}}=\lim_{n\to\infty}\frac{q+n}{p+n}=\lim_{n\to\infty}\frac{1+q\backslash\mathrm{n}}{1+p\backslash\mathrm{n}}=1$$

So D'Alembert's test fail Now applying Raabe's test

$$\lim_{n\to\infty}\left\{\boldsymbol{n}\left(\frac{\boldsymbol{a}_{\mathbf{n}}}{\boldsymbol{a}_{\mathbf{n+1}}}-\mathbf{1}\right)\right\}=\lim_{n\to\infty}\mathrm{n}\left(\frac{q+n}{p+n}-1\right)=\lim_{n\to\infty}\mathrm{n}\left(\frac{q-p}{p+n}\right)=q-p$$

By Raabe's test the series is convergent if $q-p>1$ i.e. $\mathrm{q}>1+\mathrm{p}$ and divergent if $q-p<1\ \ i.e. q>1+p$. The test fail if $q-p=1\ \ i.e.\ \ q=1+p$.

$$a_n=\frac{p(p+1)(p+1)\ldots(p+n-1)}{(p+1)(p+2)\ldots(p+n-1)(p+n)}$$

$a_n = \frac{p}{(p+n)}$ Let $b_n = \frac{1}{n}$

Then $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{pn}{p+n} = \lim_{n\to\infty} \frac{pn}{n(1+p\backslash n)} = \lim_{n\to\infty} \frac{p}{1+p\backslash n} = p > 0$

So by comparison test $\sum \boldsymbol{a_n}$ converges or diverges. Here series $\sum \boldsymbol{b_n} = \sum \frac{\mathbf{1}}{\boldsymbol{n}}$ divergent.

Hence $\sum \boldsymbol{a_n}$ diverges when q=1+p.

Therefore finally the series is convergent if q>1+p and diverges if $q \leq 1 + p$.

Logarithmic Test:

Suppose $\sum a_n$ be series of positive term such that

$\lim_{n\to\infty} \left(n \log \frac{a_n}{a_{n+1}}\right) = l$, **then the series is**

(i) **Convergent if $l > 1$.**

(ii) **Divergent if $l < 1$.**

(iii) **The test fail to describe the nature of the series if $l = 1$.**

Remarks:

(i) Logarithmic test applied only when ratio test fails and it involves the exponential ‘e’.

(ii) Logarithmic test is alternative form of Raabe’s test.

De Morgans and Bertrand's Test (D&B Test):

Suppose $\sum a_n$ be series of positive term such that

$$\lim_{n\to\infty}\left[\left\{n\left(\frac{a_n}{a_{n+1}}-1\right)-1\right\}\log n\right]=l, \text{ then the series is}$$

(i) **Convergent if $l > 1$.**

(ii) **Divergent if $l < 1$.**

Remark:

This test is applied only when D'Alembert's Ratio test and Raab's test fail to describe nature of the series.

Higher(Second) Logarithmic Ratio Test:

OR

Altenative To Bertrand's Test:

Suppose $\sum a_n$ be series of positive term such that

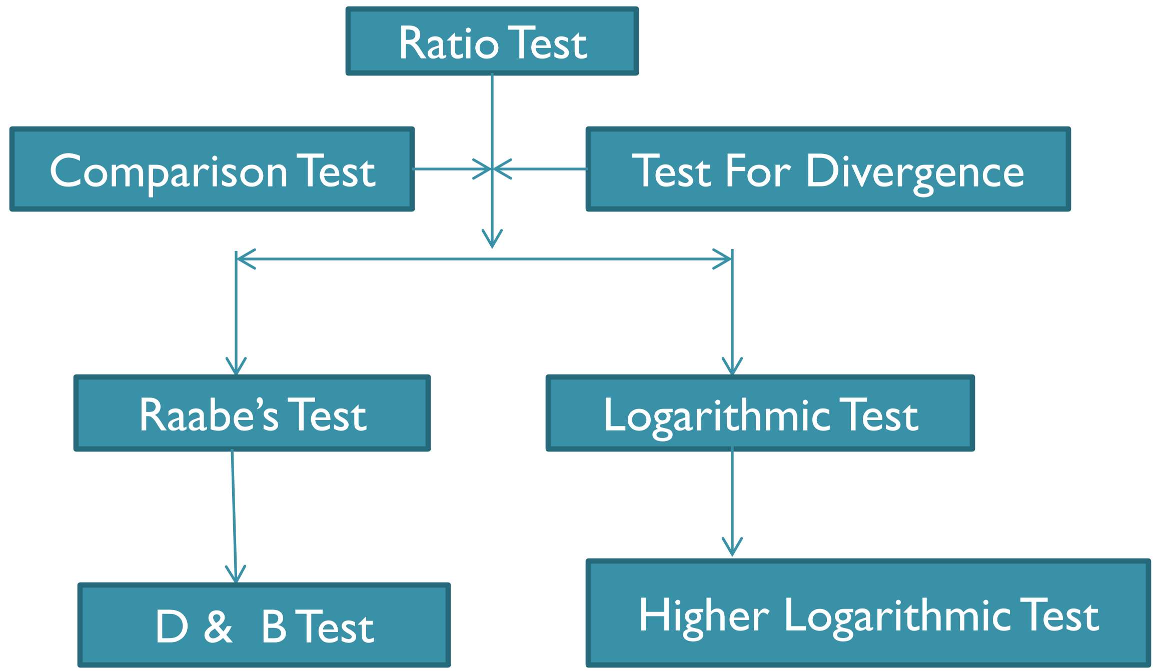
$$\lim_{n\to\infty}\left[\left\{n\left(\log\frac{a_n}{a_{n+1}}-1\right)\right\}\log n\right]=l, \text{ then the series is}$$

(i) **Convergent if $l > 1$.**

(ii) **Divergent if $l < 1$.**

Remark:

This test applied when Logarithmic test failed.


Ratio Test
Comparison Test
Test For Divergence
Raabe's Test
Logarithmic Test
D & B Test
Higher Logarithmic Test

Example: Test the convergence of following series

(i) $1+\frac{x}{1!}+\frac{2^2}{2!}x^2+\frac{3^3}{3!}x^3+\cdots$

(ii) $(1)^p+\left(\frac{1}{2}\right)^p+\left(\frac{1.3}{2.4}\right)^p+\cdots$

Solution: Here $a_n=\frac{n^n x^n}{n!}$ $\{neglecting\ First\ term\}$

$$a_{n+1}=\frac{(n+1)^{n+1}x^{n+1}}{(n+1)!}$$

$$\lim_{n\to\infty}\frac{\boldsymbol{a}_{\mathbf{n}}}{\boldsymbol{a}_{\mathbf{n+1}}}=\lim_{n\to\infty}\left(\frac{\boldsymbol{n}}{\boldsymbol{n}+\mathbf{1}}\right)^{\boldsymbol{n}}\frac{\mathbf{1}}{\boldsymbol{x}}=\lim_{n\to\infty}(\mathbf{1}+\mathbf{1}\backslash\mathrm{n})^{-\boldsymbol{n}}\frac{\mathbf{1}}{\boldsymbol{x}}=\frac{\mathbf{1}}{\boldsymbol{ex}}$$

By D'Alembert's Ratio Test the series is convergent if $\frac{\mathbf{1}}{\boldsymbol{ex}}>\mathbf{1}\ \boldsymbol{i.e.x}<\mathbf{1}\backslash\mathrm{e}$ and divergent if $\frac{\mathbf{1}}{\boldsymbol{ex}}<\mathbf{1}\ \boldsymbol{i.e.x}>\frac{\mathbf{1}}{\boldsymbol{e}}$ and the test fail if x=1\e.

Now when $x=\frac{1}{e}$

$$\frac{\boldsymbol{a}_{\mathbf{n}}}{\boldsymbol{a}_{\mathbf{n+1}}}=\boldsymbol{e}(\mathbf{1}+\mathbf{1}\backslash\mathrm{n})^{-\boldsymbol{n}}$$

$$\boldsymbol{nlog}\frac{\boldsymbol{a}_{\mathbf{n}}}{\boldsymbol{a}_{\mathbf{n+1}}}=\mathrm{nlog}\,\boldsymbol{e}-\boldsymbol{n}^{\mathbf{2}}\log(\mathbf{1}+\mathbf{1}\backslash\mathrm{n})$$

$$\boldsymbol{n}\log\frac{\boldsymbol{a}_{\mathbf{n}}}{\boldsymbol{a}_{\mathbf{n+1}}}=\boldsymbol{n}-\boldsymbol{n}^{\mathbf{2}}\left(\mathbf{1}\backslash\mathrm{n}-\mathbf{1}\backslash\mathbf{2}\boldsymbol{n}^{\mathbf{2}}+\mathbf{1}\backslash\mathbf{3}\boldsymbol{n}^{\mathbf{3}}-\cdots\right)$$

$$\lim_{n\to\infty}\boldsymbol{n}\log\frac{\boldsymbol{a}_{\mathbf{n}}}{\boldsymbol{a}_{\mathbf{n+1}}}=\lim_{n\to\infty}\left(\frac{\mathbf{1}}{\mathbf{2}}-\frac{\mathbf{1}}{\mathbf{3}\boldsymbol{n}}+\cdots\right)=\frac{\mathbf{1}}{\mathbf{2}}<\mathbf{1}$$

So by logarithmic test, given series is divergent if x=1\e

Finally given series is convergent if $\boldsymbol{x}<\frac{\mathbf{1}}{\boldsymbol{e}}$ and diverges if $\boldsymbol{x}\geq\frac{\mathbf{1}}{\boldsymbol{e}}$.

(II). Here $a_n = \left[\frac{1.3.5....(2n-1)}{2.4.6....2n}\right]^p$ $\qquad a_{n+1} = \left[\frac{1.3.5....(2n+1)}{2.4.6....2n+2}\right]^p$

$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \lim_{n\to\infty} \left[\frac{(2n+2)}{2n+1}\right]^p = 1$$

So D'Alembert's test fail to describe nature of series. Now we applying Logarithmic test:

$$n \log \frac{a_n}{a_{n+1}} = n \log \left[\frac{(2n+2)}{2n+1}\right]^p = np \log \frac{(1+1\backslash n)}{(1+1\backslash 2n)}$$

$$\lim_{n\to\infty} n\, log \frac{a_n}{a_{n+1}} = \lim_{n\to\infty} np \left[\left(\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \cdots\right) - \left(\frac{1}{2n} - \frac{1}{22^2n^2} + \frac{1}{32^3n^3} - \cdots\right)\right]$$

$$= \lim_{n\to\infty} \text{p} \left[\frac{1}{2} - \frac{3}{8n} + \frac{7}{24n^2} - \cdots\right] = \frac{p}{2}$$

So by Logarithmic test, the series is convergent if $\frac{p}{2} > 1$ i.e. $\text{p} > 2$ and if $\frac{p}{2} < 1$ $i.e. p < 2$ and test fail at p=2.

Now applying Higher Logarithmic Test

$$n\, log \frac{a_n}{a_{n+1}} - 1 = 2\left(\frac{1}{2} - \frac{3}{8n} + \frac{7}{24n^2} - \cdots\right) - 1$$

$$\lim_{n\to\infty} \left(n\, log \frac{a_n}{a_{n+1}} - 1\right) \log n = \lim_{n\to\infty} \left(-\frac{3}{4} + \frac{7}{12n} - \cdots\right) \frac{\log n}{n} = 0 < 1$$

Therefore by Higher Logarithmic Test the series is divergent if p=2.

Thus finally given series converges if p>2 and diverges if p<=2.

Example: Test the convergence of the series:

(i) $\frac{1^2}{2^2} + \frac{1^2.3^2}{2^2 4^2} x + \frac{1^2.3^2.5^2}{2^2 4^2.6^2} x^2 + \cdots$

(ii) $x + x^{1+1\backslash 2} + x^{1+1\backslash 2+1\backslash 3} + x^{1+1\backslash 2+1\backslash 3+1\backslash 4} + \cdots$

Solution: Here $a_n = \frac{1^2.3^2....(2n-1)^2}{2^2 4^2....(2n)^2} x^{n-1}$ $a_{n+1} = \frac{1^2.3^2....(2n+1)^2}{2^2 4^2....(2n+2)^2} x^n$

$$\lim_{n\to\infty} \frac{\boldsymbol{a}_{\mathbf{n}}}{\boldsymbol{a}_{\mathbf{n+1}}} = \lim_{n\to\infty} \left(\frac{2n+2}{2n+1}\right)^{\mathbf{2}} \frac{\mathbf{1}}{\boldsymbol{x}} = \frac{\mathbf{1}}{\boldsymbol{x}}$$

Therefore by D'Alembert's Test the given series is convergent if $\frac{\mathbf{1}}{\boldsymbol{x}} > \mathbf{1}$ $\boldsymbol{i.e.x} < \mathbf{1}$ and divergent if $\frac{\mathbf{1}}{\boldsymbol{x}} < \mathbf{1}$ $\boldsymbol{i.e.x} > \mathbf{1}$ and the test fail if x=1

Now when x=1

$$\frac{\boldsymbol{a}_{\mathbf{n}}}{\boldsymbol{a}_{\mathbf{n+1}}} - \mathbf{1} = \left(\frac{\mathbf{2}\boldsymbol{n}+\mathbf{2}}{\mathbf{2}\boldsymbol{n}+\mathbf{1}}\right)^{\mathbf{2}} - \mathbf{1}$$

$$\lim_{n\to\infty} \boldsymbol{n}\left(\frac{\boldsymbol{a}_{\mathbf{n}}}{\boldsymbol{a}_{\mathbf{n+1}}} - \mathbf{1}\right) = \lim_{n\to\infty} \boldsymbol{n} \times \frac{\mathbf{4}\boldsymbol{n}+\mathbf{3}}{(\mathbf{2}\boldsymbol{n}+\mathbf{1})^{\mathbf{2}}} = \mathbf{1}$$

Therefore Raabe's test fail to describe nature of series, now applying DeMorgans and Bertrand test

$$\left\{\boldsymbol{n}\left(\frac{\boldsymbol{a}_{\mathbf{n}}}{\boldsymbol{a}_{\mathbf{n+1}}} - \mathbf{1}\right)\right\} - \mathbf{1} = \boldsymbol{n} \times \frac{\mathbf{4}\boldsymbol{n}+\mathbf{3}}{(\mathbf{2}\boldsymbol{n}+\mathbf{1})^{\mathbf{2}}} - 1$$

$$\lim_{n\to\infty} \left[\left\{\boldsymbol{n}\left(\frac{\boldsymbol{a}_{\mathbf{n}}}{\boldsymbol{a}_{\mathbf{n+1}}} - \mathbf{1}\right) - \mathbf{1}\right\} \log \boldsymbol{n}\right] = \lim_{n\to\infty} \frac{-\boldsymbol{n}-\mathbf{1}}{(\mathbf{2}\boldsymbol{n}+\mathbf{1})^{\mathbf{2}}} \log \boldsymbol{n} = \lim_{n\to\infty} \frac{-\mathbf{1}-\mathbf{1}\backslash\mathrm{n}}{(\mathbf{2}+\mathbf{1}\backslash\mathrm{n})^{\mathbf{2}}} \frac{\log \boldsymbol{n}}{\boldsymbol{n}} = \mathbf{0} < \mathbf{1}$$

Thus by D & B Test given series is divergent when x=1. hence finally series is convergent if x<1 and divergent if x>=1.

Cauchy's Condensation Test:

If f(a) be a positive function of positive integral value of n and f(n) is a monotonically decreasing function of n $\forall n \in N$ then the two infinite series $\sum f(n)$ and $\sum a^n f(a^n)$ converge or diverge together, here a being a positive integer greater than unity.

OR

If f(n) be a positive monotonically decreasing function of n $\forall n \in N$, then the two series f(1)+f(2)+f(3)+ ……..+f(n) and $af(a) + a^2 f(a^2) + a^3 f(a^3) + \cdots + a^n f(a^n)$ converge or diverge together, here a being a positive integer greater than unity.

Remark:

This test generally applied when a_n contains $\log n$.

Theorem:

The auxiliary series $\sum_{n=1}^{\infty} \frac{1}{n(\log n)^p}$ is convergent if $p > 1$ and divergent if $p \leq 1$.

Proof: **Case I** if $p \leq 0$

Then $\frac{1}{n(\log n)^p} \geq \frac{1}{n}$ for $n \geq 2$

So by comparison test since $\sum \frac{1}{n}$ is divergent so the series $\sum \frac{1}{n(\log n)^p}$ is also divergent.

Case II If $p > 0$ the consider

$$f(n) = \frac{1}{n(\log n)^p}$$

Here f(n) is positive for all $n \geq 2$ and since $n(\log n)^p$ is increasing sequence so $\frac{1}{n(\log n)^p}$ i.e. f(n) is decreasing sequence. Hence by Cauchy's condensation test should apply. By CCT the series $\sum f(n)$ is convergent or divergent according as $\sum a^n f(a^n)$ is convergent or divergent.

Now $a^n f(a^n) = \frac{a^n}{a^n(\log a^n)^p} = \frac{1}{(n \log a)^p} = \frac{1}{n^p}\frac{1}{(\log a)^p}$

Since $\frac{1}{(\log a)^p}$ is constant so the series $\sum a^n f(a^n)$ is converges or diverges as $\sum \frac{1}{n^p}$ is convergent or divergent.

Since $\sum \frac{1}{n^p}$ is convergent if p>1 and divergent if $p \leq 1$. hence by CCT the given series $\sum \frac{1}{n(\log n)^p}$ is convergent if p>1 and diverges $p \leq 1$.

Example: test the convergence of the series

(i) $\sum \frac{(\log n)^2}{n^2}$

Solution: The n^{th}term of the series $a_n = f(n) = \frac{(\log n)^2}{n^2}$

Which is positive for n≥ 2and monotonically decreasing as n increases.

Now $a^n f(a^n) = a^n \frac{(\log a^n)^2}{a^{2n}} = \frac{n^2(\log a)^2}{a^n}$

Where a being a positive integer greater than one .

Consider $\sum a^n f(a^n) = \sum a^n \frac{(\log a^n)^2}{a^{2n}} = \sum b_n(say)$

$b_n = \frac{n^2(\log a)^2}{a^n}$ $\quad\quad$ $b_{n+1} = \frac{(n+1)^2(\log a)^2}{a^{n+1}}$

$$\lim_{n\to\infty} \frac{b_n}{b_{n+1}} = \lim_{n\to\infty} \frac{an^2}{n^2(1 + 1\backslash n^2)} = a > 1$$

So by D'Alemberts Ratio test the series $\sum b_n$ $i.e. \sum a^n f(a^n)$ is convergent.

Hence by CCT the series $\sum \frac{(\log n)^2}{n^2}$ is convergent.

Convergence Of Infinite Integrals:

The infinite integrals $\int_1^\infty f(x)dx$ is said to be convergent (divergent) if $\lim_{t\to\infty} \int_1^t f(x)dx \quad is\ finite\ (or\ infinite)$.

Definition:

Let f(x) be real valued function with domain $[1,\infty)$ the function f(x) is said to be non negative if $f(x) \geq 0\ \forall x \geq 1$ and f(x) is said to be monotonically decreasing if $x \leq y\ implies\ f(x) \geq f(y) \quad x, y \in [1,\infty)$.

Cauchy's Integral Test:

If f(x) is nonnegative monotonically decreasing and integrable function such that f(n) $= a_n\ \forall n \in N$

Then the series $\sum_{n=1}^{\infty} a_n$ is convergent and the integral $\int_1^\infty f(x)dx$ are either both convergent or both divergent.

Example: Test the convergence of the series by using Cauchy's integral test

(i) $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

(ii) $\sum \frac{1}{n\,(\log n)^p}$

Solution: Here $f(x) = \frac{1}{x^2+x}$ so that $f(n) = a_n$

For $x \geq 1, f(x)$ is nonnegative, decreasing and integrable function. Now

$$\int_1^t \frac{dx}{x(x+1)} = \int_1^t \frac{1}{x} - \frac{1}{x+1}\,dx = \Big|_1^t\left(\log\frac{x}{x+1}\right) = \log\frac{t}{t+1} - log2$$

Therefore $\lim_{t\to\infty} \int_1^t \frac{dx}{x(x+1)} = log1 + log2 = log2 \quad (finite)$

Hence the integral $\int_1^{\infty} \frac{dx}{x(x+1)}$ is convergent and so by Cauchy's integral test the given series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is also convergent.

Alternating Series

A series of the form $a_1 - a_2 + a_3 - a_4 + \cdots + (-1)^{n-1}a_n + \cdots$ where $a_n > 0 \forall n \in N$ is called an alternating series and denoted by

$$\sum_{n=1}^{\infty} (-1)^{n-1} a_n$$

Leibnitz Test:

An alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}a_n$ where $a_n > 0 \forall n \in N$ is convergent if

(i) $a_{n+1} \leq a_n \quad \forall n \in N$

(ii) $\lim\limits_{n\to\infty} a_n = 0$

Example: Test the convergence of the series

(i) $1 - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \cdots (\mathrm{p} > 0)$

(ii) $\frac{log2}{2^2} - \frac{log3}{3^2} + \frac{log4}{4^2} - \cdots$

Solution: Here $a_n = \frac{1}{n^p}$ $\qquad a_{n+1} = \frac{1}{(n+1)^p}$

$$a_n - a_{n+1} = \frac{1}{n^p} - \frac{1}{(n+1)^p} > 0 \quad , \quad p > 0$$

$$a_n - a_{n+1} > 0 \qquad a_n > a_{n+1} \qquad \forall n$$

And $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \frac{1}{n^p} = 0 \qquad (since\ p > 0)$

Hence by Leibnitz test the given series is convergent.

(ii) Here $a_n = \frac{\log(n+1)}{(n+1)^2} \qquad \forall n \in N$

$\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \frac{\log(n+1)}{(n+1)^2} = 0 \qquad$ [since $\lim\limits_{n\to\infty} \frac{\log n}{n^2} = 0$]

Next to show that $a_{n+1} \leq a_n \quad \forall n \in N$

Let $f(x) = \frac{logx}{x^2}$ $\quad f'(x) = \frac{x^2 1\backslash x - 2xlogx}{x^4} < 0 \quad \forall\, x > e^{1\backslash 2}$

(Since $x > e^{1\backslash 2} \leftrightarrow logx > 1\backslash 2 \leftrightarrow 1 - 2logx < 0$)

Which implies that f(x) is decreasing function $\forall\, x > e^{1\backslash 2}$

Thus f(n+2) ≤f(n+1) $\quad \forall \boldsymbol{n} \in \boldsymbol{N} \qquad (\boldsymbol{n+2 > n+1} > e^{1\backslash 2})$

$$\frac{\log(n+2)}{(n+2)^2} \leq \frac{\log(n+1)}{(n+1)^2} \qquad \forall \boldsymbol{n} \in \boldsymbol{N}$$

This shows that $\boldsymbol{a_{n+1} \leq a_n} \quad \forall \boldsymbol{n} \in \boldsymbol{N}$

Hence by Leibnitz's test the given series is convergent.

Absolute convergence:

A series $\sum \boldsymbol{a_n}$ is said to be absolutely convergent if the series $\sum|\boldsymbol{a_n}|$ is convergent.

Example: Consider a series $\sum \boldsymbol{a_n} = \boldsymbol{1 - 1\backslash 2 + 1\backslash 2^2 - 1\backslash 2^3} + \cdots$ which is absolutely convergent

Since $\sum|\boldsymbol{a_n}| = \boldsymbol{1 + 1\backslash 2 + 1\backslash 2^2 + 1\backslash 2^3} + \cdots$

Which is geometric series with common ratio r=1\2<1 then the series $\sum|\boldsymbol{a_n}|$ is convergent. Hence $\sum \boldsymbol{a_n}$ is absolutely convergent.

Note: The given series is also convergent

$$\sum \boldsymbol{a_n} = \boldsymbol{1 - 1\backslash 2 + 1\backslash 2^2 - 1\backslash 2^3} + \cdots$$

Clearly $\boldsymbol{a_{n+1} \leq a_n} \quad \forall \boldsymbol{n} \in \boldsymbol{N}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{2^n} = 0$$

So by Leibnitz's test the given series is convergent.

<u>Remark</u>: Every absolutely convergent series is always convergent but converse need not be true.

<u>Example</u>: consider a series

$\sum a_n = 1 - 1\backslash 2 + 1\backslash 3 - 1\backslash 4 + \cdots$

Clearly $a_{n+1} < a_n \quad \forall n \in N$

And $\lim_{n\to\infty} a_n = \lim_{n\to\infty} 1\backslash n = 0$

So by Leibnitz's test the given series is convergent.

But $\sum|a_n| = 1 + 1\backslash 2 + 1\backslash 3 + 1\backslash 4 + \cdots = \sum \frac{1}{n}$, which is divergent

Hence $\sum a_n$ is not absolutely convergent.

<u>Conditional Convergence:</u>

A series $\sum a_n$ is said to be conditionally convergent, if

(i) $\sum a_n$ is convergent.

(ii) $\sum a_n$ is not absolutely convergent.

Example $\sum a_n = 1 - 1\backslash 2 + 1\backslash 3 - 1\backslash 4 + \cdots$ is conditionally convergent.

Since $\sum a_n$ is convergent, but $\sum|a_n|$ is not convergent.

Summery of tests:

Let us take a series $\sum a_n$ of positive term, now to check the convergence of series we proceed as follows:

1. First find $\lim\limits_{n\to\infty} a_n$.

i. If $\lim\limits_{n\to\infty} a_n \neq 0$ then series is divergent.

ii. If $\lim\limits_{n\to\infty} a_n = 0$ then series may or may not be convergent.

2. In this case we apply comparison test if a_n is algebraic function in n

First comparison Test **Limit for Comparison Test**

3. If in a_n, n as an exponent form then Cauchy's nth root test applied .

4. If above test fail and a_n contains the term of logn then Cauchy's Condensation tesr must be applied.

5. Next to find the nature of series D'Alembert's Ratio Test should be applied.

6. If this test fails then apply Raabe's test.

7. Again if Raabe's test fails for l=1, then immediately DeMorgan's and Bertrand test must be applied.

8. If $\frac{a_n}{a_{n+1}} - 1$ cannot be easily calculated then evaluate $\log\frac{a_n}{a_{n+1}}$, and apply Logarithmic test and after failure if this test we always use Higher Logarithmic Test.