Numerical Analysis

Course:- B.Sc. III

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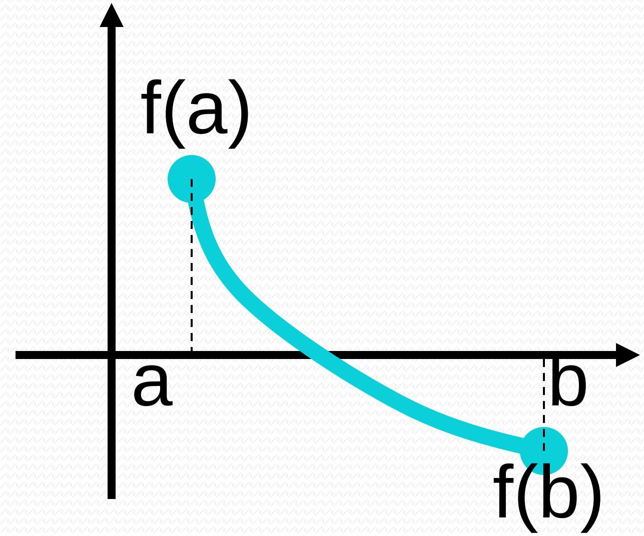
OUTLINES

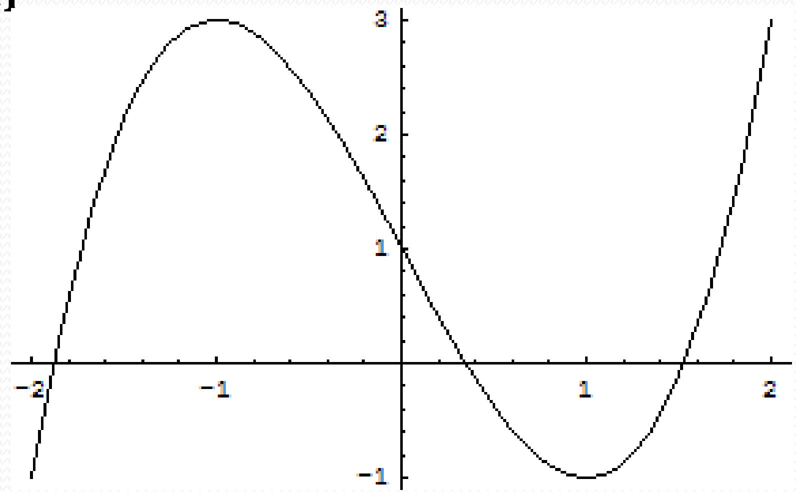
Numerical Methods

- Bisection Method
- Regula-falsi Methods
- Newton-Raphson Method

Intermediate Value Theorem

If a function f(x) is continuous on some interval [a,b] and f(a) and f(b) have different signs then the equation f(x)=0 has at least one real root (zero) or an odd number of real roots in the interval [a,b].



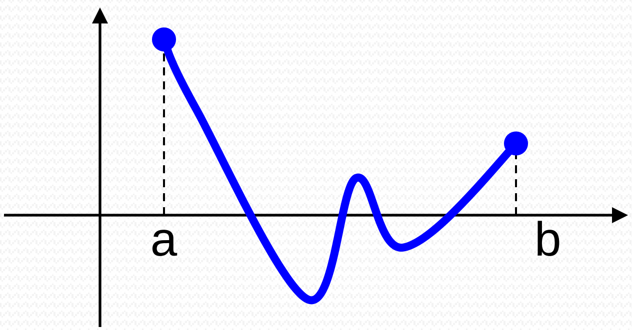


Bisection Method

- The **Bisection method** is one of the simplest methods to find a zero of a nonlinear function.
- To use the Bisection method, one needs an initial interval that is known to contain a zero of the function.
- The method systematically reduces the interval. It does this by dividing the interval into two equal parts, performs a simple test and based on the result of the test half of the interval is thrown away.
- The procedure is repeated until the desired interval size is obtained.

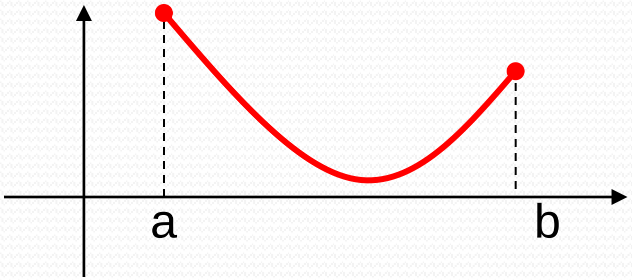
Examples

- If f(a) and f(b) have the same sign, the function may have an even number of real zeros or no real zeros in the interval [a, b].



The function has four real zeros

- Bisection method can not be used in these cases.



The function has no real zeros

Bisection Method

<u>Assumptions:</u>

Given an interval [a,b]

f(x) is continuous on [a,b]

f(a) and f(b) have opposite signs.

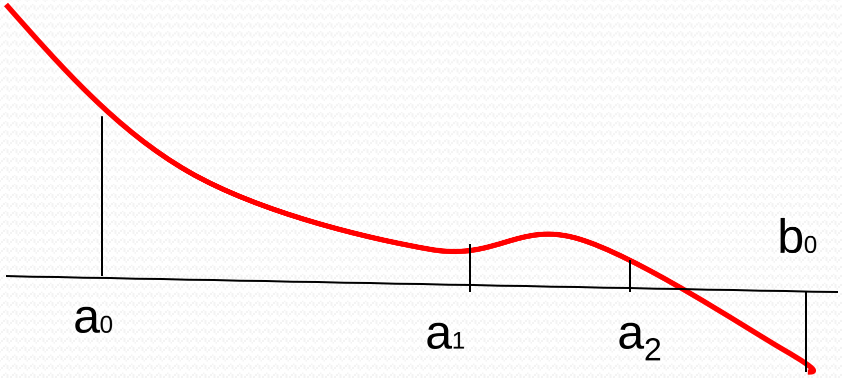
These assumptions ensures the existence of at least one zero in the interval [a,b] and the bisection method can be used to obtain a smaller interval that contains the zero.

Stopping Criteria

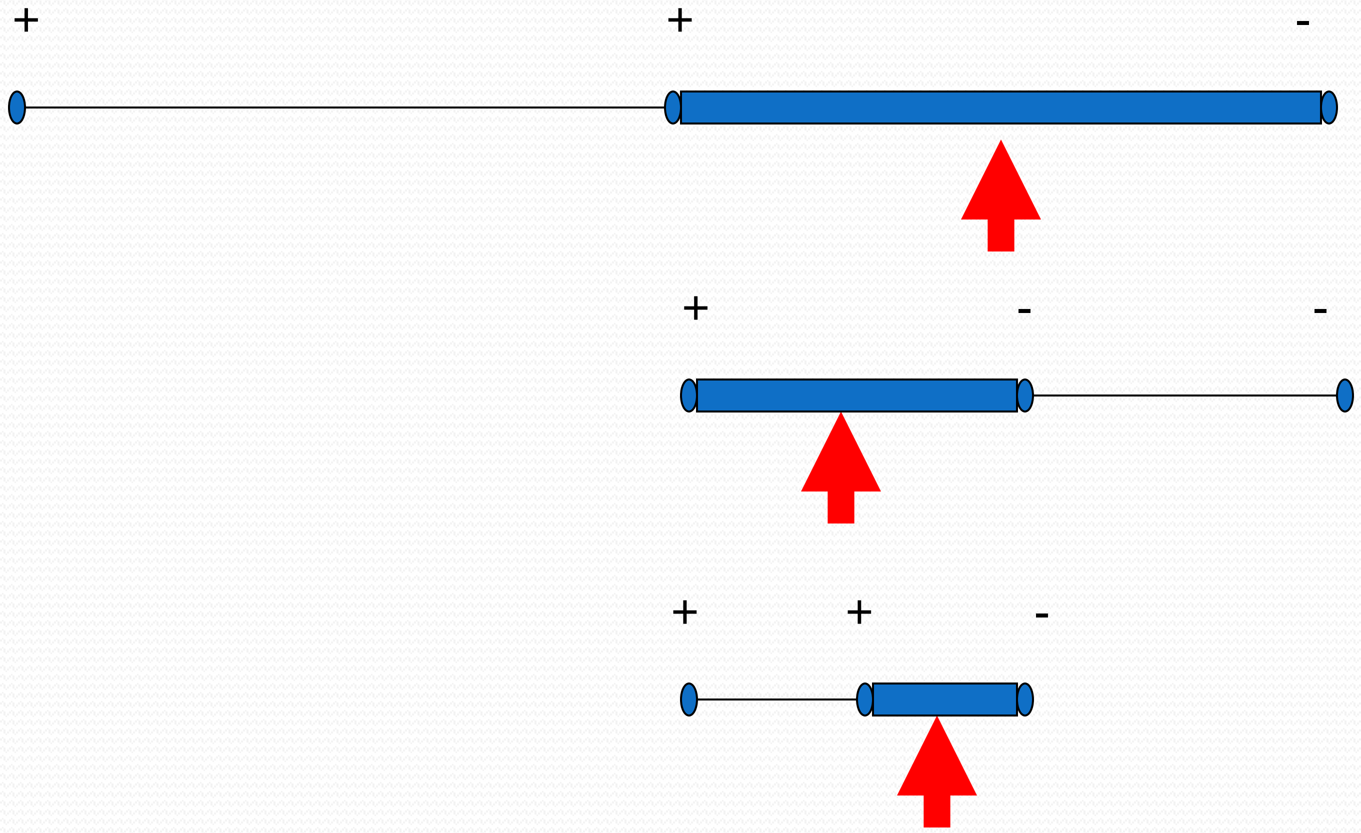
Two common stopping criteria

1. Stop after a fixed number of iterations.
2. Stop when the two approximate values x_n and x_{n+1} are equal.

Bisection Method



Example



Example

Can you use Bisection method to find a zero of :

$f(x) = x^3 - 3x + 1$ in the interval [0,2]?

Answer:

$f(x)$ is continuous on [0,2]

and $f(0) * f(2) = (1)(3) = 3 > 0$

$\Rightarrow$ Assumptions are not satisfied

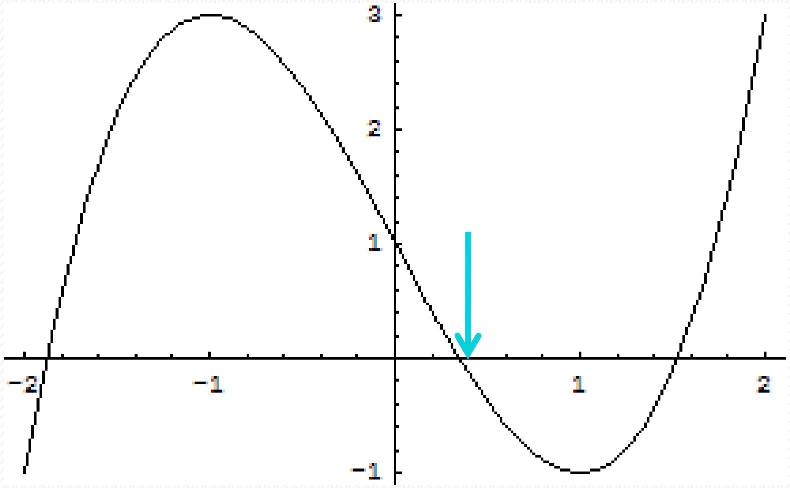
$\Rightarrow$ Bisection method can not be used

Example:

Can you use Bisection method to find a zero of

$f(x) = x^3 - 3x + 1$ in the interval [0,1]?

Answer:



$f(x)$ is continuous on [0,1]

$f(0) * f(1) = (1)(-1) = -1 < 0$

Assumptions are satisfied

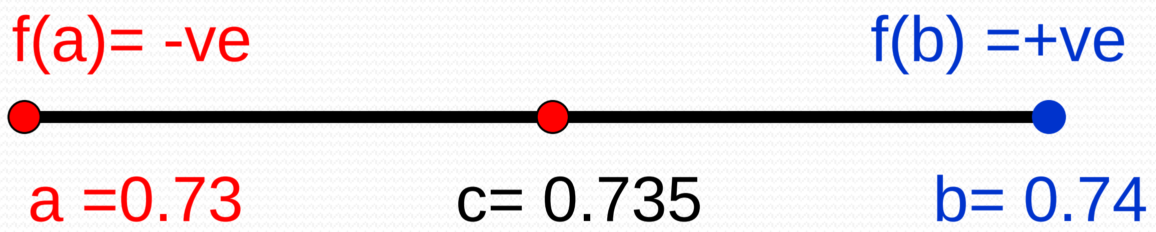
Bisection method can be used

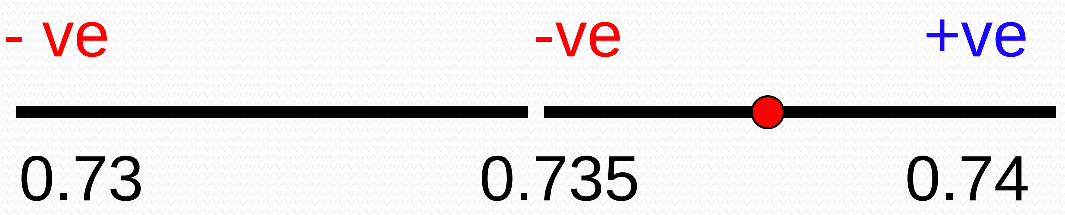
Example

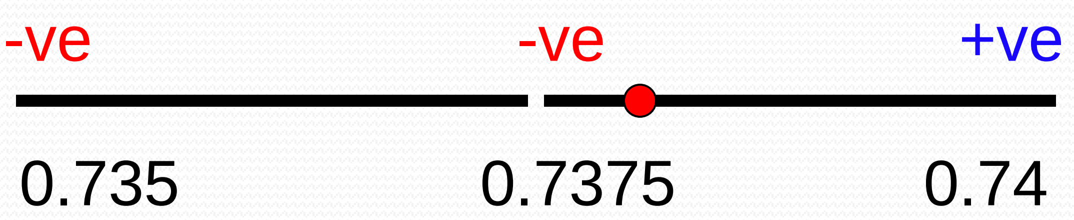
- Use Bisection method to find a root of the equation
  x = cos (x).
  (assume the initial interval [0.73,0.74])

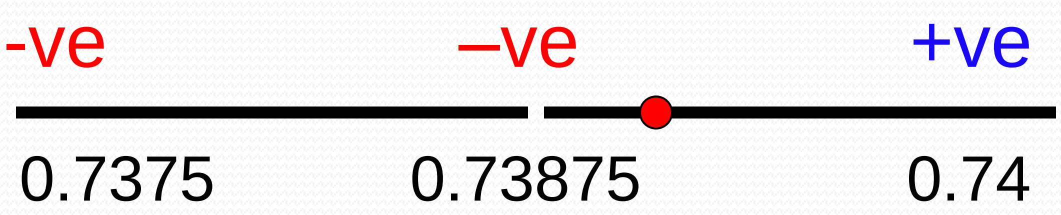
Question 1: What is $f(x)$?
Question 2: Are the assumptions satisfied ?

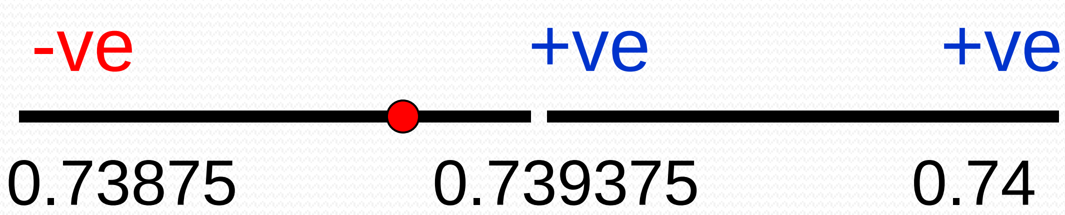
Bisection Method
Initial Interval




- ve
-ve
+ve
0.73
0.735
0.74


-ve
-ve
+ve
0.735
0.7375
0.74


-ve
–ve
+ve
0.7375
0.73875
0.74


-ve
+ve
+ve
0.73875
0.739375
0.74

Summary

- Initial interval containing the root [0.73,0.74]
- After 8 iterations
  - Interval containing the root [0.7390625 ,0.73914]
  - Best estimate of the root is 0.73910.

Bisection Method

Advantages

- Simple and easy to implement
- One function evaluation per iteration
- The size of the interval containing the zero is reduced by 50% after each iteration
- No knowledge of the derivative is needed
- The function does not have to be differentiable

Disadvantage

- Slow to converge
- Good intermediate approximations may be discarded

Regula Falsi Method

- The convergce process in the bisection method is very slow.
- It depends only on the choice of end points of the interval [a,b].
- The function f(x) does not have any role in finding the point c (which is just the mid-point of a and b).
- It is used only to decide the next smaller interval [a,c] or [c,b].

Consider the equation f(x)=0 and let a and b be two values of x that f(a) and f(b) are of opposite signs. Also let a<b. the graph of y=f(x) will meet the x-axis at the same point between a and b, the equation chord joining the two points [a, f(a)] and [b, f(b)] is

$$\frac{y-f(a)}{x-a}=\frac{f(b)-f(a)}{b-a}$$

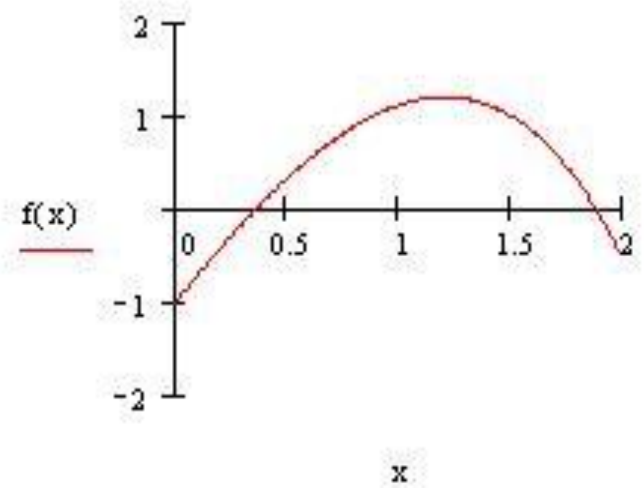
in the small interval (a, b) the graph of the function can be considered as a straight line. So that x-coordinate of the point of intersection of the chord joining [a, f(a)] and [b, f(b)] with the x-axis will give an approximate value of the root. So putting y=0.

$$\frac{f(a)}{x-a}=\frac{f(b)-f(a)}{b-a} \quad \rightarrow x=a-\frac{f(a)}{f(b)-f(a)}(b-a)$$

$$or \quad x=\frac{a\,f(b)-b\,f(a)}{f(b)-f(a)}$$

Find a root of 3x + sin(x) - exp(x) = 0.

- The graph of this equation is given in the figure.
- From this it's clear that there is a root between 0 and 0.5 and also another root between 1.5 and 2.0.
- Now let us consider the function f (x) in the interval [0, 0.5] where f (0) * f (0.5) is less than zero and use the regula-falsi scheme to obtain the zero of f (x) = 0.



Iteration No.	a	b	c	f(a) * f(c)
1	0	0.5	0.376	1.38 (+ve)
2	0.376	0.5	0.36	-0.102 (-ve)
3	0.376	0.36	**0.36**	-0.085 (-ve)

Newton-Raphson Method
(also known as Newton's Method)

Given an initial guess of the root x_0 , Newton-Raphson method uses information about the function and its derivative at that point to find a better guess of the root.

Assumptions:

- f (x) is continuous and first derivative is known
- An initial guess x_0 such that $f'(x_0) \neq 0$ is given

Derivation of Newton's Method

Let x_0 be an approximate root of equation $f(x) = 0. If$

$x_1 = x_0 + \mathrm{h}$ be the exact root, then $\mathrm{f}(x_1) = 0$, then $\mathrm{f}(x_0 + \mathrm{h}) = 0$

The Taylor's expansion -

$$f(x_0) + hf'(x_0) + \frac{h^2}{2} f''(x_0) + \ldots\ldots = 0$$

since h is small, neglecting h^2 and higher power of h.

$$f(x_0) + hf'(x_0) = 0 \Rightarrow h \approx -\frac{f(x)}{f'(x)}$$

$\therefore$ A closer approximation to the root is given by

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

In general, $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

which is known as the Newton-Raphson formula or Newton's iteration formula.

Example

Find a zero of the function $f(x) = x^3 - 2x^2 + x - 3.$

Solution : $x_0 = 4,\ f'(x) = 3x^2 - 4x + 1$

Iteration 1: $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)} = 4 - \dfrac{33}{33} = 3$

Iteration 2: $x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)} = 3 - \dfrac{9}{16} = 2.4375$

Iteration 3: $x_3 = x_2 - \dfrac{f(x_2)}{f'(x_2)} = 2.4375 - \dfrac{2.0369}{9.0742} = 2.2130$

Summary

Bisection, Regula Falsi Method	Reliable, Slow One function evaluation per iteration Needs an interval [a,b] containing the root, f(a) f(b)<0 No knowledge of derivative is needed
Newton Raphson Method	Fast (if near the root) but may diverge Two function evaluation per iteration Needs derivative and an initial guess x_0, $f'(x_0)$ is nonzero